18.155 LECTURE 3: 12 SEPTEMBER, 2013

- We showed $\mathcal{F}:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n)$ is continuous, in fact
$$\|\hat{\phi}\|_N \le C\|\phi\|_{N+n+1}.$$
- To prove it is an isomorphism we start with two Lemmas – the first is very standard

Lemma 1. *There exists $\psi\in\mathcal{S}(\mathbb{R}^n)$, $\psi(x)\ge 0$, $\psi(x)=1$ in $|x|<\frac{1}{2}$ $\psi(x)=0$ in $|x|>1$.*

Proof. In one variable consider the function
$$\mu(x)=\begin{cases}\exp(-1/x) & x>0\\ 0 & x\le 0.\end{cases}$$
This is an example of a non-analytic but infinitely differentiable function. It is certainly smooth in $x>0$ and the derivatives are of the form

(1) $$\frac{d^k}{dx^k}\mu(x)=\frac{p_k(x)}{x^{2k}}\exp(-1/x),$$

with p_k a polynomial. The convergence of the power series for e^s, all terms in which are positive for $s>0$ shows that for each N, $e^s\ge s^n/n!$ – and hence
$$\frac{d^k}{dx^k}\mu(x)\le C_{k,N}x^N \text{ in } 0<x\le 1.$$
Thus all the derivatives, defined in $x>0$, extend by continuity down to 0 where they vanish to infinite order. This includes μ itself which is therefore given by the integral from -1 of its own derivative extended to be 0 in $x<0$. This argument iterates to show – or you could do it directly anyway – that μ is infinitely differentiable across 0.

From this we can construct

(2) $$\psi'(x)=\mu(1-|x|^2)\in\mathcal{S}(\mathbb{R}^n),\ \eta(x)=\mu(|x|^2-\frac{1}{2})\in\mathcal{C}^\infty(\mathbb{R}^n)$$

which are respectively positive in $|x|<1$ but zero in $|x|\ge 1$ and zero near $x=0$ and positive in $|x|>3/4$. Thus $\psi+\eta$ is strictly positive everywhere and then

(3) $$0\le\psi(x)=\frac{\psi'(x)}{\psi'(x)+\eta(x)}\in\mathcal{S}(\mathbb{R}^n)$$

satisfies the conditions we want, since it is equal to 1 where $\eta(x)=0$ and 0 where $\psi'(x)=0$. □

- **Lemma 2.** *Any $\phi\in\mathcal{S}(\mathbb{R}^n)$ can be written in the form*

(4) $$\phi(x)=\phi(0)\exp(-\frac{|x|^2}{2})+\sum_{j=1}^n x_j\psi_j(x),\ \psi\in\mathcal{S}(\mathbb{R}^n).$$

Proof. Taylor's formula and then use of the cutoff from the previous lemma. □

- Now, to back to the Forier transform. We consider the composite and then restrict to 0 and claim that

$$\mathcal{G}(\hat{\phi})(0) = c\phi(0) \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{5}$$

where c is a fixed constant independent of ϕ. Indeed to see this insert (4) into the left side to get

$$\mathcal{G}(\hat{\phi})(0) = c\phi(0) + \sum_j \mathcal{G}(\widehat{x_j\psi_j})(0),\ c = \mathcal{G}(\hat{\gamma})(0),\ \gamma = \exp(-\frac{|x|^2}{2}). \tag{6}$$

However, $\widehat{x_j\psi_j}(\xi) = i\partial_{\xi_j}\hat{\psi}_j(\xi)$ as we showed last time and since

$$\mathcal{G}(f)(0) = (2\pi)^{-n}\int f, \tag{7}$$

so each of the terms in the sum vanishes. Thus we arrive at (5) where we even no the constant in terms of the Gaussian.

- Now, we can also work out formulæ for the Fourier transform of translates and multiples by exponentials:-

$$\mathcal{F}(\phi(\bullet + y))(\xi) = e^{iy\cdot\xi}\hat{\psi},\ \mathcal{G}(e^{-iy\cdot\bullet}f)(x) = \mathcal{G}(f)(x+y). \tag{8}$$

