

18.155 LECTURE 11, 10 OCTOBER, 2013

Last time I talked about hypoellipticity, today I want to talk about ellipticity for constant coefficient operators. By definition (from last week) a polynomial $P(\xi)$ of degree m in n variables is elliptic if its leading part has no real, non-zero, roots:-

$$(1) \quad P(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha, \quad P_m(\xi) = \sum_{|\alpha|=m} c_\alpha \xi^\alpha,$$

P elliptic of order m iff $P_m(\xi) \neq 0$ on $\mathbb{R}^n \setminus \{0\}$.

Now, the ellipticity of the principal part is equivalent to

$$(2) \quad |P_m(\xi)| \geq c|\xi|^m, \quad c > 0$$

since $|P_m(\hat{\xi})|$ is a continuous, non-vanishing, function on the unit sphere.

Lemma 1. $P(\xi)$ is elliptic of order m if and only if there exists $\delta > 0$ such that

$$(3) \quad |\xi| \geq 1/\delta, \quad \xi \in \mathbb{R}^n \setminus \{0\} \implies |P(\xi)| \geq \delta|\xi|^m.$$

Proof. Since $P(\xi) - P_m(\xi)$ is of degree at most $m - 1$,

$$(4) \quad |P(\xi) - P_m(\xi)| \leq C(|\xi|^{m-1} + 1).$$

Since the right side is bounded above by $\frac{1}{2}c|\xi|^m$ for any $c > 0$ for $|\xi|$ large enough, (3) follows from (2) by choosing $\delta > 0$ small enough. Conversely, (2) follows from (3) in $|\xi| \geq C$ for some sufficiently large C . Then by homogeneity it holds in general. \square

In particular of course, if P is elliptic then $\{\xi \in \mathbb{R}^n; P(\xi) = 0\}$ is compact. So, for an elliptic polynomial if $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is one in $|x| \leq 1$ then for $\delta > 0$ small enough

$$(5) \quad F(\xi) = \begin{cases} \frac{1-\chi(\delta\xi)}{P(\xi)} & |\xi| > 1/\delta \\ 0 & |\xi| \leq 1/\delta \end{cases} \in \mathcal{C}^\infty(\mathbb{R}^n).$$

In fact we can estimate the derivatives in a straightforward way, since for $|\xi| \geq C/\delta$ for some $C > 0$, $F = 1/P$ and the by the usual inductive proof:

$$(6) \quad D^\alpha F = \frac{Q_\alpha}{P^{1+|\alpha|}}, \quad Q_\alpha \text{ polynomial of degree at most } (m-1)|\alpha| \\ \implies |D^\alpha F(\xi)| \leq C_\alpha(1 + |\xi|)^{-m-|\alpha|}.$$

Exactly as from the discussion last time it follows that if $E \in \mathcal{S}'(\mathbb{R}^n)$ is defined by $\hat{E} = F$ then

$$(7) \quad |x|^{2N} E(x) \in H^k(\mathbb{R}^n), \quad \text{if } 2N > k + \frac{n}{2} - m$$

but in any case if N is large enough. So $E(x)$ is smooth outside the origin, i.e.

$$(8) \quad \text{singsupp}(E) \subset \{0\}.$$

It follows that $P(D)$ is hypoelliptic, in terms of the definition from last lecture. However, for elliptic operators we get something stronger:-

Proposition 1. *If $P(D)$ is elliptic of order m then for any open subset $\Omega \subset \mathbb{R}^n$, and any $s \in \mathbb{R}$,*

$$(9) \quad u \in C^{-\infty}(\Omega), P(D)u \in H_{\text{loc}}^s(\Omega) \implies u \in H_{\text{loc}}^{s+m}(\Omega).$$

This is *elliptic regularity* where any solution u has m more (Sobolev) derivatives than the ‘forcing term’ $P(D)u$.

Proof. This follows from the same proof as last time, which I will write down here since I did not write it down in the notes for last lecture. The big difference is that

$$(10) \quad E* : H^s(\mathbb{R}^n) \longleftrightarrow H^{s+m}(\mathbb{R}^n) \quad \forall s \in \mathbb{R}$$

where E is constructed above, $\hat{E} = F$. That convolution by E is well-defined can be seen by writing it as the sum $\chi_\epsilon E + (1 - \chi_\epsilon)E$ where $\chi_\epsilon(x) = \chi(x/\epsilon)$ and $\chi \in C_c^\infty(\mathbb{R}^n)$ is equal to 1 near 0 – we take χ to have support in the unit ball. Alternatively, we can just see that

$$(11) \quad \widehat{E * u} = \hat{E} \cdot \hat{u} = F(\xi)\hat{u}(\xi)$$

is well-defined on L^2 since F is a bounded (and smooth) function. In fact $|F| \leq C(1 + |\xi|)^{-m}$ from which it follows that

$$(12) \quad (1 + |\xi|)^s \hat{u} \in L^2 \implies (1 + |\xi|)^{m+s} \widehat{E * u} \in L^2$$

and this is the content of (10).

So, now to the argument from last time. We only need to show that the Sobolev regularity holds for ψu where $\psi \in C_c^\infty(\Omega)$ is arbitrary, but fixed from now on. So, choose $\epsilon > 0$ such that $\text{supp}(\psi) + B(0, \epsilon) \Subset \Omega$ and then select $\phi \in C_c^\infty(\Omega)$ which is equal to 1 in a neighbourhood of this larger compact subset of Ω . Thus

$$(13) \quad P(D)\phi u = f + g, \quad f = \phi P(D)u \in H^s(\mathbb{R}^n), \quad \text{supp}(g) \cap \text{supp}(\psi) + B(0, \epsilon) = \emptyset.$$

This last condition follows from the fact that g arises from all the terms in the expansion of the action of the differential operator on the product in which ϕ is differentiated at least once and these vanish where $\phi \equiv 1$ in a neighbourhood of $\text{supp}(\psi) + B(0, \epsilon)$.

