LECTURE 17, 18.155, 5 NOVEMBER 2013

The first thing I want to talk about in relation to half-spaces and bounded domains is the restriction theorem for Sobolev spaces. So consider the embedding map

$$E : \mathbb{R}^{n-1} \ni x' \longmapsto (x', 0) \in \mathbb{R}^n. \tag{1}$$

Pullback under this is the restriction map, $R = E^*$, $Rf(x') = f(x', 0) = f \circ E$.

Proposition 1. *The restriction map*

$$R : \mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto \phi(\cdot, 0) \in \mathcal{S}(\mathbb{R}^{n-1}) \tag{2}$$

extends by continuity to a surjective bounded map

$$H^m(\mathbb{R}^n) \longrightarrow H^{m-\frac{1}{2}}(\mathbb{R}^{n-1}) \ \forall \ m > \frac{1}{2}. \tag{3}$$

Proof. The Fourier transform of the restriction to $x_n = 0$ of a Schwartz function can be written in terms of the Fourier transform:

$$\widehat{R\phi}(\xi') = \int e^{-ix'\cdot\xi'} \phi(x', 0) dx' = (2\pi)^{-1} \int \hat{\phi}(\xi', \xi_n) d\xi_n. \tag{4}$$

Now, for functions in $H^m(\mathbb{R}^n)$ with Fourier tranform supported in $\{|\xi| \le 1\}$ the result is clear, since the restriction also has Fourier tranform with support in the ball and so is in $H^\infty(\mathbb{R}^{n-1})$. So, we may assume that $\hat{u} = 0$ in $\{|\xi| \le 1\}$ and that the same is true of an approximating sequence in $\mathcal{S}(\mathbb{R}^n)$. Thus it suffices to estimate the norm of (4) in $H^{m-1}(\mathbb{R}^{n-1})$ under this assumption.

The integral may be estimated by Cauchy Schwartz' inequality, aiming at the H^m norm:

$$|\int \hat{\phi}(\xi', \xi_n) d\xi_n|^2 \le \int |\hat{\phi}(\xi', \xi_n)|^2 (|\xi|^2 + |\xi_n|^2)^m d\xi_n \int (|\xi|^2 + |\xi_n|^2)^{-m} d\xi_n \tag{5}$$

where the second integral is finite proved $m > \frac{1}{2}$ as we are assuming. Then it can be evaluated by scaling

$$\int (|\xi|^2 + |\xi_n|^2)^{-m} d\xi_n = c|\xi'|^{-2m+1}, \ c > 0. \tag{6}$$

Inserting this in (5) shows that

$$\|R\phi\|_{H^{m-\frac{1}{2}}} \le \int |\xi'|^{2m-1} |\int \hat{\phi}(\xi', \xi_n) d\xi_n|^2 \le C\|\phi\|_{H^m} \tag{7}$$

using the support property of the Fourier tranform. This proves (3).

For the converse we will construct a right inverse to R. If $v \in H^{m-\frac{1}{2}}(\mathbb{R}^{n-1})$ then

$$\int_{\mathbb{R}^{n-1}} (1+|\xi'|^2)^{2m-1}|\hat{v}(\xi')|^2 d\xi' < \infty \tag{8}$$

and all we need to do is to construct $w \in L^2(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^{2m}|w(\xi)|^2 d\xi < \infty,\ v(\xi') = (2\pi)^{-1}\int w(\xi', \xi_n) d\xi_n.$$

Choose $0 \le \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ with $\int \phi = 1$. For part of $\hat{v}$ supported in $|\xi'| \le 2\pi$ such an extension is given by

$$w(\xi', \xi) = \phi(\xi_n)\chi_{\{|\xi'|\le 1\}}\hat{v}(\xi')$$

since this is in $H^\infty(\mathbb{R}^n)$. So, really just for notation convenience, we can assume that $\hat{v}(\xi') = 0$ in $|\xi'| \le 1$. Then we use the same idea, but 'spread the support':

$$w(\xi', \xi_n) = \phi(\frac{\xi_n}{|\xi'|})|\xi'|^{-1}\hat{v}(\xi'). \tag{9}$$

Then

(10)

$$\int_{\mathbb{R}} w(\xi', \xi_n) d\xi_n = |\xi'|^{-1}\hat{v}(\xi')\int \phi(\frac{\xi_n}{|\xi'|}) d\xi_n = \hat{v}(\xi'),$$

$$\int_{\mathbb{R}^n} |\xi|^{2m}|w(\xi', \xi_n)|^2 d\xi_n d\xi' = \int_{\mathbb{R}^{n-1}} |\hat{v}(\xi')|^2 \int_{\mathbb{R}} (|\xi_n|^2 + |\xi'|^2)^m |\xi'|^{-2}\phi(\frac{\xi_n}{|\xi'|})|^2 d\xi_n d\xi'$$

where the inner integral is actually a constant multiple of $|\xi'|^{2m-1}$. $\square$

