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As last time we consider $B \subset \mathbb{R}^n$ be a smoothly bounded domain.

Starting from the Sobolev spaces on $\mathbb{R}^n$ we define various Sobolev spaces associated to B. In fact it is convenient to consider the 'unbounded domain' $\tilde{B} = \overline{\mathbb{R}^n \setminus B}$ which has $B \cap \tilde{B} = \partial B = \partial \tilde{B}$ as well. Then for each $m \in \mathbb{R}$ we can define

(1)
$$\begin{aligned} \dot{H}^m(B) &= \{u \in H^m(\mathbb{R}^n); \operatorname{supp}(u) \subset B\}, \\ \dot{H}^m(\tilde{B}) &= \{u \in H^m(\mathbb{R}^n); \operatorname{supp}(u) \subset \tilde{B}\}, \\ H^m(B) &= \{u \in \mathcal{C}^{-\infty}(B \setminus \partial B); \exists\ \tilde{u} \in H^m(\mathbb{R}^n),\ u(\phi) = \tilde{u}(\phi)\ \forall\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(B \setminus \partial B)\}, \\ H^m(\tilde{B}) &= \{u \in \mathcal{C}^{-\infty}(\mathbb{R}^n \setminus B); \exists\ \tilde{u} \in H^m(\mathbb{R}^n),\ u(\phi) = \tilde{u}(\phi)\ \forall\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n \setminus B)\}. \end{aligned}$$

Then we have short exact sequences

$$\begin{gathered} \dot{H}^m(B) \longrightarrow H^m(\mathbb{R}^n) \longrightarrow H^m(B) \\ \dot{H}^m(\tilde{B}) \longrightarrow H^m(\mathbb{R}^n) \longrightarrow H^m(\tilde{B}) \end{gathered} \tag{2}$$

since if $u \in H^m(\mathbb{R}^n)$ restricts to be zero in $\mathbb{R}^n \setminus B$ (resp. $B \setminus \partial B$) it has support in B (resp. $\tilde{B}$). The 'supported' subspaces are closed, and hence the 'extension' subspaces are also Hilbert spaces, given as the quotients.

Proposition 1. *There are dense subspaces of smooth dense functions*

$$\begin{gathered} \mathcal{C}^\infty_c(B \setminus \partial B) \subset \dot{H}^m(B), \\ \mathcal{C}^\infty_c(\mathbb{R}^n \setminus B) \subset \dot{H}^m(\tilde{B}) \\ \mathcal{C}^\infty(B) \subset H^m(B) \\ \mathcal{C}^\infty_c(\tilde{B}) \subset H^m(\tilde{B}) \end{gathered} \tag{3}$$

and duality pairing

$$H^m(B) \times \dot{H}^{-m}(B) \longrightarrow \mathbb{C},\ \forall\ m \in \mathbb{R} \tag{4}$$

which extends the pairing between $\mathcal{C}^\infty(B)$ and $\mathcal{C}^\infty_c(B \setminus \partial B)$ and allows us to identify $H^m(B)$ as the dual of $\dot{H}^{-m}(B)$.

Proof. For the extension spaces this follows directly – extend the distribution, approximate the extension by compactly supported smooth functions and restrict.

For the supported spaces, more is involved. Use a partition of unity centred on boundary points as discussed last lecture we may write $u \in \dot{H}^m(B)$ as a finite sum of terms each supported in the interior or in a coordinate neighbourhoods. Approximation by convolution works for the first term, with supports staying in the interior. Using a 'straightening' diffeomorphism the other terms are transformed to have support in $y_n \geq 0$. Now translation in x_n by $\delta_n \to 0$ approximates these by a sequence with support in the interior of B when fted nback by thge diffeomorphism, so again convolution with an approximate identity gives a smooth approximating sequence. □

We are particularly interested in the case of $\dot{H}^1(B)$. This is a Hilbert space with the usual inner product:

$$\langle u, v\rangle = \int u\bar{v} + \sum_{i=1}^{n} \int D_i u \overline{D_i v}.$$

It is also a Hilbert space with the 'homogeneous' inner product where the L^2 term is dropped

$$\langle Du, Dv\rangle = \sum_{i=1}^{n} \int D_i u \overline{D_i v}. \tag{5}$$

Lemma 1 (Poincaré)**.** *For each smoothly bounded domain there is a constant C such that*

$$\|u\|_{L^2}^2 \le C^2 \sum_{i=1}^{n} \int |D_i u|^2 \ \forall \ u \in \dot{H}^1(B). \tag{6}$$

Proof. Since we are not looking for the best constant here, we can replace B by any bigger domain, since $\dot{H}^1(B)$ increases. So it is enough to prove (6) for $u \in \dot{H}^1(\mathbb{R}^n)$ with support in a fixed ball $\{|x| \le R\}$. Now any point except the origin is of the form $s\omega$ for a unit vector $\omega \in \mathbb{S}^{n-1}$ and we can integrate outwards along the radial line to see that

$$u(s\omega) = -\int_s^R \frac{du(r\omega)}{dr} dr. \tag{7}$$

Since $du(r\omega)/dr = \omega \cdot \nabla u(r\omega)$ we can apply Cauchy-Schwartz to see that

$$|u(s\omega)|^2 \le C_R \int_s^R |\nabla u(r\omega)|^2 dr, \ \nabla u = (\partial_1 u, \dots, \partial_n u). \tag{8}$$

