18.155 LECTURE 7, 26 SEPTEMBER, 2013

Last time I showed that if $K \Subset \Omega$ is a compact subset of an open subset of $\mathbb{R}^n$ and $K \subset \circ K' \subset K' \Subset \Omega$ is another compact subset which contains K in its interior then there exists $0 \le \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \subset \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ which has support in K' and $\phi = 1$ in some neighbourhood of K, which we can express as

$$\operatorname{supp}(\phi) \subset K',\ \operatorname{supp}(1-\phi) \cap K = \emptyset. \tag{1}$$

So in fact Ω plays no rôle here, except that we an find such a K'.

Lemma 1. *Suppose $\Omega = \sum\limits_{j=1}^{N} \Omega_j$ is a finite union of open sets and $\psi \in \mathcal{C}^\infty_c(\Omega)$ then there exist $\psi_j \in \mathcal{C}^\infty_c(\Omega_j)$ such that*

$$\psi = \sum_{j=1}^{N} \psi_j. \tag{2}$$

Proof. I did this last time for $j = 2$. Namely, each point of $\operatorname{supp}(\psi) \Subset \Omega_1 \cup \Omega_2$ is the centre of an open ball contained in either Ω_1 or Ω_2. By compactness, a finite collection of the balls of half the radius cover K. Take the finite number of points at the centres of those balls which lie in Ω_1 and let K_1 be the intersection of K with the union of the closures of these balls (of half the original radius) and similarly form K_2 from the remaining balls. Thus $K_i \Subset \Omega_i$ and $K = K_1 \cup K_2$. Then choose $\phi_1 \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega_1)$ with $\phi_1 = 1$ in a neighbourhood of K_1. It follows that

$$\psi = \phi_1\psi + (1-\phi_1)\psi \tag{3}$$

is a decomposition as desired, since the first term has compact support in Ω_1 and the second has support in $K_2 \subset \Omega_2$ because the second factor vanishes outside K and the first factor vanishes outside a neighbourhood of K_1.

The general case follows by induction over the number of open sets, since we can apply this result to Ω_1 and $\sum\limits_{j=2}^{N} W_j$. □

As a corollary of this we find that

(4) For each $u \in \mathcal{S}'(\mathbb{R}^n)$ there exists a unique largest open set on which $u = 0$.

Recall that $u = 0$ on an open set Ω means that $u(\phi) = 0$ for all $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$. The largest open set in (4) is the union of all open sets on which u vanishes provided we show that u does indeed vanish on this set. However, if $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ has support in this union, then by compactness, its support is contained in a finite subunion. Thus it can be split up as in (2) and it follows that

$$u(\psi) = \sum_{j=1}^{N} u(\psi_j) = 0 \tag{5}$$

since by assumption u vanishes on each of the sets in the union.

Thus, for a tempered distribution we can *define* the support by

$$\mathbb{R}^n \setminus \operatorname{supp}(u) = \text{The largest open set on which } u = 0. \tag{6}$$

Notice that this is consistent with the definition for continuous (and actually for integrable) functions and has properties such as

$$\phi \in \mathcal{S}(\mathbb{R}^n),\ \operatorname{supp}(\phi) \cap \operatorname{supp}(u) = \emptyset \Longrightarrow u(\phi) = 0 \tag{7}$$

as follows directly from the definition if $\operatorname{supp}(\phi)$. If $\operatorname{supp}(\phi)$ is not compact then it is not quite so obvious but if $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ then $u(\psi\phi) = 0$. However, we can choose a sequence $\psi_n \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ such that $\psi_n\phi \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$ so by the continuity of u it follows that $u(\phi) = 0$.

You might well take this proof to heart in thinking about the problem for this week where you seem to have to invert a polynomial which is just known to be non-zero on $\mathbb{R}^n$. In fact the inverse *is* of slow growth, which is what you need, but it is not necessary to know this! Just multiply by any smooth function of compact support, you can certainly invert the function on a neighourhood of its support, and then pass to the limit as above.

Thus $\operatorname{supp}(u)$ is closed. Let me use the notation, somewhat suggestively

$$\mathcal{C}^{-\infty}_{\mathrm{c}}(\Omega) = \{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{supp}(u) \Subset \Omega\}. \tag{8}$$

So we are thinking of these as 'infinitely *un*differentiable functions' of compact support in Ω.