We do not actually need $m > \frac{1}{2}$ to find a right inverse in the last part of the argument – even for $m \le \frac{1}{2}$ if $v \in H^{m-\frac{1}{2}}(\mathbb{R}^{n-1})$ there is a distribution $u \in H^m(\mathbb{R}^n)$ which happens to have the property that $u(\cdot, x_n)$ is continuous in x_n with values in distributions, which restricts to v at $x_n = 0$. If I have some time later I will discuss this sort of thing a bit more.

The remainder of this lecture is reconstructed after the event.

A diffeomorphism between open sets, U, $U' \subset \mathbb{R}^n$, is a smooth map with a smooth two-sided inverse, $F : U \longrightarrow U'$, $G : U' \longrightarrow U$, $F(x) = (f_1, \dots, f_n x)$, $G(x) = (g_1(x), \dots, g_n(x))$ with $f_i \in \mathcal{C}^\infty(U)$, $g_i \in \mathcal{C}^\infty(U')$ (real-valued of course) and

$$F(G(y)) = y\ \forall\ y \in U',\ G(F(x)) = x\ \forall\ x \in U. \tag{11}$$

For any smooth map, the pull-back operation is defined by composition:

$$F^*:\mathcal{C}^\infty(U')\longrightarrow\mathcal{C}^\infty(U),\ F^*f(x)=f(F(x)). \tag{12}$$

Then F is a diffeomorphism if and only if (12) is a bijection – since the components of G are the functions which pull-back to the coordinate functions on U.

The tangent space of $\mathbb{R}^n$ at a pont p may be defined as the space of derivations of $\mathcal{C}^\infty(O)$ for any open $O\ni p$, the linear maps

$$T_p\mathbb{R}^n=\{\delta:\mathcal{C}^\infty(O)\longrightarrow\mathbb{C},\ \text{s.t. }\delta(fg)=f(p)\delta(g)+g(p)\delta(f)\}. \tag{13}$$

Such a derivation if just a sum of the basic derivations

$$\partial_i:\mathcal{C}^\infty(O)\ni f\longmapsto\frac{\partial f}{\partial x_i}(p),\ \delta=\sum_i c_i\partial_i. \tag{14}$$

Thus the standard coordinates give a natural trivialization $T_p\mathbb{R}^n=\mathbb{R}^n$. If $F:U\longrightarrow U'$ is smooth then its differential at p is

(15)

$$F_*:T_p\mathbb{R}^n\longrightarrow T_{F(p)}\mathbb{R}^n,\ F_*(\delta)=\delta',\ \delta':\mathcal{C}^\infty(U')\longrightarrow\mathbb{R},\ \delta'(g)=\delta(F^*g).$$

Clearly as a map in terms of the coordinate trivialization this is given by the Jacobian matrix

$$F_*(\partial_i(p))=\sum_j\frac{\partial F_j}{\partial x_i}(p)\partial_j(f(p)). \tag{16}$$

If F is a diffeomorphism, then F_* must be invertible at each point, with inverse $G_*(f(p))$. Conversely, the inverse function theorem implies that if $F:O\longrightarrow\mathbb{R}^n$ is smooth and $F_*(p)$ is invertible then $F:B(p,\epsilon)\longrightarrow F(B(p,\epsilon))$ is a diffeomorphism of open sets for $\epsilon>0$ small enough.

Now, if $F:U\longrightarrow U'$ is a diffeomorphism then, not only does (12) hold, but also

$$F^*:\mathcal{C}^\infty_{\mathrm{c}}(U')\longrightarrow\mathcal{C}^\infty(U) \tag{17}$$

is an isomorphism, since F maps compact subsets of U onto (all) compact subsets of U'. These two spaces are dense in the distribution spaces so it makes sense to claim:

Proposition 2. *For any diffeomorphism $F : U \longrightarrow U'$. the maps* (17) *and* (12) *extend by continuity to isomorphisms*

$$
\begin{aligned}
F^* &: H^m_{\mathrm{loc}}(U') \longrightarrow H^m_{\mathrm{loc}}(U),\\
F^* &: H^m_c(U') \longrightarrow H^m_c(U)\ \forall\ m,\\
F^* &: \mathcal{C}^{-\infty}(U') \longrightarrow \mathcal{C}^{-\infty}(U),\\
F^* &: \mathcal{C}^{-\infty}_c(U') \longrightarrow \mathcal{C}^{-\infty}_c(U).
\end{aligned}
\tag{18}
$$

