CHAPTER 7

Suspended families and the resolvent

For a compact manifold, M, the Sobolev spaces $H^s(M;E)$ (of sections of a vector bundle E) are defined above by reference to local coordinates and local trivializations of E. If M is not compact (but is paracompact, as is demanded by the definition of a manifold) the same sort of definition leads either to the spaces of sections with compact support, or the "local" spaces:

$$H^s_{\mathrm{c}}(M;E) \subset H^s_{\mathrm{loc}}(M;E),\ s \in \mathbb{R}. \tag{0.1}$$

Thus, if $F_a : \Omega_a \to \Omega'_a$ is a covering of M, for $a \in A$, by coordinate patches over which E is trivial, $T_a : (F_a^{-1})^* E \cong \mathbb{C}^N$, and $\{\rho_a\}$ is a partition of unity subordinate to this cover then

$$\mu \in H^s_{\mathrm{loc}}(M;E) \Leftrightarrow T_a(F_a^{-1})^*(\rho_a \mu) \in H^s(\Omega'_a;\mathbb{C}^N)\ \forall\ a. \tag{0.2}$$

Practically, these spaces have serious limitations; for instance they are not Hilbert or even Banach spaaces. On the other hand they certainly have their uses and differential operators act on them in the usual way,

$$\begin{aligned} &P \in \mathrm{Diff}^m(M;\mathbb{E}) \Rightarrow \\ &\qquad P : H^{s+m}_{\mathrm{loc}}(M;E_+) \to H^s_{\mathrm{loc}}(M;E_-), \\ &\qquad P : H^{s+m}_{\mathrm{c}}(M;E_+) \to H^s_{\mathrm{c}}(M;E_-). \end{aligned} \tag{0.3}$$

However, without some limitations on the growth of elements, as is the case in $H^s_{\mathrm{loc}}(M;E)$, it is not reasonable to expect the null space of the first realization of P above to be finite dimensional. Similarly in the second case it is not reasonable to expect the operator to be even close to surjective.

1. Product with a line

Some corrections from Fang Wang added, 25 July, 2007.

Thus, for non-compact manifolds, we need to find intermediate spaces which represent some growth constraints on functions or distributions. Of course this is precisely what we have done for $\mathbb{R}^n$ in defining the weighted Sobolev spaces,

$$H^{s,t}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); \langle z\rangle^{-t}u \in H^s(\mathbb{R}^n)\right\}. \tag{1.1}$$

However, it turns out that even these spaces are not always what we want.

To lead up to the discussion of other spaces I will start with the simplest sort of non-compact space, the real line. To make things more interesting (and useful) I will conisider

$$X = \mathbb{R} \times M \tag{1.2}$$

where M is a compact manifold. The new Sobolev spaces defined for this product will combine the features of $H^s(\mathbb{R})$ and $H^s(M)$. The Sobolev spaces on $\mathbb{R}^n$ are associated with the translation action of $\mathbb{R}^n$ on itself, in the sense that this fixes the "uniformity" at infinity through the Fourier transform. What happens on X is quite similar.

First we can define "tempered distributions" on X. The space of Schwartz functions of rapid decay on X can be fixed in terms of differential operators on M and differentiation on $\mathbb{R}$.

$$\mathcal{S}(\mathbb{R}\times M) = \left\{u : \mathbb{R} \times M \to \mathbb{C};\ \sup_{\mathbb{R}\times M}\left|t^l D_t^k Pu(t,\cdot)\right| < \infty \ \forall\ l,\ k,\ P \in \operatorname{Diff}^*(M)\right\}. \tag{1.3}$$

Exercise 1. Define the corresponding space for sections of a vector bundle E over M lifted to X and then put a topology on $\mathcal{S}(\mathbb{R} \times M; E)$ corresponding to these estimates and check that it is a complete metric space, just like $\mathcal{S}(\mathbb{R})$ in Chapter 3.

