As last time we consider $B \subset \mathbb{R}^n$ be a smoothly bounded domain.

Starting from the Sobolev spaces on $\mathbb{R}^n$ we define various Sobolev spaces associated to $B$. In fact it is convenient to consider the ‘unbounded domain’ $\tilde{B} = \mathbb{R}^n \setminus B$ which has $B \cap \tilde{B} = \partial B = \partial \tilde{B}$ as well. Then for each $m \in \mathbb{R}$ we can define

(1) $\dot{H}^m(B) = \{ u \in H^m(\mathbb{R}^n); \text{supp}(u) \subset B \}$,

(2) $\dot{H}^m(\tilde{B}) = \{ u \in H^m(\mathbb{R}^n); \text{supp}(u) \subset \tilde{B} \}$,

$H^m(B) = \{ u \in \mathcal{C}^{-\infty}(B \setminus \partial B); \exists \tilde{u} \in H^m(\mathbb{R}^n), u(\phi) = \tilde{u}(\phi) \forall \phi \in \mathcal{C}_c^\infty(B \setminus \partial B) \}$,

$H^m(\tilde{B}) = \{ u \in \mathcal{C}^{-\infty}(\mathbb{R}^n \setminus B); \exists \tilde{u} \in H^m(\mathbb{R}^n), u(\phi) = \tilde{u}(\phi) \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus B) \}$.

Then we have short exact sequences

(2) \[
\begin{align*}
\dot{H}^m(B) & \rightarrow H^m(\mathbb{R}^n) \rightarrow H^m(B) \\
\dot{H}^m(\tilde{B}) & \rightarrow H^m(\mathbb{R}^n) \rightarrow H^m(\tilde{B})
\end{align*}
\]

since if $u \in H^m(\mathbb{R}^n)$ restricts to be zero in $\mathbb{R}^n \setminus B$ (resp. $B \setminus \partial B$) it has support in $B$ (resp. $\tilde{B}$). The ‘supported’ subspaces are closed, and hence the ‘extension’ subspaces are also Hilbert spaces, given as the quotients.

**Proposition 1.** There are dense subspaces of smooth dense functions

(3) \[
\begin{align*}
\mathcal{C}_c^\infty(B \setminus \partial B) & \subset \dot{H}^m(B), \\
\mathcal{C}_c^\infty(\mathbb{R}^n \setminus B) & \subset \dot{H}^m(\tilde{B}) \\
\mathcal{C}^\infty(B) & \subset H^m(B) \\
\mathcal{C}_c^\infty(\tilde{B}) & \subset H^m(\tilde{B})
\end{align*}
\]

and duality pairing

(4) \[
H^m(B) \times \dot{H}^{-m}(B) \rightarrow \mathcal{C}, \ \forall \ m \in \mathbb{R}
\]

which extends the pairing between $\mathcal{C}^\infty(B)$ and $\mathcal{C}_c^\infty(B \setminus \partial B)$ and allows us to identify $H^m(B)$ as the dual of $\dot{H}^{-m}(B)$.

**Proof.** For the extension spaces this follows directly – extend the distribution, approximate the extension by compactly supported smooth functions and restrict.

For the supported spaces, more is involved. Use a partition of unity centred on boundary points as discussed last lecture we may write $u \in H^m(B)$ as a finite sum of terms each supported in the interior or in a coordinate neighbourhoods. Approximation by convolution works for the first term, with supports staying in the interior. Using a ‘straightening’ diffeomorphism the other terms are transformed to have support in $y_n \geq 0$. Now translation in $x_n$ by $\delta_n \to 0$ approximates these by a sequence with support in the interior of $B$ when tided back by the diffeomorphism, so again convolution with an approximate identity gives a smooth approximating sequence. \hfill \Box
We are particularly interested in the case of $\dot{H}^1(B)$. This is a Hilbert space with the usual inner product:

$$\langle u, v \rangle = \int u\overline{v} + \sum_{i=1}^{n} \int D_i u D_i \overline{v}.$$ 

It is also a Hilbert space with the ‘homogeneous’ inner product where the $L^2$ term is dropped

$$(5) \quad \langle Du, Dv \rangle = \sum_{i=1}^{n} \int D_i u D_i \overline{v}.$$ 

**Lemma 1 (Poincaré).** For each smoothly bounded domain there is a constant $C$ such that

$$(6) \quad \|u\|_{L^2}^2 \leq C^2 \sum_{i=1}^{n} \int |D_i u|^2 \ \forall \ u \in \dot{H}^1(B).$$

**Proof.** Since we are not looking for the best constant here, we can replace $B$ by any bigger domain, since $\dot{H}^1(B)$ increases. So it is enough to prove (6) for $u \in \dot{H}^1(\mathbb{R}^n)$ with support in a fixed ball $\{|x| \leq R\}$. Now any point except the origin is of the form $s\omega$ for a unit vector $\omega \in S^{n-1}$ and we can integrate outwards along the radial line to see that

$$(7) \quad u(s\omega) = -\int_{s}^{R} \frac{du(r\omega)}{dr} dr.$$ 

Since $du(r\omega)/dr = \omega \cdot \nabla u(r\omega)$ we can apply Cauchy-Schwartz to see that

$$(8) \quad |u(s\omega)|^2 \leq C_R \int_{s}^{R} |\nabla u(r\omega)|^2 dr, \ \nabla u = (\partial_1 u, \ldots, \partial_n u).$$