Now, as I said last time we want to consider the space $\mathcal{C}^\infty(\Omega)$ of smooth functions on the open set Ω. We can always choose a compact *exhaustion* of Ω. That is an increasing sequence of compact subsets $K_n \Subset \overset{\circ}{K}_{n+1} \subset K_{n+1} \Subset \Omega$ such that

$$\Omega = \bigcup_n K_n. \tag{9}$$

Indeed, let K_n consist of the points satisfying $|x| \le n$ and at distance at least $1/n$ from $\complement\Omega$. For n large this is non-empty and clearly (9) holds. A compact exhaustion has the property that any $K \Subset \Omega$ is contained in one of the K_n since the interiors of the K_n must cover K and so have a finite subcover.

Now, given such a compact exhaustion we can consider the seminorms on $\mathcal{C}^\infty(\Omega)$:

$$\|\phi\|_n = \sum_{0\le|\alpha|\le n,\ x\in K_n} |D^\alpha\phi(x)|. \tag{10}$$

These are never norms, since ϕ might be non-zero with support disjoint from K_n. However, if all the norms vanish, then ϕ vanishes identically.

Proposition 1. *The seminorms* (10) *for any exhaustion make* $\mathcal{C}^\infty(\Omega)$ *into a Fréchet space with the distance*

$$d(\phi_1, \phi_2) = \sum_n 2^{-n} \frac{\|\phi_1 - \phi_2\|_n}{1 + \|\phi_1 - \phi_2\|_n} \tag{11}$$

The topology does not depend on the choice of compact exhaustion. Indeed, each of the norms for one exhaustion is bounded by the norms from some point onwards of the other. We know that a set is open if and only if it contains a ball around each of its points with respect to one of the seminorms (possibly depending on the point). Similarly we know that a linear function

$$U : \mathcal{C}^\infty(\Omega) \longrightarrow \mathbb{C} \text{ is continuous iff } \exists\ C, N \text{ s.t. } |U(\phi)| \le C\|\phi\|_N. \tag{12}$$

I did mention this when I carried out the proof for $\mathcal{S}(\mathbb{R}^n)$ – it is a general fact for Fréchet spaces (and the completeness is not used for this either).

So, I leave it up to you to at least mumble a proof to yourself.

Now, as I mentioned last time we can map

$$I : \mathcal{C}_{\mathrm{c}}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^{\infty}(\Omega)'. \tag{13}$$

Here I am reusing I since we have dropped its earlier use! To define $U = I(u)$ for $u \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\Omega)$, recall this means $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathrm{supp}(u) \Subset \Omega$. So we can choose $\psi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\Omega)$ such that $\psi = 1$ in a neighbourhood of $\mathrm{supp}(u)$. Thus $\psi u = u$ which means that for any $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$u(\phi) = u(\psi\phi). \tag{14}$$

But $\times\psi : \mathcal{C}^{\infty}(\Omega) \longleftrightarrow \mathcal{C}_{\mathrm{c}}^{\infty}(\Omega) \subset \mathcal{S}(\mathbb{R}^n)$ so we can legally define

$$I(u)(\phi) = u(\psi\phi) \ \forall \ \phi \in \mathcal{C}^{\infty}(\Omega). \tag{15}$$

Moreover, this does not depend on the choice of ψ since if $\psi' \in \mathcal{C}_{\mathrm{c}}^{\infty}(\Omega)$ also has $\psi = 1$ in a neighbourhood of $\mathrm{supp}(u)$ then the same is true of $\psi\psi'$ and $\psi u = \psi' u = \psi\psi' u$ allows one to shift from one to the other in (15):

$$u(\psi'\phi) = \psi u(\psi'\phi) = u(\psi\psi'\phi) = \psi' u(\psi\phi) = u(\psi\phi) \tag{16}$$

where the argument is always in $\mathcal{S}$ (in fact $\mathcal{C}_{\mathrm{c}}^{\infty}(\Omega)$) so this makes sense.

It also follows that $I(u) \in \mathcal{C}^{\infty}(\Omega)'$ since

$$|I(u)(\phi)| = |u(\psi\phi)| \le \sup_{|\alpha|+|\beta|\le m} \sup |x^{\alpha} D^{\beta} \psi\phi| \le C_{\psi} \|\phi\|_N \tag{17}$$

for one of the seminorms on $\mathcal{C}^{\infty}(\Omega)$, where $K_N \supset \mathrm{supp}\,\psi$ and N is large enough to control the derivatives. Thus I in (13) is well-defined.

Proposition 2. *The map I is a bijection.*

Proof. If $I(u) = 0$ then certainly $u(\phi) = 0$ for all $\phi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\Omega)$. This means that $\mathrm{supp}(u) \subset \mathbb{R}^n \setminus \Omega$, by the definition of support. Since $\mathrm{supp}(u) \Subset \Omega$ by assumption, this means $\mathrm{supp}(u) = \emptyset$ which is another way of saying $u = 0$ in $\mathcal{S}'(\mathbb{R}^n)$. Thus I is injective.