There are several different ways to look at

$$\mathcal{S}(\mathbb{R} \times M) \subset \mathcal{C}^\infty(\mathbb{R} \times M).$$

Namely we can think of either $\mathbb{R}$ or M as "coming first" and see that

$$\mathcal{S}(\mathbb{R} \times M) = \mathcal{C}^\infty(M; \mathcal{S}(\mathbb{R})) = \mathcal{S}(\mathbb{R}; \mathcal{C}^\infty(M)). \tag{1.4}$$

The notion of a $\mathcal{C}^\infty$ function on M with values in a topological vector space is easy to define, since $\mathcal{C}^0(M; \mathcal{S}(\mathbb{R}))$ is defined using the metric space topology on $\mathcal{S}(\mathbb{R})$. In a coordinate patch on M higher derivatives are defined in the usual way, using difference quotients and these definitions are coordinate-invariant. Similarly, continuity and differentiability for a map $\mathbb{R} \to \mathcal{C}^\infty(M)$ are easy to define and then

$$\mathcal{S}(\mathbb{R}; \mathcal{C}^\infty(M)) = \left\{u : \mathbb{R} \to \mathcal{C}^\infty(M);\ \sup_t \left\|t^k D_t^p u\right\|_{\mathcal{C}^l(M)} < \infty,\ \forall\ k,\ p,\ l\right\}. \tag{1.5}$$

Using such an interpretation of $\mathcal{S}(\mathbb{R}\times M)$, or directly, it follows easily that the 1-dimensional Fourier transform gives an isomorphism $\mathcal{F}:\mathcal{S}(\mathbb{R}\times M)\to\mathcal{S}(\mathbb{R}\times M)$ by

$$\mathcal{F}:u(t,\cdot)\longmapsto\hat{u}(\tau,\cdot)=\int_{\mathbb{R}}e^{-it\tau}u(t,\cdot)\,dt. \tag{1.6}$$

So, one might hope to use $\mathcal{F}$ to define Sobolev spaces on $\mathbb{R}\times M$ with uniform behavior as $t\to\infty$ in $\mathbb{R}$. However this is not so straightforward, although I will come back to it, since the 1-dimensional Fourier transform in (1.6) does *nothing* in the variables in M. Instead let us think about $L^2(\mathbb{R}\times M)$, the definition of which requires a choice of measure.

Of course there is an obvious class of product measures on $\mathbb{R}\times M$, namely $dt\cdot\nu_M$, where ν_M is a positive smooth density on M and dt is Lebesgue measure on $\mathbb{R}$. This corresponds to the functional

$$\int:\mathcal{C}^0_{\mathrm{c}}(\mathbb{R}\times M)\ni u\longmapsto\int u(t,\cdot)\,dt\cdot\nu\in\mathbb{C}. \tag{1.7}$$

The analogues of (1.4) correspond to Fubini's Theorem.

(1.8)

$$L^2_{\mathrm{ti}}(\mathbb{R}\times M)=\left\{u:\mathbb{R}\times M\to\mathbb{C}\text{ measurable; }\int|u(t,z)|^2\,dt\,\nu_z<\infty\right\}/\sim\ \text{ a.e.}$$
$$L^2_{\mathrm{ti}}(\mathbb{R}\times M)=L^2(\mathbb{R};L^2(M))=L^2(M;L^2(\mathbb{R})).$$

Here the subscript "ti" is supposed to denote translation-invariance (of the measure and hence the space).

