18.155 LECTURE 8, 1 OCTOBER, 2013

Last time I defined the support of a distribution and showed that the) tempered distributions of compact support in an open set Ω form the dual space of $\mathcal{C}^\infty(\Omega)$ with the topology 'of uniform convergence of all derivatives on compact subsets'. Then the space $\mathcal{C}^{-\infty}(\Omega) = \mathcal{D}'(\Omega)$ of distributions on Ω is the dual of $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ with the inductive limit topology of the subspaces $\mathcal{C}^\infty_{\mathrm{c}}(K_n) \subset \mathcal{S}(\mathbb{R}^n)$ over any exhaustion of Ω by compact sets.

The first thing I want to do today is to identify the distributions supported at one point.

Proposition 1. *Any distribution with support contained in $\{p\} \subset \Omega$ for any open set Ω is a finite sum of derivatives of Dirac deltas:*

$$\sum_{|\alpha| \le N} c_\alpha D^\alpha \delta_p. \tag{1}$$

Suppose $u \in \mathcal{C}^{-\infty}(\Omega)$ has support in $\{p\}$. What this means is that if $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ and $p \notin \mathrm{supp}(\phi)$ then $u(\phi) = 0$. I showed last time that (using a cutoff in a neighbourhood of the support) we can always interpret $u \in \mathcal{S}'(\mathbb{R}^n)$ with the same support. In fact it is easier to think of $u \in \mathcal{C}^{-\infty}_{\mathrm{c}}(B(p,1))$, the open ball around p. So what we know is that if $\phi \in \mathcal{C}^\infty(B(p,1))$ and $\phi \equiv 0$ in $|x-p| < \epsilon$ for any $\epsilon > 0$ then $u(\phi) = 0$. To get further we need to use the continuity of u as a distribution – which means here that

$$|u(\phi)| \le \sup_{|\alpha| \le M,\ |x-p| \le 1/2} |D^\alpha \phi| \ \forall\ \phi \in \mathcal{C}^\infty(B(p,1)) \tag{2}$$

where the $1/2$ can be replaced by any fixed $\epsilon > 0$.

The norm on the right is the $\mathcal{C}^M$ norm on the closed ball of radius $1/2$. What we know is that $u(\phi) = 0$ if $\phi = 0$ in $|x-p| < \delta$ for any $\delta > 0$.

Lemma 1. *The closure with respect to the $\mathcal{C}^M$ norm of smooth functions which vanish in a neighbourhood of 0 is the subspace of $\mathcal{C}^M(\overline{B(p,1/2)})$ consisting of the functions which vanish with all derivatives up to order M at p.*

Proof. In fact all we are really interested in is that this closure contains the infinitely differentiable functions which vanish at p with their derivatives up to order M.

So, suppose $\psi \in \mathcal{C}^\infty(B(p,1))$ is such a function. It follows that the derivatives $D^\beta_x \psi$ vanish at p to order $M+1-|\beta|$ and so by Taylor's theorem, or the mean-value theorem,

$$|D^\beta \psi(x)| \le C|x-p|^{M+1-|\beta|} \text{ in } |x-p| \le 1/2, \text{ for } |\beta| \le M. \tag{3}$$

Choose a cut-off function $\mu \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ which has support in $|x| \le 1$ and is identically equal to 1 in $|x| \le 1/2$. Then consider

$$\psi_j = (1 - \mu((x-p)j))\psi \in \mathcal{C}^\infty(B(p,1)). \tag{4}$$

Thus $\psi_j = 0$ in a neighbourhood of p (depending on j of course) and the difference $\psi - \psi_j = \mu((x-p)j)\psi$ vanishes in $|x-p| > 1/j$. To estimate the difference, differentiate out and use (3):

$$|D^\gamma(\psi-\psi_j)| \le C_\gamma \sum_{\beta\le\gamma} j^{|\beta|}|(D^\beta\mu)((x-p)j)||x-p|^{M+1-|\gamma|+|\beta|}.|\gamma|\le M. \tag{5}$$

The support condition on μ means this is bounded by C/j so $\psi_j \to \psi$ in the $\mathcal{C}^M$ norm on $|x-p| \le 1/2$.

The general case just requires approximation of M times differentiable functions by $\mathcal{C}^\infty$ functions plus a little care with the derivatives at p. □

So this is enough to prove the Proposition. If $\psi \in \mathcal{C}^\infty(B(p,1))$ then Taylor's formula lets us write

$$\psi(x) = \sum_{|a|\le M} c_\alpha D^\alpha\psi(p)(x-p)^\alpha + \psi' \tag{6}$$

where ψ' and its derivatives up to order M vanish at p. By the continuity of u and the Lemma, $u(\psi') = 0$ so

$$u(\psi) = \sum_{|\alpha|\le M} c_\alpha c'_\alpha D^\alpha\psi(p),\ c'_\alpha = u((x-p)^\alpha). \tag{7}$$

Since $(-1)^{|\alpha|}D\delta_p$ is the distribution which evaluates to $D^\alpha\psi(p)$ this is (1).

Said another way, the distributions with support at a point $p\in\mathbb{R}^n$ are tempered and precisely those with Fourier transforms of the form $p(\xi)\exp(-i\xi\cdot p)$.

