18.155 LECTURE 5, 19 SEPTEMBER, 2013

So, today I first want to prove the Schwartz' structure theorem. Let me first remind you of the Sobolev embedding theorem. What we notices is that

$$\text{For } v \in \mathcal{S}'(\mathbb{R}^n),\ (1+|\xi|^2)^{s/2} v \in L^2(\mathbb{R}^n) \Longrightarrow v \in L^1(\mathbb{R}^n) \text{ if } s > n/2.$$

Applying this to the Fourier transform of $u \in H^s(\mathbb{R}^n)$ we concluded that

$$s > n/2 \Longrightarrow u(x)(= u'(x)) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(\xi) d\xi \in \mathcal{C}^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \tag{1}$$

is a bounded and continuous function. You should be careful to understand what is behind this – it really says that $u \in H^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ has a representative $I(u')$ where u' is the function in (1). Why is this true? The absolute convergence of the integral in (1) means that $u'\mathcal{C}^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is defined from $\hat{u}$ (use Dominated Convergence to get the continuity). So $I(u')$ is well defined and from the absolute convergence of the integrals and the Fourier inversion formula on $\mathcal{S}(\mathbb{R}^n)$,

$$I(u')(\hat{\phi}) = (2\pi)^{-n} \int\int e^{ix\cdot\xi} \hat{u}(\xi) \hat{\phi}(x) = \hat{u}(\phi)\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{2}$$

Thus u and $I(u')$ represent the same distribution, which we now write as $u = u'$.

If $s > \frac{n}{2} + 1$ then we can use the convergence of the difference quotient and bound

$$\left|\frac{e^{i(x+te_j)\cdot\xi} - e^{ix\cdot\xi}}{t} - i\xi_j e^{ix\cdot\xi}\right| \le |t||\xi| \tag{3}$$

to see that $u'(x)$ has continuous and bounded first partial derivatives and that these satisfy $\hat{D}_j u' = \xi_j \hat{u}$. Now induction on the integer k shows the full Sobolev embedding theorem

$$u \in H^s(\mathbb{R}^n),\ s > \frac{n}{2} + k,\ k \in \mathbb{N} \Longrightarrow D^\alpha u \in \mathcal{C}^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \tag{4}$$

You should remind yourself that these derivatives actually vanish at infinity as well.

So, in short if u has Sobolev regularity $\frac{n}{2} + k$ then it has classical continouous, bounded derivatives up to order k.

Let's try the first form of the Schwartz Structure Theorem – that any $u \in \mathcal{S}'(\mathbb{R}^n)$ can be written in the form

$$u = \sum_{|\alpha|+|\beta| \le N} x^\alpha D_x^\beta u_{\alpha,\beta},\ u_{\alpha,\beta} \in L^2(\mathbb{R}^n). \tag{5}$$

All we have to start from is that u is a tempered disbribution. This means that for some k

$$|u(\phi)| \le C \sup_{x \in \mathbb{R}^n} \sum_{|\alpha|+|\beta| \le k} |D^\beta x^\alpha \phi| \tag{6}$$

since it is a continuous linear functional (and I used the Pset1 to reverse the multicplication and differentiation). However, from the Sobolev embedding theorem we know that for $s > n/2$, which we will take to be an integer here

$$\sup |D^\beta \phi| \le C \|\phi|_{H^{s+|\beta|}},\ \phi \in \mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n). \tag{7}$$

Now, we can apply this to $x^\alpha\phi$ to see that

$$\sup |D^\beta(x^\alpha\phi)| \le C\|x^\alpha\phi|_{H^{s+|\beta|}}. \tag{8}$$

So we conclude that any distribution satisfies an estimate

$$|u(\phi)| \le C \sum_{|\alpha|\le N} \|x^\alpha\phi\|_{H^N}. \tag{9}$$

Now, let's multiply u by $(1+|x|^2)^{-N}$. Since

$$(1+|x|^2)^{-N}u(\phi) = u((1+|x|^2)^{-N}\phi)$$

we can apply (9) to see that

$$|(1+|x|^2)^{-N}u(\phi)| \le C \sum_{|\alpha|\le N} \|x^\alpha(1+|x|^2)^{-N}u(\phi)x^\alpha\phi\|_{H^N} \tag{10}$$

