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Last time I showed that we needed some sort of additional argument to see that the Laplace-Beltrami operator, which I will recall again later, satisfies global elliptic regularity in the following sense. Namely for any $k \in \mathbb{Z}$ (and in fact for any real order),

(1)
$$u \in \mathcal{C}^{-\infty}(M),\ (\Delta+L)u \in H^k(M) \Longrightarrow u \in H^{k+2}(M) \text{ and } \|u\|_{k+2} \le C(k)\|(\Delta+L)u\|_k.$$

Now, from the original construction what we know

(2)
$$(\Delta + L) : H^1(M) \longrightarrow H^{-1}(M) \text{ is an isomorphism.}$$

The basic 'regularization' we need is just convolution in local coordinates. So remember (changing dimension to p) that if $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^p)$ has $\phi \ge 0$, $\int \phi = 1$ and $\phi = 0$ in $|x| > 1$ then

(3)
$$\phi_\epsilon * v \to v \in H^k(\mathbb{R}^p)\ \forall\ k,\ \phi_\epsilon(x) = \epsilon^{-p}\phi(\frac{x}{\epsilon}).$$

We also know that if v has compact support in K then the support of $\phi_\epsilon v$ lies in $K + \{|x| \le \epsilon\}$.

The main thing we need is

Lemma 1. *With $K \Subset \mathbb{R}^p$ if $g \in \mathcal{C}^\infty(U)$ where U is open and contains $K + \{|x| \le 2\epsilon_0\}$ then*

(4)
$$[\phi_\epsilon, g] : \{v \in H^k(\mathbb{R}^p); \operatorname{supp}(v) \subset K\} \longrightarrow H^{k+1}(\mathbb{R}^p)$$

is uniformly bounded for $0 < \epsilon < \epsilon_0$ for all $k \in \mathbb{Z}$.

Proof. If we write out the convolution operator explicitly and put in the multiplier the commutator is

(5)
$$A_\epsilon v(x) = \epsilon^{-p}\int_{\mathbb{R}^p} \phi(\frac{x-y}{\epsilon})(g(x) - g(y))v(y)dy.$$

Here the convolution kernel vanishes outside $|x - y| \le \epsilon$ and v vanishes outside K so everything is well-defined for ϵ small. In fact we can expand the difference using Taylor's formula and we get

(6)
$$g(x) - g(y) = \sum_{0<|\alpha|<N} g_\alpha(x)(x-y)^\alpha + \sum_{|\alpha|=N} (x-y)^\alpha G_\alpha(x, y)$$

where the $G_\alpha(x, y)$ are smooth functions in $|x - y| \le \epsilon$.

Inserting (6) in (5) each term in the first sum gives an operator

(7)
$$B_{\alpha,\epsilon}v = \epsilon^{|\alpha|}g_\alpha(x)\Phi_{\alpha,\epsilon} * v,\ \Phi_{\alpha,\epsilon} = \epsilon^{-p}\Phi_\alpha(\frac{x}{\epsilon}),\ \Phi_\alpha(x) = x^\alpha\phi(x).$$

So each of these is a convolution operator followed by multiplication by a smooth function. Now, if we differentiate up to α times we cancel off at most $|\alpha|$ factors of ϵ so the estimates we proved for convolution show that

(8)
$$B_{\alpha,\epsilon} : \{v \in H^k(\mathbb{R}^p); \operatorname{supp}(v) \subset K\} \longrightarrow H^{k+|\alpha|}(\mathbb{R}^p),\ \|B_{\alpha,\epsilon}\| < C(\alpha),\ \forall\ \alpha(|\alpha| \ge 1).$$

So these terms uniformly smooth by at least one derivative. We only need to look at many of them to make the remainder term simpler.

So consider the terms in the second sum in (6) inserted into (5). Now we are free to take N very large and

$$N = |\alpha| > p + l \Longrightarrow \epsilon^{-p}\phi(\frac{x-y}{\epsilon})(x-y)^\alpha G_\alpha(x,y) \in \mathcal{C}^l(\omega) \tag{9}$$

where Ω includes the region of integration, where $y \in K + \{|x| \le \epsilon\}$ and $|x-y| \le \epsilon$ for $\epsilon < \epsilon_0$.

