18.155 LECTURE 21: 21 NOVEMBER, 2013

Reconstructed.

To define the Laplacian acting on function and distributions on the manifold $M = \partial B$ realized as the boundary of a smoothly bounded domain, I need to go through some of the basic material of differential geometry − last time I defined the tangent and cotangent bundles, and we need to consider sections of these, the notion of a metric, densities (so we can integrate) and distributions. All this works perfectly well on an abstractly defined manifold, but I will limit myself here to those which can be embedded in this way.

The definition of the tangent space to M at p in terms of derivations shows that if $F : O \longrightarrow \Omega' \subset \mathbb{R}^{n-1}$ is a smooth map from an open neighbourhood $p \in O \subset M$ then

$$F_* : T_pM \longrightarrow T_{F(p)}\mathbb{R}^{n-1} = \mathbb{R}^{n-1},\ F_*(\delta)u = \delta(u \circ F) = \delta(F^*u). \tag{1}$$

In particular this applies to our 'adapted coordinates'. The tangent bundle of M is just the disjoint union of all the tangent spaces

$$TM = \bigcup_{p\in M} T_pM \tag{2}$$

and a section of TM over an open set $O \subset M$ is a map $v : O \longrightarrow TM$ such that $v(p) \in T_pM$. Such a section is smooth if F_*v is smooth as a map from $F(O)$ into $\mathbb{R}^{n-1}$ for all, or equivalently a covering of M by, adapted coordinates. Such a smooth section is a smooth vector field on M; we write the space of global smooth sections as $\mathcal{C}^\infty(M; TM)$. Each such defines a map $v : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ since $v(p)$ is a derivation at p and in local coordinates if of the form

$$F_*(v) = \sum_1^{n-1} a_i(y')\partial_{y_i},\ a_i \in \mathcal{C}^\infty(\Omega'). \tag{3}$$

Exercise 1. Show that if one has a 'local' vector field as in (3) for each Ω'_i forming a covering of M by adapted coordinates then summing over a partition of unity gives a global smooth section of TM. Use this to show that the restriction map from global sections

$$\mathcal{C}^\infty(M; TM) \longrightarrow T_pM \tag{4}$$

is surjective.

A linear differential opertor of order m on functions on M can now be defined to be a linear map

$$P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M) \tag{5}$$

which is local, so $u = 0$ in an open set implies $Pu = 0$ in that set, and which can be written as a finite sum of up to m-fold products of vector fields acting on $\mathcal{C}^\infty(M)$ (where 0-factors means multiplication by a smooth function).

Exercise 2. Show that in local adapted coordinates a differential operator is the obvious thing:

$$G^*PF^*u = \sum_{|\alpha|\le m} p_\alpha(y')D^\alpha_{y'}u,\ u \in \mathcal{C}^\infty(\Omega'). \tag{6}$$

The coefficients are determined by P and F (recall $G = F^{-1}$) so knowing them in one coordinate system determines them on the overlap with any other coordinate system. The transformation law is rather complicated! However, you can check that the principal part

$$p_m(p,\eta) = \sum_{|\alpha|=m} p_\alpha(y')\xi^\alpha,\ \eta = f^*\xi \in T^*_pM \tag{7}$$

is actually a globally well-defined function on T^*M which is a homogeneous polynomial on each fibre.

We are after the particular example of such an operator given by the Laplacian associated to a metric on M.

A metric on a real vector space V is just a positive-definite symmetric bilinear form. A metric on M is such a metric $\langle\cdot,\cdot\rangle$ on each T_pM such that if v and w are any smooth sections of TM, as defined above, then $\langle v, w\rangle \in \mathcal{C}^\infty(M)$. You should check that in adapted local coordinates this means that

$$\langle v, w\rangle = \sum_{i,j=1}^{n-1} g_{ij}(y')v_i(y')w_j(y'),\ F_*v = \sum_i v_i(y')\partial_{y_i},\ F_*w = \sum_{j=1}^{n-1} w_j(y')\partial_{y_j} \tag{8}$$

where the symmetric real matrix g_{ij} is positive definite. The transformation law between coordinate systems is not so complicated.

Now a metric in this sense also induces a fibre metric on T^*M. This is just the algebraic fact that metrics on V and V^* are in 1-1 correspondence since a metric on V determines an isomorphism $l : V \longrightarrow V^*$ by

$$l(v) = \alpha \Longleftrightarrow \alpha(w) = \langle v, w\rangle\ \forall\ w \in V. \tag{9}$$

In local coordinates, using the basis dy_i of T^*M, given by an adapted coordinate system the dual metric to (8) is

$$\begin{gathered}\langle\alpha,\beta\langle = \sum_{i,i=1}^{n-1} g^{ij}\alpha_i\beta_j,\\ \alpha = F^*(\sum_{i=1}^{n-1}\alpha_i dy_i),\ \beta = F^*(\sum_{j=1}^{n-1}\beta_j dy_j),\\ g^{ij}(y') = (g_{ij}(y')^{-1}.\end{gathered} \tag{10}$$

Now, recall that we have a direct way of constructing a metric on M, namely by restricting the Euclidean metric on $\mathbb{R}^n$ to each $T_pM \subset T_p\mathbb{R}^n$. This is a very special metric, but I will use none of its properties here – we can simply take any metric on M.

Exercise 3. Show how to construct a metric on M given a metric on each of the Ω'_i corresponding to a covering of M by adapted coordinate systems.

Using a chosen metric – fixed from now on – we have the first ingredient for the Dirichlet form on M. Namely, if f, $g \in \mathcal{C}^\infty(M)$ (say real-valued) then the metric inner product on T^*M defines a function

$$\langle df, dg \rangle \in \mathcal{C}^\infty(M). \tag{11}$$

What we want to do next is to integrate this over M.

(1) Densities and distributions on M.
(2) Localization.
(3) Recall Laplacian on $M = \partial B$.
(4) $(\Delta + L)^{-1} : H^{-1}(M) \longrightarrow H^1(M)$ bounded.
(5) Lower regularity, $(\Delta + L)^{-1} : H^{-1-k}(M) \longrightarrow H^{1-k}(M)$ using vector fields.
(6) A priori estimates.
(7) Regularity
(8) Self-adjointness of Δ and eigenbasis – estimates.
(9) What sort of kernel?
(10) Elliptic operators in general.
(11) Functions of Δ, $\exp(-t\Delta)$.