Integrating over $s$ and using the fact $s \leq r$ and then changing the order of integration on the right gives

$$(9) \quad \int |u(s\omega)|^2 s^{n-1} ds \leq C'_R \int_{s}^{R} |\nabla u(r\omega)|^2 r^{n-1} dr$$

Now integrating over the unit sphere gives the Poincaré inequality

$$(10) \quad \int_{\mathbb{B}(R)} |u|^2 \leq C(R) \int_{\mathbb{B}(R)} |Du|^2.$$ 

□

This allows us to approach the Dirichlet problem via ‘abstract functional analysis’. Namely, suppose $f \in (\dot{H}^1(B))' = H^{-1}(B)$ is in the dual space. Then

$$(11) \quad \dot{H}^1(B) \ni u \mapsto \langle u, f \rangle$$

given by the distributional pairing (i.e. extension of the $L^2$ pairing) is a continuous linear functional. Thus, by Riesz’ representation theorem there exists $v \in \dot{H}^1(B)$ such that

$$(12) \quad \langle u, v \rangle_D = \int_B \sum_{i} D_i u D_i \overline{v} = \langle u, f \rangle.$$
Now $v$ is uniquely determined by $f$ and $\|v\|_{H^1} \leq C\|f\|_{H^{-1}}$ so this defines a bounded linear map

$$A : H^{-1}(B) \longrightarrow \dot{H}^1(B).$$

(13)

Notice that if $v = \phi \in C_c^\infty(B \setminus \partial B)$ then in terms of the distributional pairing over $B \setminus \partial B$, 

$$\Delta Af = f \text{ on } B \setminus \partial B, \ f \in H^{-1}(B) \subset C^{-\infty}(B \setminus \partial B).$$

Thus we have constructed at least a right inverse of the Laplacian, with $Af$ satisfying the Dirichlet condition in the sense of restriction of Sobolev spaces to the boundary.

Consider $A$ restricted to a bounded operator

$$A : \dot{H}^1(B) \hookrightarrow \dot{H}^1(B).$$

(15)

If we restrict to $f \in \dot{H}^1(B)$ in (12) then

$$\langle Af, v \rangle_D = \langle f, v \rangle_{L^2} = \langle v, f \rangle_{L^2} = \langle Av, f \rangle_D = \langle f, Av \rangle_D.$$ 

This shows that as an operator (15), $A$ is self-adjoint.

Now, as an operator $L^2(B) \longrightarrow \dot{H}^1(B) \hookrightarrow L^2(B)$, by restriction, $A$ is compact since it maps into $\dot{H}^1(B)$ which is compactly included into $L^2(B)$. Thus the spectrum of $A$ on $L^2(B)$ is discrete, and of finite algebraic multiplicity, outside 0. If $A - t, t \neq 0$ is invertible on $L^2$ then solving $(A - t)g = f$ with $f \in \dot{H}^1(B)$ and $g \in L^2(B)$ then $Lg \in \dot{H}^1(B)$ and hence $tg = Ag - f \in \dot{H}^1(B)$. So $A$ also has discrete spectrum outside 0 as a self-adjoint operator on $\dot{H}^1(B)$. In fact the same argument shows that the range of $A - t$ is always closed on $\dot{H}^1(B)$, as it is on $L^2(B)$ for $t \neq 0$. From this we conclude that $A$ is actually compact on $\dot{H}^1(B)$ and has a complete orthonormal basis of eigenfunctions in $\dot{H}^1(B)$.

Using the definition of $A$ again, if $e_i$ and $e_j$ correspond to different eigenvalues, and hence are orthogonal in $\dot{H}^1(B)$ they are orthogonal in $L^2(B)$

$$\langle e_i, e_j \rangle_{L^2} = \langle Ae_i, e_j \rangle_D = s_i \langle e_i, e_j \rangle_D = 0.$$ 

From the density of their span in $\dot{H}^1(B)$ it follows that, renormalized to have

$$\|e_i\|_{L^2} = 1, \ Ae_i = s_i e_i$$

these eigenvectors form an orthonormal basis of $L^2(B)$.

So in fact we have shown that for any smoothly bounded domain, $L^2(B)$ has a complete orthonormal basis of eigenfunctions each in $\dot{H}^1(B)$ and satisfying $\Delta e_j = \lambda_j e_j$ in the interior of $B$. What we have not shown is that these eigenfunctions are smooth. The regularity result we want is that

$$A : H^k(B) \longrightarrow H^{k+2}(B) \cap \dot{H}^1(B) \text{ for } k = 0, 1, 2, \ldots.$$ 

By locally elliptic regularity for $\Delta$ – which is a constant coefficient elliptic operator – we do know that the range of $A$ on $H^k(B)$ lie in $H^{k+2}_{loc}(B \setminus \partial B)$. It is regularity up to the boundary that we need to work for.