Integrating over s and using the fact $s \le r$ and then changing the order of integration on the right gives

$$\int |u(s\omega)|^2 s^{n-1} ds \le C_R' \int_s^R |\nabla u(r\omega)|^2 r^{n-1} dr \tag{9}$$

Now integrating over the unit sphere gives the Poincaré inequality

$$\int_{\mathbb{B}(R)} |u|^2 \le C(R) \int_{\mathbb{B}(R)} |Du|^2. \tag{10}$$

□

This allows us to approach the Dirichlet problem via 'abstract functional analysis'. Namely, suppose $f \in (\dot{H}^1(B))' = H^{-1}(B)$ is in the dual space. Then

$$\dot{H}^1(B) \ni u \longrightarrow \langle u, f\rangle \tag{11}$$

given by the distributional pairing (i.e. extension of the L^2 pairing) is a continuous linear functional. Thus, by Riesz' representation theorem there exists $v \in \dot{H}^1(B)$ such that

$$\langle u, v\rangle_D = \int_B \sum_i D_i u \overline{D_i v} = \langle u, f\rangle. \tag{12}$$

Now v is uniquely determined by f and $\|v\|_{H^1} \le C\|f\|_{H^{-1}}$ so this defines a bounded linear map

$$A : H^{-1}(B) \longrightarrow \dot{H}^1(B). \tag{13}$$

Notice that if $v = \phi \in \mathcal{C}^\infty_{\mathrm{c}}(B \setminus \partial B)$ then in terms of the distributional pairing over $B \setminus \partial B$,

$$\Delta Af = f \text{ on } B \setminus \partial B,\ f \in H^{-1}(B) \subset \mathcal{C}^{-\infty}(B \setminus \partial B). \tag{14}$$

Thus we have constructed at least a right inverse of the Laplacian, with Af satisfying the Dirichlet condition in the sense of restriction of Sobolev spaces to the boundary.

Consider A restricted to a bounded operator

$$A : \dot{H}^1(B) \hookrightarrow \dot{H}^1(B). \tag{15}$$

If we restrict to $f \in \dot{H}^1(B)$ in (12) then

$$\langle Af, v\rangle_D = \langle f, v\rangle_{L^2} = \overline{\langle v, f\rangle_{L^2}} = \overline{\langle Av, f\rangle_D} = \langle f, Av\rangle_D. \tag{16}$$

This shows that as an operator (15), A is self-adjoint.

Now, as an operator $L^2(B) \longrightarrow \dot{H}^1(B) \hookrightarrow L^2(B)$, by restriction, A is *compact* since it maps into $\dot{H}^1(B)$ which is compactly included into $L^2(B)$. Thus the spectrum of A on $L^2(B)$ is discrete, and of finite algebraic multiplicity, outside 0. If $A - t$, $t \neq 0$ is invertible on L^2 then solving $(A - t)g = f$ with $f \in \dot{H}^1(B)$ and $g \in L^2(B)$ then $Lg \in \dot{H}^1(B)$ and hence $tg = Ag - f \in \dot{H}^1(B)$. So A also has discrete spectrum outside 0 as a self-adjoint operator on $\dot{H}^1(B)$. In fact the same argument shows that the range of $A-t$ is always closed on $\dot{H}^1(B)$, as it is on $L^2(B)$ for $t \neq 0$. From this we conclude that A is actually compact on $\dot{H}^1(B)$ and has a complete orthonormal basis of eigenfunctions in $\dot{H}^1(B)$.

Using the definition of A again, if e_i and e_j correspond to different eigenvalues, and hence are orthogonal in $\dot{H}^1(B)$ they are orthogonal in $L^2(B)$

$$\langle e_i, e_j\rangle_{L^2} = \langle Ae_i, e_j\rangle_D = s_i\langle e_i, e_j\rangle_D = 0.$$

From the density of their span in $\dot{H}^1(B)$ it follows that, renormalized to have

$$\|e_i\|_{L^2} = 1,\ Ae_i = s_i e_i \tag{17}$$

these eigenvectors form an orthonormal basis of $L^2(B)$.

So in fact we have shown that for any smoothly bounded domain, $L^2(B)$ has a complete orthonormal basis of eigenfunctions each in $\dot{H}^1(B)$ and satisfying $\Delta e_j = \lambda_j e_j$ in the interior of B. What we have not shown is that these eigenfunctions are smooth. The regularity result we want is that

$$A : H^k(B) \longrightarrow H^{k+2}B) \cap \dot{H}^1(B) \text{ for } k = 0, 1, 2, \dots.$$

By locally elliptic regularity for Δ – which is a constant coefficient elliptic operator – we do know that the range of A on $H^k(B)$ lie in $H^{k+2}_{\mathrm{loc}}(B \setminus \partial B)$. It is regularity up to the boundary that we need to work for.