Combining these two and (5) shows that

$$\mathcal{G}\mathcal{F} = c\,\mathrm{Id}\,. \tag{9}$$

- So we are reduced to working out the constant. This amounts to working out the Fourier transform of the Gaussian and this is pretty standard. First we only need do the 1-D case since

$$\mathcal{F}(\exp(-\frac{|x|^2}{2}) = \prod_j \mathcal{F}(\exp(-x_j^2/2) \tag{10}$$

and then we can check that

$$(\frac{d}{d\xi} + \xi)\mathcal{F}(\exp(-x^2/2)(\xi) = 0 \Longrightarrow \mathcal{F}(\exp(-x^2/2)(\xi) = c'\exp(-\xi^2/2) \tag{11}$$

where the constant is the value of the Fourier transform at 0, i.e.

$$c'\int_{\mathbb{R}} \exp(-x^2/2) = \sqrt{2\pi}. \tag{12}$$

Going back to (5) it follows that $c = 1$ and we have the Fourier inversion formula

$$\mathcal{G}\circ\mathcal{F} = \mathrm{Id} = \mathcal{F}\circ\mathcal{G} \text{ on } \mathcal{S}(\mathbb{R}^n) \tag{13}$$

where the second follows from the first by changing signs.

- So the Fourier transform is an isomorphism on $\mathcal{S}(\mathbb{R}^n)$, a continuous linear bijection with a continuous inverse.
- Computing with absolutely convergent Lebesgue integrals it also follows directly that for any ϕ, $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \hat{\phi}\psi = \int_{\mathbb{R}^{2n}} e^{-ix\cdot\xi}\phi(x)\psi(\xi) = \int_{\mathbb{R}^n} \phi\hat{\psi}. \tag{14}$$

This gives a weak formulation of the Fourier transform which we can write

$$I(\hat{\phi})(psi) = I(\phi)(\hat{\psi}) \tag{15}$$

and so for general $u \in \mathcal{S}'(\mathbb{R}^n)$ it is consistent to *define*

$$\hat{u}(\psi) = u(\hat{\psi}) \ \forall \ u \in \mathcal{S}'(\mathbb{R}^n), \ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{16}$$

This gives a linear bijection (so far we don't have a topology to measure continuity)

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{17}$$

- If we apply the identity (14) to $\psi = \overline{\hat{\eta}}$ for some $\eta \in \mathcal{S}(\mathbb{R}^n)$ then $\hat{\eta} = \overline{\psi}$ so by the inversion formula,

$$\eta(\xi) = \mathcal{G}(\overline{\psi}) = (2\pi)^{-n}\mathcal{F}(\overline{\psi})(-\xi) \Longrightarrow \overline{\eta}(\xi) = (2\pi)^{-n}\hat{\psi}(\xi).$$

The result is Parseval's formula

$$\int_{\mathbb{R}^n} \hat{\phi}\overline{\hat{\eta}} = (2\pi)^{-n} \int_{\mathbb{R}^n} \phi\overline{\eta} \tag{18}$$

the essential unitarity of the Fourier transform, i.e. $(2\pi)^{-n/2}\mathcal{F}$ is preserves the L^2 inner product on $\mathcal{S}(\mathbb{R}^n)$ and so, if we accept that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$,

Proposition 1. *The operators $(2\pi)^{-n/2}\mathcal{F}$ extends by continuity to be unitary on $L^2(\mathbb{R}^n)$.*

- So we have seen that the Fourier transform is an isomorphism on $\mathcal{S}(\mathbb{R}^n)$, and on $\mathcal{S}'(\mathbb{R}^n)$ which restricts to an isomorphism on the subspace $L^2(\mathbb{R}^n)$.
- The L^2-based Sobolev spaces are then defined for each *real* number $s \in \mathbb{R}$:

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); (1 + |\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n). \tag{19}$$

Note that in the homework this week you showed that $(1 + |x|^2)^{s/2}$ is a 'multiplier' on $\mathcal{S}(\mathbb{R}^n)$, and hence on $\mathcal{S}'(\mathbb{R}^n)$ so the definition makes sense, $L^2(\mathbb{R}^n)$ being a well-defined subspace of $\mathcal{S}'(\mathbb{R}^n)$.

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Lemma 3. *For $s = k \in \mathbb{N}$ a positive integer,*

$$H^k(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); D^\alpha u \in L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n), \ |\alpha| \le k\} \tag{20}$$

is also the space of L^2 functions with strong *derivatives up to order k in L^2 where successive strong derivatives are defined by convergence of the difference quotient in L^2 :*

$$\partial_j u = \lim_{s\to 0} \frac{u(x + se_j) - u(x)}{s} \ \text{in } L^2(\mathbb{R}^n). \tag{21}$$