Now, all the derivatives of $x^\alpha(1+|x|^2)^{-N}$ are bounded (since $|\alpha|\le N$ and the Sobolev norm is equivalent to the sum of the L^2 norms of the derivatives so in fact

$$|(1+|x|^2)^{-N}u(\phi)| \le C'\|\phi\|_{H^N} \ \forall\ \phi\in\mathcal{S}(\mathbb{R}^n). \tag{11}$$

for some N. We showed last time, from Riesz' Representation Theorem, that this implies that $(1+|x|^2)^{-N}u \in H^{-N}(\mathbb{R}^n)$. Then from the characterization of the negative Sobolev spaces it follows that

$$u = (1+|x|^2)^N \sum_{|\beta|\le N} D^\alpha u_\alpha. \tag{12}$$

Multiplying out the polynomial we get (5) (with N replaced by $2N$ but I have not been counting anyway.

To get other 'even more classical' forms of the structure theorem, we can use the fact that $(1+|D|^2)^k : H^{2k}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is an isomorphism. Using this to replace each of the u_α in (12) by sums of derivatives of functions in $H^{2k}(\mathbb{R}^n)$ and choosing $2k > n/2$ we find that any tempered distribution can also be written in the form (for a yet larger N)

$$u = \sum_{|\alpha|+|\beta|\le N} x^\alpha D_x^\beta u_{\alpha,\beta},\ u_{\alpha,\beta}\in\mathcal{C}^0(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n). \tag{13}$$

Next week I will start working out some examples. I want to devote the rest of the lecture today to explaining the other big theorem about (tempered) distributions and at least outlining the proof. We don't really use this result later, but it is important to know about it.

Namely we need fairly soon to start thinking about operators. We have in mind (partial) differential operators with constant, polynomial or more general variable coefficients – for instance if P is any polynomial in n variables then

$$p(D) = \sum_{0\le|\alpha|\le m} c_\alpha D^\alpha : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{14}$$

It also maps $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. As an operator on $\mathcal{S}(\mathbb{R}^n)$ it is continuous and so it is on tempered distributions if we define an appropriate topology on them (at last).

The simplest topology to consider is the *weak* topology. This is defined by a collection of seminorms, just not a countable number. Namely if $\phi\in\mathcal{S}(\mathbb{R}^n)$ then

$$\mathcal{S}'(\mathbb{R}^n) \ni u \longmapsto |u(\phi)| \in \mathbb{R} \tag{15}$$

is a seminorm. We want to choose a topology on $\mathcal{S}'(\mathbb{R}^n)$ so that these are continuous functions. So the inverse images of open subsets of $\mathbb{R}$ under each of them should be open. The coarsest topology on $\mathcal{S}'(\mathbb{R}^n)$ with this property is the weak topology – an open subset of $\mathcal{S}'(\mathbb{R}^n)$ is an arbitrary union of finite intersections of such open sets. Clearly this collection of (open) sets is closed under arbitrary unions and is easily seen to be closed under finite intersections. So this is the weak topology.

The most general 'operator' one is likely to encounter in the setting of tempered distributions is a linear map

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^m) \tag{16}$$

(maybe mapping to distributions on a space of different dimension) which is continuous in terms of the metric topology on $\mathcal{S}(\mathbb{R}^n)$ and the weak topology on $\mathcal{S}'(\mathbb{R}^m)$. This actually just means that all the composites

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto (A\phi, \psi) \in \mathbb{C},\ \ \psi \in \mathcal{S}(\mathbb{R}^m) \tag{17}$$

are continous. It is a lot of conditions, but each of them is pretty feeble! I leave it as an exercise to check that this is actually equivalent to the continuity.

Now, the question arises as to how such operators might arise. There is a fairly natural way you might construct some based on the fact that the exterior product is a continous bilinear map

$$\mathcal{S}(\mathbb{R}^m) \times \mathcal{S}(\mathbb{R}^n) \ni (\psi, \phi) \longmapsto \psi(x)\phi(y) \in \mathcal{S}(\mathbb{R}^{m+n}). \tag{18}$$

If you go back to the properties of bilinear maps, this is easy to check.

