To define the Laplacian acting on function and distributions on the manifold $M = \partial B$ realized as the boundary of a smoothly bounded domain, I need to go through some of the basic material of differential geometry – last time I defined the tangent and cotangent bundles, and we need to consider sections of these, the notion of a metric, densities (so we can integrate) and distributions. All this works perfectly well on an abstractly defined manifold, but I will limit myself here to those which can be embedded in this way.

The definition of the tangent space to $M$ at $p$ in terms of derivations shows that if $F : O \rightarrow \Omega' \subset \mathbb{R}^{n-1}$ is a smooth map from an open neighbourhood $p \in O \subset M$ then

\[ F_* : T_p M \rightarrow T_{F(p)} \mathbb{R}^{n-1} = \mathbb{R}^{n-1}, \quad F_*(\delta)u = \delta(u \circ F) = \delta(F^*u). \]

In particular this applies to our ‘adapted coordinates’. The tangent bundle of $M$ is just the disjoint union of all the tangent spaces

\[ TM = \bigcup_{p \in M} T_p M \]

and a section of $TM$ over an open set $O \subset M$ is a map $v : O \rightarrow TM$ such that $v(p) \in T_p M$. Such a section is smooth if $F_*v$ is smooth as a map from $F(O)$ into $\mathbb{R}^{n-1}$ for all, or equivalently a covering of $M$ by, adapted coordinates. Such a smooth section is a smooth vector field on $M$; we write the space of global smooth sections as $C^\infty(M; TM)$.

Exercise 1. Show that if one has a ‘local’ vector field as in (3) for each $\Omega_i'$ forming a covering of $M$ by adapted coordinates then summing over a partition of unity gives a global smooth section of $TM$. Use this to show that the restriction map from global sections

\[ C^\infty(M; TM) \rightarrow T_p M \]

is surjective.

A linear differential operator of order $m$ on functions on $M$ can now be defined to be a linear map

\[ P : C^\infty(M) \rightarrow C^\infty(M) \]

which is local, so $u = 0$ in an open set implies $Pu = 0$ in that set, and which can be written as a finite sum of up to $m$-fold products of vector fields acting on $C^\infty(M)$ (where 0-factors means multiplication by a smooth function).
Exercise 2. Show that in local adapted coordinates a differential operator is the obvious thing:

\[ G^*PF^*u = \sum_{|\alpha| \leq m} p_\alpha(y') D_y^\alpha u, \; u \in C^\infty(\Omega'). \]

The coefficients are determined by \( P \) and \( F \) (recall \( G = F^{-1} \)) so knowing them in one coordinate system determines them on the overlap with any other coordinate system. The transformation law is rather complicated! However, you can check that the principal part

\[ p_m(p,\eta) = \sum_{|\alpha|=m} p_\alpha(y')\xi^\alpha, \; \eta = f^*\xi \in T^*_pM \]

is actually a globally well-defined function on \( T^*M \) which is a homogeneous polynomial on each fibre.

We are after the particular example of such an operator given by the Laplacian associated to a metric on \( M \).

A metric on a real vector space \( V \) is just a positive-definite symmetric bilinear form. A metric on \( M \) is such a metric \( \langle \cdot, \cdot \rangle \) on each \( T_pM \) such that if \( v \) and \( w \) are any smooth sections of \( TM \), as defined above, then \( \langle v, w \rangle \in C^\infty(M) \). You should check that in adapted local coordinates this means that

\[ \langle v, w \rangle = \sum_{i,j=1}^{n-1} g_{ij}(y')v_i(y')w_j(y'), \; F^*v = \sum_i v_i(y')\partial_{y_i}, \; F^*w = \sum_{j=1}^{n-1} w_j(y')\partial_{y_j} \]

where the symmetric real matrix \( g_{ij} \) is positive definite. The transformation law between coordinate systems is not so complicated.

Now a metric in this sense also induces a fibre metric on \( T^*M \). This is just the algebraic fact that metrics on \( V \) and \( V^* \) are in 1-1 correspondence since a metric on \( V \) determines an isomorphism \( l : V \to V^* \) by

\[ l(v) = \alpha \iff \alpha(w) = \langle v, w \rangle \forall w \in V. \]

In local coordinates, using the basis \( dy_i \) of \( T^*M \), given by an adapted coordinate system the dual metric to \( \xi_i \) is

\[ \langle \alpha, \beta \rangle = \sum_{i,i=1}^{n-1} g^{ij} \alpha_i \beta_j, \]

\[ \alpha = F^*(\sum_{i=1}^{n-1} \alpha_i dy_i), \; \beta = F^*(\sum_{j=1}^{n-1} \beta_j dy_j), \]

\[ g^{ij}(y') = (g_{ij}(y'))^{-1}. \]

Now, recall that we have a direct way of constructing a metric on \( M \), namely by restricting the Euclidean metric on \( \mathbb{R}^n \) to each \( T_pM \subset T_p\mathbb{R}^n \). This is a very special metric, but I will use none of its properties here – we can simply take any metric on \( M \).

Exercise 3. Show how to construct a metric on \( M \) given a metric on each of the \( \Omega'_i \) corresponding to a covering of \( M \) by adapted coordinate systems.
Using a chosen metric – fixed from now on – we have the first ingredient for the Dirichlet form on $M$. Namely, if $f, g \in C^\infty(M)$ (say real-valued) then the metric inner product on $T^*M$ defines a function

$$\langle df, dg \rangle \in C^\infty(M).$$

What we want to do next is to integrate this over $M$.

1. Densities and distributions on $M$.
2. Localization.
3. Recall Laplacian on $M = \partial B$.
4. $(\Delta + L)^{-1} : H^{-1}(M) \rightarrow H^1(M)$ bounded.
5. Lower regularity, $(\Delta + L)^{-1} : H^{-1-k}(M) \rightarrow H^{1-k}(M)$ using vector fields.
6. A priori estimates.
7. Regularity
8. Self-adjointness of $\Delta$ and eigenbasis – estimates.
9. What sort of kernel?
10. Elliptic operators in general.
11. Functions of $\Delta, \exp(-t\Delta)$. 