To prove surjectivity just use the continuity condition which means that if $U \in \mathcal{C}^{\infty}(\Omega)'$ then

$$|U(\phi)| \le C\|\phi\|_N \tag{18}$$

for some N corresponding to a compact set $K_N \Subset \Omega$. In particular, if $\psi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\Omega)$ and $\psi = 1$ in a neighbourhood of K_N (and we know that such a function exists) then $\|(1-\psi)\phi\|_N = 0$ so

$$U(\phi) = U(\psi\phi), \ \phi \in \mathcal{C}^{\infty}(\Omega). \tag{19}$$

However, (19) makes sense for any $\phi \in \mathcal{S}(\mathbb{R}^n)$ as well, since $\psi\phi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ so we can *define*

$$u(\phi) = U(\psi\phi). \tag{20}$$

I leave it to you to check that $u \in \mathcal{S}'(\mathbb{R}^n)$, that $\mathrm{supp}(u) \Subset \Omega$ and that $I(u) = U$, but none of these is hard. Thus I is also surjective. ∎

This is all very amusing, but we really want to define distributions on arbitrary open sets. To do this we need to consider the space

$$\mathcal{C}^\infty_{\mathrm{c}}(\Omega) = \{\phi \in \mathcal{S}(\mathbb{R}^n); \operatorname{supp}(\phi) \Subset \Omega\} \tag{21}$$

the space of test functions with compact support in Ω. As I mentioned last time, we can think of these functions as only defined on Ω, so

$$\mathcal{C}^\infty_{\mathrm{c}}(\Omega) \subset \mathcal{C}^\infty(\Omega) \tag{22}$$

but since each of them vanishes outside a compact subset of Ω, we can unambiguously extend it as zero outside Ω and arrive at (21). It is convention to ignore this nicety and think of these functions either way as the mood takes us.

Now, although (21) gives an inclusion into $\mathcal{S}(\mathbb{R}^n)$ we will want the topology on $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ to be stronger than the induced, subspace, topology so that it is appropriately complete. In particular we need a topology which for a convergent (or Cauchy) sequence will force the supports to remain in some compact subset of Ω. I hope by this stage you can at least guess what this topolgy might be – it cannot be metrizable, but that is just the way things are. The topology corresponds to writing

$$\mathcal{C}^\infty_{\mathrm{c}}(\Omega) = \bigcup_j \mathcal{C}^\infty_{\mathrm{c}}(K_j),\ \mathcal{C}^\infty_{\mathrm{c}}(K_j) = \{\phi \in \mathcal{S}(\mathbb{R}^n); \operatorname{supp}(\phi) \subset K_j\} \tag{23}$$

for a compact exhaustion K_j. So we have replaced the one open set by many compact sets (with non-empty interior of large j.) However, $\mathcal{C}^\infty_{\mathrm{c}}(K_j)$ is actually a closed subspace of $\mathcal{S}(\mathbb{R}^n)$, since a convergent sequence with respect to the topology of $\mathcal{S}(\mathbb{R}^n)$ certainly converges uniformly on compact sets, so the limit must also have support in $\mathcal{C}^\infty_{\mathrm{c}}(K_j)$.

So it is reasonably to give $\mathcal{C}^\infty_{\mathrm{c}}(K_j)$ the metric topology from $\mathcal{S}(\mathbb{R}^n)$ and this just corresponds to uniform convergence of all derivatives – it is the same topology as inherited from $\mathcal{C}^\infty(\Omega)$ for any $\Omega \Supset K_j$. Then we say (as in inductive limit) that a subset $\mathcal{O} \subset \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ is open if and only if each of the intersections $\mathcal{O} \cap \mathcal{C}^\infty_{\mathrm{c}}(K_n)$ is open. I leave it to you in the problem set next week to check that

$$v_k \to v \text{ in } \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \Longleftrightarrow \operatorname{supp}(v_k) \subset K_j \text{ for some } j \text{ and } v_k \to v \text{ in } \mathcal{C}^\infty_{\mathrm{c}}(K_j). \tag{24}$$