So, suppose that $K \in \mathcal{S}'(\mathbb{R}^{m+n})$ is a tempered distribution on the product space. The continuity above means that if we take $\phi \in \mathcal{S}(\mathbb{R}^n)$ and look at

$$\mathcal{S}(\mathbb{R}^m) \ni \psi \longmapsto K(\psi, \phi) \in \mathbb{C} \tag{19}$$

it is continuous, so defines a distribution in $\mathcal{S}'(\mathbb{R}^m)$ and hence a linear map

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^m). \tag{20}$$

Then A is said to have 'Schwartz kernel' K. The bilinear estimates give

$$|A(\phi)(\psi)| = |K(\psi \boxtimes \phi)| \le C\|\psi\|_N \|\phi\|_N \tag{21}$$

for some N. You will easily check that this implies the continuity of (20) in the weak topology on the range space – in fact it looks much stronger! However Schwartz' Kernel theorem says:

Theorem 1. *There is a linear bijection between operators A in* (16)*, continuous into the weak topology, and distributions in* $\mathcal{S}'(\mathbb{R}^{m+n})$.

Proof. We have to show that $K \longmapsto A$ is a bijection. The surjectivity follows from the density of the linear span of products in $\mathcal{S}(\mathbb{R}^m) \boxtimes \mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^{m+n}$ which is straightfoward and I will mention below. The converse is the 'hard part' – how to construct K from A. In fact we do get a bilinear map

$$\mathcal{S}(\mathbb{R}^m) \times \mathcal{S}(\mathbb{R}^n) \ni (\psi, \phi) \longmapsto (A\phi)(\psi) \in \mathbb{C}. \tag{22}$$

The 'hard part' is to show that this is continous in the metric topology. In fact I believe I outlined this in an notes for an earlier lecture. What we have is 'separate continuity'. Namely if we hold either ϕ or ψ fixed in (22) then we have continuity as a lienar map in the other variable. One way round this is the fact that A takes values in $\mathcal{S}'(\mathbb{R}^m)$ and the other is the weak continuity.

The problem then is to show that a separately continuous bilinear form (4) is 'jointly continuous' – meaning continuous in the metric topology on the product. I hope I proved this in the earlier notes, it is basically Baire's Theorem.

There is a bit more to do even after we show joint continuity. I will add it to the notes when I get a chance, but do not plan to go through it in lecture (unless I have a lot of spare time ...). □

One way of thinking about the Schwartz Kernel Theorem, which is indeed sometimes important, is that it says operators are just distributions. So the properties of operators can be related to the properties of their kernels – even though in plactice this can be decidedly tricky.

Last time I, probably unwisely, talked about isotropic Sobolev spaces. One reason for doing that was to warm you that there are othe 'global' Sobolev spaces that you might need to consider in a particular context. Here I will at least describe some of the properties. For the moment, we only have integral order spaces and for $k \in \mathbb{N}$ we define

$$H^k_{\text{iso}}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); x^\alpha D^\beta u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| + |\beta| \le k\|\}. \tag{23}$$

The 'isotropic' refers to the fact the the multiplication operators x_j and differentiation D_i are 'on the same footing'. This is a Hilbert space for each k with norm

$$\|u\|^2_{H^k_{\text{iso}}} = \sum_{0 \le |\alpha|+|\beta| \le k} \int |x^\alpha D^\beta u|^2. \tag{24}$$

Completeness follows as usual – for a Cauchy sequence all the $x^\alpha D^\beta u_n$ are Cauchy in L^2 and hence converge and the limits can be shown to be weak, and also strong, derivatives of the limit of the sequence u_n in L^2.

Since the conditions are stronger,

$$H^k_{\text{iso}}(\mathbb{R}^n) \subset H^k(\mathbb{R}^n),\ \forall\ k \in \mathbb{N}. \tag{25}$$

Use of the Sobolev embedding theorem allows one to see that

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_k H^k_{\text{iso}}(\mathbb{R}^n). \tag{26}$$

One can also define spaces of negative integral order but analogy with the result shown above for ordinary Sobolev spaces and set

$$H^{-k}_{\text{iso}}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); u = \sum_{0 \le |\alpha|+|\beta| \le k} x^\alpha D^\beta u_{\alpha,\beta},\ u_{\alpha,\beta} \in L^2(\mathbb{R}^n). \tag{27}$$