Proof. We need first to recall the behaviour of integrable functions under diffeomorphisms. If $f \in L^1_{\mathrm{c}}(U')$ then indeed, $F^*f \in L^1_{\mathrm{c}}(U)$, and the integrals are related by

$$
\int_U F^*f J_F = \int_{U'} f,\ \ J_F = |\det(\frac{\partial F_i}{\partial x_j})|. \tag{19}
$$

In particular there is no sign change in the Lebesgue integral if one reverses one of the variables.

Perhaps I should recall a little where (19) comes from, but it is of course a very standard formula.

This immediately extends to L^2 since a function $u \in L^2_{\mathrm{c}}(U')$ is just one such that $u,\ |u|^2 \in L^1_{\mathrm{c}}(U')$. This gives the second result in (18) for $m = 0$.

Continuing with this case, consider $0 < m < 1$. Since we are looking at functions with compact support, $u \in H^m_{\mathrm{c}}(U')$ then means that $u \in L^2_{\mathrm{c}}(U')$ and

$$
\int_{U'\times U'} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2m}} dxdy < \infty.
$$

In fact, if $\delta > 0$ then for an L^2 function of compact support, () is equivalent to

$$
\int_{U'\times U',\ |x-y|<\delta} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2m}} dxdy < \infty. \tag{20}
$$

Indeed, the integral over $|x-y| \ge \delta$ can be bounded by twice

$$
2\int_{U'\times U',\ |x-y|>\delta} \frac{|u(x)|^2}{|x-y|^{n+2m}} dxdy \tag{21}
$$

which is indeed finite. So, given $u \in H^m_{\mathrm{c}}(U')$ to show that $F^*u \in H^m_{\mathrm{c}}(U)$ it remains only to show that

$$
\int_{K\times K,\ |x-y|<\delta} \frac{|u(F(x)) - u(F(y))|^2}{|x-y|^{n+2m}} dxdy < \infty \tag{22}
$$

where $K \Subset U$. Using Taylor's formula

$$F(x)-F(y) = (x-y)\cdot\frac{\partial F}{\partial x}+E,\ |E(x,y)| \le C|x-y|^2 \text{ in } |x-y| \le \delta \tag{23}$$

uniformly over $x \in K$ if $\delta > 0$ is small enough. Since the Jacobian matrix is invertible it follows that

$$|x-y| \ge c|F(x)-F(y)| \text{ on } K \times K \cap \{|x-y| < \delta\}.$$

Thus instead of (22) it is enough to show that

$$\int_{K\times K,\ |x-y|<\delta} \frac{|u(F(x))-u(F(y))|^2}{|F(x)-F(y)|^{n+2m}} J_f(x)J_f(y)dxdy < \infty \tag{24}$$

since the Jacobian factors are strictly positive. Now we simply change variable as in (19) and the finiteness follows from ().

Now suppose that $k \le m < k+1$ for $k \in \mathbb{N}$. We can proceed by induction over k using the fact that $u \in H^m_c(U')$ is equivalent to

$$u,\ D_i u \in H^{m-1}_c(U'),\ i = 1, \dots, n.$$

Thus, by the inductive hypothesis, it follows that F^*u, $F^*(D_iu) \in H^{m-1}_c(U)$. However, the behaviour of derivations is simple, in that

$$D_i F^* u = \sum_{j=1}^n a_{ij}(x) F^*(D_j u) \tag{25}$$

where the coefficients are again essentially the Jacobian matrix, in any case are smooth. Since we know the compactly-supported Sobolev spaces are modules over $\mathcal{C}^\infty(U)$, we conclude that $u \in H^m_c(U)$ and the result follows for all $m \ge 0$.

The proof for $m < 0$ is similar, since if $-k < m < -k+1$, $k \in \mathbb{N}$, then $u \in H^m_c(U')$ is equivalent to being able to decompose it as a sum

$$u = v_0 + \sum_{i=1}^n D_i v_i,\ v_p \in H^{m+1}_c(U').$$

The same sort of inductive argument therefore applies.

Thus we have proved the second statement in (18). The last is a consequence since each compactly supported distribution is in some Sobolev space. The first and third identifications then follow from the second and last by a suitable localization argument. $\square$