So, with the same idea of trying to understand some concrete distributions let us consider the wider question of *homogeneous* distributions. For a function what we mean here is 'positive' homogeneity, meaning that

$$f(rx) = r^z f(x),\ x\in\mathbb{R}^n,\ r>0. \tag{8}$$

In fact it is better to ignore the origin altogether for the moment and think about homogeous functions on $\mathbb{R}^n\setminus\{0\}$. While most of the obvious examples have z real there is no great ambiguity for $z\in\mathbb{C}$ since $r>0$ so $r^z = \exp(z\log r)$ where $\log : (0,\infty) \longrightarrow \mathbb{R}$ is the 'standard branch'.

You worked on this in a problem set in one dimension which is somewhat special but very important. In higher dimensions there is a question which is absent in dimension 1, namely how smooth the function f is outside the origin. Let us just consider the case

$$f \text{ is smooth outside the origin.} \tag{9}$$

Then it is easy to see when it 'defines' a distribution $I(f)$ in our old notation – namely when it is locall integrable near 0 since it is certainly of polynomial growth. Again, unless it vanishes identically, this is when $\operatorname{Re} z > -n$ since then

$$|f(x)| \le C|x|^{-n+\epsilon},\ \epsilon>0 \tag{10}$$

and since the power is locally integrable and the function is measurable, it is locally integrable too. So then we can define f as a tempered distribution

$$f(\phi) = \int f(x)\phi(x)dx,\ \phi\in\mathcal{S}(\mathbb{R}^n) \tag{11}$$

since the integral is in $L^1(\mathbb{R}^n)$.

Now, this allows us to state the condition of homogeneity in a *weak* form by changing variable from x to $y = x/r$, for $r > 0$:

$$f(\phi) = \int f(x)\phi(x)dx = \int f(ry)\phi(ry)r^n dy = r^{n+z} \int f(y)\phi(ry)dy \\ i.e.\ I(f)(\phi(r\cdot) = r^{-n-z} I(f)(\phi). \tag{12}$$

Conversely, if f is such that $(1 + |x|^2)^{-N} f \in L^1$ for some N and (12) holds then (8) holds by the injectivity of the inclusion I.

The point then, is that we can take the second part of (12) as the definition of homogeneity of a distribution

$$u(\phi(r\cdot) = r^{-n-z} u(\phi)\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n)$$

For the moment we will also assume that in addition

$$u = f \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}) \text{ on } \mathbb{R}^n \setminus \{0\}. \tag{13}$$

The objective is to find all u satisfying () and (13) so we at least have some distributions to play with.

Now, first note that the function f must itself be homogenous of degree z outside the origin. Indeed we can deduce from

$$I(f)(\phi(r)) = r^{-n-z} I(f)(\phi) \tag{14}$$

for $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n \setminus \{0\})$, that(8) does hold on $\mathbb{R}^n \setminus \{0\}$. Thus we can restate the conditions in the stronger form

$$u(\phi(r\cdot) = r^{-n-z} u(\phi)\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \text{ and } u = |x|^z f(\hat{x}) \text{ in } |x| > 0 \text{ with } f \in \mathcal{C}^\infty(\mathbb{S}^{n-1}) \tag{15}$$

From this point of view the question becomes: When does the homogeneous function in $|x| > 0$ have a homogeneous distributional 'extension across 0'. So we know the answer when $\operatorname{Re} z > -n$ since then the RHS is locally integrable and $I(f)$ is such a homogenous extension! Moreover this extension is the only homogeneous extension. Were there two, the difference $u_2 - u_1$ would have to be supported at 0. Now, we can easily check from the definitions that

$$D^\alpha \delta_0 \text{ is homogeneous of degree } -n - |\alpha|. \tag{16}$$

A non-trival sum of homogenuous distributions of different homogeneities cannot be homogeneous and certainly the homogeneity here does not have $\operatorname{Re} z > -n$ so uniqueness follows from our earlier result.

So, we need to consider what happens if $\operatorname{Re} z \le -n$. The question of whether $|x|^z f(\hat{x})$ for a smooth function on the sphere has a homogeneous extension breaks into two pieces which are hidden a little by the result. The first question is: Does it have a distributional extension at all. The second is whether this extension can be chosen/arranged to be homogeneous. In fact the answer to the first question is always YES, it is the second that involves numerology.

Proposition 2. *If $f \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ then there exists a unique $u \in \mathcal{S}'(\mathbb{R}^n)$ which is homogenous of degree $z \in \mathbb{C}$ and equal to $|x|^z f(\hat{x})$ in $|x| > 0$ provided $z \neq -n - k$, with k a non-negative integer. For $z = -n - k$ such an extension exists if and only if*

$$\int_{\mathbb{S}^{n-1}} f(\hat{x})p(\hat{x}) = 0 \tag{17}$$

for any homogeneous polynomial of degree k and then the extension is unique up to a sum of k-fold derivatives of δ_0.

Even in case (17) fails to hold we can say what happens. In that case $|x|^{-n-k}f(\hat{x})$ has a 'quasi-homogeneous' extension, u which satisfies

$$u(\phi(r\cdot) = r^k u(\phi) + r^k \log r v(\phi),\ v \in \mathcal{S}'(\mathbb{R}^n) \tag{18}$$

where I am being a big disingenuous in that v has support at the origin. You can see more by replacing r by rs with r and $s > 0$ and applying (18) twice.

Notice that despite the exceptional behaviour at these special points $z = -n-k$, there is a certain elegance to what happens. Namely we only get a continuous extension if $f \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ satisfies $N(k)$ constraints, where $N(k)$ is the dimension of the homogeneous polynomials of degree k on $\mathbb{R}^n$. Then, there are magically an extra $N(k)$ homogeneous distributions of this degree which vanish outside the origin and are just of the form $p(D)\delta_0$ for the same homogeneous polynomials.

Proof. □