It is not quite clear that this is a Hilbert, or even a normed, space since the expression as a sum like this is by no means unique. However, one can check that

$$u \in H^{-k}(\mathbb{R}^n) \Longleftrightarrow |u(\phi)| \le C\|\phi\|_{H^k_{\text{iso}}}\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{28}$$

although it is not quite trivial to do so (as far as I can see). However, one can now from the Schwartz structure theorem that

$$\mathcal{S}'(\mathbb{R}^n) = \bigcup_{k \in \mathbb{Z}} H^k_{\text{iso}}(\mathbb{R}^n). \tag{29}$$

Note that (25) is reversed for negative orders

$$H^{-k}(\mathbb{R}^n) \subset H^{-k}_{\text{iso}}(\mathbb{R}^n),\ \forall\ k \in \mathbb{N}.$$

So in some sense these isotropic spaces are more closely related to $\mathcal{S}$ and $\mathcal{S}'$ than the usual Sobolev spaces. This is closely related to the (fairly straightforward) result that

$$\mathcal{F} : H^k(\mathbb{R}^n) \longrightarrow H^k(\mathbb{R}^n)\ \forall\ k \in \mathbb{Z}. \tag{30}$$

For the ordinary Sobolev spaces we used the Fourier transform to define non-integral order spaces but that does not work here. Still they can be defined! One way to do so is to use the Spectral Theorem (from later in the semester) which allows us to define powers of the operator which is essentially the harmonic oscillator:

$$P = 1 + |D|^2 + |x|^2 \tag{31}$$

and then to define

$$u \in H^s_{\mathrm{iso}}(\mathbb{R}^n) \Longleftrightarrow u \in \mathcal{S}'(\mathbb{R}^n),\ P^{s/2}u \in L^2(\mathbb{R}^n),\ s \in \mathbb{R}. \tag{32}$$

Of course to see that this makes much sense – and is for instance consistent with the definitions above – we need to do quite a bit of work. It is better to leave such things until we have some more machinery to make life easier.

Just to reënforce this idea that there are more 'global' Sobolev spaces than just the basic ones, although for the moment I cannot easily convince you of there significance (but that will come later) note that the 'ordinary' Sobolev spaces are defined by decay (and some regularity) of the Fourier transform. Said another way, we can define 'growth/decay' spaces by

$$H^{0,t}(\mathbb{R}^n) = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n); u = (1 + |x|^2)^{t/2}v,\ v \in L^2(\mathbb{R}^n)\}. \tag{33}$$

This is actually the standard convention (but not standard notation) but it might be (or have been) more logical to change the sign of t since the definition of the Sobolev spaces is just

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); \hat{u} \in H^{0,-s}(\mathbb{R}^n)\} \tag{34}$$

where you see the sign issue. Anyway, one can then *define*

$$H^{s,t}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); u = (1 + |t|^2)^{t/2}v,\ v \in H^s(\mathbb{R}^n) \tag{35}$$

which one could write somewhat dangerously as

$$H^{s,t}(\mathbb{R}^n) = (1 + |x|^2)^{t/2}H^s(\mathbb{R}^n). \tag{36}$$

Now one needs to do some work, pretty each if at least one of s and t is an integer, and show that

$$\begin{gathered}\mathcal{F} : H^{s,t}(\mathbb{R}^n) \longrightarrow H^{-t,-s}(\mathbb{R}^n) \text{ is an isomorphism}\\ \mathcal{S}(\mathbb{R}^n) = \bigcap_{s,t} H^{s,t}(\mathbb{R}^n)\\ \mathcal{S}'(\mathbb{R}^n) = \bigcup_{s,t} H^{s,t}(\mathbb{R}^n).\end{gathered} \tag{37}$$

These spaces will reappear later – assuming they appeared at all in my lecture which is not particularly likely.

Exercise 1. Think about the proof of (37). What would you need to do to show that the operator

$$Q_{s,t} = (1 + |x|^2)^t(1 + |D|^2)^{-s/2} = (1 + |x|^2)^t\mathcal{G}(1 + |\xi|^2)^{-s/2}\mathcal{F} \tag{38}$$

is an isomorphism on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ and also that

$$Q_{s,t} : H^{a,b}(\mathbb{R}^n) \longrightarrow H^{a+s,b+t}(\mathbb{R}^n) \tag{39}$$

is an isomorphism for all a, b, s and $t \in \mathbb{R}$?