□

Remark added after lecture:

Using these commutator methods we conclude that $(\varDelta + L) : H^{k+1}(M) \longrightarrow H^{k-1}(M)$ is an isomorphism for any $k \in \mathbb{N}_0$.

We can use this to get 'full elliptic regularity' meaning

$$u \in \mathcal{C}^{-\infty}(M) \text{ and } \varDelta u \in H^p(M) \Longrightarrow u \in H^{p+2}(M) \tag{10}$$

First we prove the same thing with $\varDelta$ replaced by $\varDelta + L$. Thus suppose $u \in \mathcal{C}^{-\infty}(M)$ and that $(\varDelta u + L) \in H^p(M)$ for some $p \in \mathbb{Z}$, $p < -1$. Then for $f \in \mathcal{C}^\infty(M)$, there exists $v \in \mathcal{C}^\infty(M)$ such that

(11)

$$(u,f) = (u, (\varDelta{+}L)v) = ((\varDelta{+}L)u, v) \Longrightarrow |(u,f)| \le |((\varDelta{+}L)u,v)| \le C\|v\|_{H^{-p}} \le C'\|f\|_{H^{-p-2}}$$

which shows that u extends by contuity to $f \in H^{-p-2}(M)$ so $u \in H^{p+2}(M)$. For $p \ge 0$ this implies $u \in L^2$ and so from the regularity for positive order above it follows that $u \in H^{p+2}(M)$ for any $p \in \mathbb{Z}$.

To get the same result for $\varDelta$ it is enough to bootstrap. Thus $u \in \mathcal{C}^{-\infty}(M)$ means $u \in H^q(M)$ for some q and $\varDelta u \in H^p(M)$ implies $(\varDelta + L)u \in H^{p'}$ where p' is the smaller of q and p. Then $u \in H^{p'+2}(M)$ which allows the argument to be repeated until $p' = p$ and we see that $u \in H^{p+2}(M)$.

I did not actually go over the following in lecture and maybe it is out of place here.

Let me recall the definition of the Laplace-Beltrami operator on the boundary, $M = \partial B$, of a smoothly bounded domain B, either using the metric and density induced from the Eulidean metric on $\mathbb{R}^n$, or any other smooth metric and corresponding smooth density (if you use a different density this all still works except you don't have the Laplace-Beltrami operator).

n brief we have a positive section $\nu \in \mathcal{C}^\infty(M;\Omega)$ of the density bundle and a smooth family of positive definite (real) inner products on the tangent, and hence cotangent, spaces which we can turn into an inner product on functions using the differential on M :

$$\langle u, v\rangle_D = \int_M \langle d^M u, \overline{d^M v}\rangle \nu,\ u,\ v \in \mathcal{C}^\infty(M). \tag{12}$$

If we use a local straightening of B near a point $p \in M$ so that ∂B is mapped into $x_n = 0$ then for u and v supported in the coordinate ball near p (12) becomes

$$\langle u, v\rangle = \sum_{p,q=1}^{n-1} \int_M a_{pq}(x')(\partial_{x'_p}\tilde{u}\partial_{x'_q}\overline{\tilde{v}}\nu(x'),\ \nu = \nu(x')|dx'|. \tag{13}$$

Here $a_{pq}(x')$ is the smooth positive-definite symmetric matrix expressing the metric in local coordinates and $\nu(x')$ is the local expression for the Riemannian density – in fact

$$\nu(x') = \det(a_{pq}(x'))^{-\frac{1}{2}}. \tag{14}$$

Now, the inner product $\langle u, v\rangle_D$ cannot be definite, since the constants are in the null space. However, as I indicated last time

Lemma 2. *For* $L > 0$

$$\langle \cdot, \cdot \rangle_D + L \int_u \overline{v}\nu \tag{15}$$

is positive definite and induces a Hilbert space structure on $H^1(M)$.