We can rather easily see when a maps on $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ is continuous into a topological space. Namely if the inverse image of each open set is open. But this just means that the inverse of each open set when the map is restricted to $\mathcal{C}^\infty_{\mathrm{c}}(K_j)$ is open there. So, this just means that a map is continuous if restricted to each $\mathcal{C}^\infty_{\mathrm{c}}(K_j)$ it is continous with respect to the metric topology. For a linear maps

$$v : \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \longrightarrow \mathbb{C} \tag{25}$$

this means that for each $K \Subset \Omega$ there exist $N = N(K)$ and $C = C(K)$ such that

$$|v(\phi)| \le C \sup_{|\alpha| \le N, x} |D^\alpha \phi(x)|. \tag{26}$$

I denote the space of such, the dual of $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ as $\mathcal{C}^{-\infty}(\Omega)$. The traditional notation (which seems to me to waste perfectly useful letters) is do denote

$$\begin{aligned} \mathcal{E}(\Omega) &= \mathcal{C}^\infty(\Omega) \text{ with dual } \mathcal{E}'(\Omega) = \mathcal{C}^{-\infty}_{\mathrm{c}}(\Omega) \\ \mathcal{D}(\Omega) &= \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \text{ with dual } \mathcal{D}'(\Omega) = \mathcal{C}^{-\infty}(\Omega). \end{aligned} \tag{27}$$

Notice that there is a duality between compact and unrestricted supports as well as smoothness and the quality of being a distribution.

There is a long list of things one should check. Let me write a short list of some truths, there are many more.

(1) $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$ each and inclusion and dense.
(2) Dually $\mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n)$ each an inclusion and appropriately weakly dense – note that duality flips the inclusions too.
(3) In fact we can aggrandize this to a commutative diagram with all maps inclusions and appropriately dense

$$\begin{tikzcd} \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \arrow[r] \arrow[d] & \mathcal{S}(\mathbb{R}^n) \arrow[r] \arrow[d] & \mathcal{C}^\infty(\mathbb{R}^n) \arrow[d] \\ \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \arrow[r] & \mathcal{S}'(\mathbb{R}^n) \arrow[r] & \mathcal{C}^{-\infty}(\mathbb{R}^n). \end{tikzcd} \tag{28}$$

(4) For a general open set we have a bit less of course, but still

$$\begin{tikzcd} \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \arrow[r] \arrow[d] & \mathcal{S}(\mathbb{R}^n) \arrow[r] \arrow[d] & \mathcal{C}^\infty(\Omega) \arrow[d] \\ \mathcal{C}^{-\infty}_{\mathrm{c}}(\Omega) \arrow[r] & \mathcal{S}'(\mathbb{R}^n) \arrow[r] & \mathcal{C}^{-\infty}(\Omega) \end{tikzcd} \tag{29}$$

where the vertical maps are dense inclusions and the left horizontal maps are, non-dense, inclusions while the right ones are have dense ranges but are restriction maps, not inclusions and are neither injective nor surjective.
(5) There is an (obvious) action of D^α on each space in (29) which commutes with all the maps.
(6) Multiplication by an element of $\mathcal{C}^\infty(\Omega)$ is defined on the spaces in the smaller square (consistent with (29)) commuting with the maps

$$\begin{tikzcd} \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \arrow[r] \arrow[d] & \mathcal{C}^\infty(\Omega) \arrow[d] \\ \mathcal{C}^{-\infty}_{\mathrm{c}}(\Omega) \arrow[r] & \mathcal{C}^{-\infty}(\Omega). \end{tikzcd} \tag{30}$$

(7) Combining these, a linear differential operator with smooth coeficients

$$P(x,D) = \sum_{|\alpha|\le m} p_\alpha(x) D^\alpha,\ p_\alpha \in \mathcal{C}^\infty(\Omega) \tag{31}$$

acts on all the spaces in (30) commuting with the maps.

Now, let's talk about something more practical. We know that the delta 'function' at a point p makes sense as an element of $\mathcal{S}'(\mathbb{R}^n)$ and it also makes sense as an element of $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ provided $p \in \Omega$:

$$\delta_p : \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \ni \phi \longmapsto \phi(p) \in \mathbb{C}. \tag{32}$$

I put this in to confuse you a little, because clearly it is defined on all smooth functions on Ω :

$$\delta_p : \mathcal{C}^\infty(\Omega) \ni \phi \longmapsto \phi(p) \in \mathbb{C} \Longrightarrow \delta_p \in \mathcal{C}^{-\infty}(\Omega). \tag{33}$$

Now, we can see that the support of a distribution on Ω is well-defined as a closed subset of Ω and that

$$\operatorname{supp}(\delta_p) = \{p\} \subset \Omega. \tag{34}$$

Theorem 1. *Any distribution in $u \in \mathcal{C}^{-\infty}(\Omega)$ with $\operatorname{supp}(u) = \{p\}$ for some $p \in \Omega$ is a finite sum*

$$u = \sum_{|\alpha| \le N} c_\alpha D^\alpha \delta_p,\ \ c_\alpha \in \mathbb{C}. \tag{35}$$

That is we can say that any distribution with support a point is the image of the delta function under a constant coefficient differential operator.