Now, since ϕu has compact support we can write

$$(14) \quad \phi u = (\phi u) * \delta_0 = (\phi u) * (P(D)E - G) = (P(D)(\phi u)) * E - g', \quad g' \in C^\infty(\mathbb{R}^n)$$

since the error term $G \in C^\infty(\mathbb{R}^n)$. Now using (13),

$$(15) \quad \phi u = f * E + g * (\chi_\epsilon E) + g * ((1 - \chi_\epsilon)E) - g'.$$

The third and fourth terms on the right are smooth. The first term we know to be in $H^{s+m}(\mathbb{R}^n)$ and because of

$$(16) \quad \text{supp}(g * (\chi_\epsilon E)) \subset \text{supp}(g) + B_\epsilon$$

the second term has support not meeting $\text{supp}(\psi)$. So in fact

$$(17) \quad \psi u = \psi f * E + \psi(g * ((1 - \chi_\epsilon)E)) - \psi g' \in H^{s+m}(\mathbb{R}^n) \quad \forall \psi \in C_c^\infty(\Omega)$$

which is what we were trying to prove. \square

In fact the proof gives a bit more than just this regularity, since it also gives estimates.

Lemma 2. *If $P(D)$ is elliptic of order m , $\Omega \subset \mathbb{R}^n$ is open and $\psi \in C_c^\infty(\Omega)$ then for any $\phi \in C_c^\infty(\Omega)$ with $\phi = 1$ in a neighbourhood of $\text{supp}(\psi)$ and any $s, t \in \mathbb{R}$, there exists a constant such that*

$$(18) \quad \|\psi u\|_{H^{s+m}} \leq C(\|\phi(P(D)u)\|_{H^s} + \|\phi u\|_{H^t}) \quad \forall u \in \mathcal{C}^{-\infty}(\Omega).$$

The convention here is that the inequality is ‘true’ if the right side is infinite. So, if both terms on the right are finite – and for any distribution the second term IS finite for some t , then the left is finite so ψu is in H^{s+m} .

Proof. The inequality (18) follows by using (17) to estimate $\|\psi u\|_{H^s}$. The first term on the right in (17) is bounded by the first term on right in (18). Moreover, once one fixes k convolution with a smooth function maps compactly supported elements of $H^t(\mathbb{R}^n)$ into $H_{\text{loc}}^{s+m}(\mathbb{R}^n)$ for any t and any s so the norm of the second term on the right in (17) is bounded by the second term on the right in (18). The same is true of the third term. \square

Now, what does this do for us? After our little Columbus Day vacation I will talk for a while about operators on Hilbert space including Fredholm operators and the Spectral Theorem for self-adjoint operators. Let me try to indicate how this links up with our present concerns.

As a guide to where we will go next, consider the periodic Sobolev spaces. These are really spaces of distributions or functions on the torus but we can define them directly as

$$(19) \quad H_{\text{per}}^s(\mathbb{R}^n) = \{u \in H_{\text{loc}}^s(\mathbb{R}^n); u(x + 2\pi k) = u(x) \quad \forall k \in \mathbb{Z}^n\}.$$

It is not quite obvious that these are Hilbert spaces but one can do this quite elegantly using Fourier series and the Poisson summation formula, or else one can proceed directly as follows.

Choose a function $0 \leq \phi \in C_c^\infty(\mathbb{R}^n)$ with $\phi(x) > 0$ on $\{x; 0 \leq x_i \leq 2\pi\}$ which is a ‘fundamental cube’ (meaning every point in \mathbb{R}^n can be translated into it by a shift by an integral multiple of 2π in each direction). Then set

$$(20) \quad \|u\|_{H_{\text{per}}^s} = \|\phi u\|_{H^s}.$$

This is a preHilbert norm, since its vanishing implies $u = 0$ in a neighbourhood of the fundamental cube and hence $u = 0$ everywhere by periodicity. To see that the space is complete note that one can recover u from ϕu . Namely define

$$(21) \quad \Phi(x) = \sum_{k \in \mathbb{Z}^n} \phi(x - 2\pi k).$$

Although this is an infinite sum, only a finite number of terms have support meeting any given compact set so $\Phi \in C^\infty(\mathbb{R}^n)$. It is also periodic and strictly positive. Now, one can recover u from ϕu by using the same sort of sum:

$$(22) \quad u = \Phi^{-1} \sum_{k \in \mathbb{Z}^n} (\phi u)(\cdot - 2\pi k).$$

From this one can see that if $\phi u_k \rightarrow v$ in H^s then $\psi u_k \rightarrow \psi u$ where u is defined from v using (22). The completeness of the space follows.

So, seeing that these are Hilbert spaces, we want to analyze an operator such as

$$(23) \quad P(D) : H_{\text{per}}^{s+m}(\mathbb{R}^n) \longrightarrow H_{\text{per}}^s(\mathbb{R}^n), \quad s \in \mathbb{R}.$$

Once we have discussed the appropriate notions for bounded and unbounded operators on a Hilbert space we can see that, if elliptic, $P(D)$ is a Fredholm operator – this follows quite directly from the estimates above – and that if in addition $P(-\xi) = \overline{P(\xi)}$ then it is self-adjoint and has a complete orthonormal basis of eigenfunctions.

This is true for instance for the Laplacian $\Delta = |D|^2$ and extends to the Laplace-Beltrami operator for a Riemann metric on any compact manifold.