We can now easily define the Sobolev spaces of positive integer order:

$$\begin{aligned}H^m_{\mathrm{ti}}(\mathbb{R}\times M)=\Big\{&u\in L^2_{\mathrm{ti}}(\mathbb{R}\times M);\\ &D^j_tP_ku\in L^2_{\mathrm{ti}}(\mathbb{R}\times M)\ \forall\, j\le m-k,\ 0\le k\le m,\ P_k\in\mathrm{Diff}^k(M)\Big\}.\end{aligned} \tag{1.9}$$

In fact we can write them more succinctly by defining

(1.10)

$$\mathrm{Diff}^k_{\mathrm{ti}}(\mathbb{R}\times M)=\left\{Q\in\mathrm{Diff}^m(\mathbb{R}\times M);\ Q=\sum_{0\le j\le m}D^j_tP_j,\ P_j\in\mathrm{Diff}^{m-j}(M)\right\}.$$

This is the space of "t-translation-invariant" differential operators on $\mathbb{R}\times M$ and (1.9) reduces to

(1.11)

$$H^m_{\mathrm{ti}}(\mathbb{R}\times M)=\left\{u\in L^2_{\mathrm{ti}}(\mathbb{R}\times M);\ Pu\in L^2_{\mathrm{ti}}(\mathbb{R}\times M),\ \forall\ P\in\mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M)\right\}.$$

I will discuss such operators in some detail below, especially the elliptic case. First, we need to consider the Sobolev spaces of non-integral order, for completeness sake if nothing else. To do this, observe that on $\mathbb{R}$ itself (so for $M = \{\mathrm{pt}\}$), $L^2_{\mathrm{ti}}(\mathbb{R} \times \{\mathrm{pt}\}) = L^2(\mathbb{R})$ in the usual sense. Let us consider a special partition of unity on $\mathbb{R}$ consisting of integral translates of *one* function.

Definition 1.1. *An element $\mu \in \mathcal{C}^\infty_c(\mathbb{R})$ generates a "**ti-partition of unity**" (a non-standard description) on $\mathbb{R}$ if $0 \le \mu \le 1$ and $\sum_{k\in\mathbb{Z}} \mu(t-k) = 1$.*

It is easy to construct such a μ. Just take $\mu_1 \in \mathcal{C}^\infty_c(\mathbb{R})$, $\mu_1 \ge 0$ with $\mu_1(t) = 1$ in $|t| \le 1/2$. Then let

$$F(t) = \sum_{k\in\mathbb{Z}} \mu_1(t-k) \in \mathcal{C}^\infty(\mathbb{R})$$

since the sum is finite on each bounded set. Moreover $F(t) \ge 1$ and is itself invariant under translation by any integer; set $\mu(t) = \mu_1(t)/F(t)$. Then μ generates a ti-partition of unity.

Using such a function we can easily decompose $L^2(\mathbb{R})$. Thus, setting $\tau_k(t) = t-k$,

(1.12)

$$f \in L^2(\mathbb{R}) \iff (\tau_k^* f)\mu \in L^2_{\mathrm{loc}}(\mathbb{R})\ \forall\, k \in \mathbb{Z} \text{ and } \sum_{k\in\mathbb{Z}} \int |\tau_k^* f\mu|^2\, dt < \infty.$$

Of course, saying $(\tau_k^* f)\mu \in L^2_{\mathrm{loc}}(\mathbb{R})$ is the same as $(\tau_k^* f)\mu \in L^2_{\mathrm{c}}(\mathbb{R})$. Certainly, if $f \in L^2(\mathbb{R})$ then $(\tau_k^* f)\mu \in L^2(\mathbb{R})$ and since $0 \le \mu \le 1$ and $\operatorname{supp}(\mu) \subset [-R, R]$ for some R,

$$\sum_k \int |(\tau_k^* f)\mu|^2 \le C \int |f|^2\, dt.$$

Conversely, since $\sum_{|k|\le T} \mu = 1$ on $[-1,1]$ for some T, it follows that

$$\int |f|^2\, dt \le C' \sum_k \int |(\tau_k^* f)\mu|^2\, dt.$$

Now, $D_t \tau_k^* f = \tau_k^*(D_t f)$, so we can use (1.12) to rewrite the definition of the spaces $H^k_{\mathrm{ti}}(\mathbb{R} \times M)$ in a form that extends to *all* orders. Namely

(1.13)

$$u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M) \iff (\tau_k^* u)\mu \in H^s_{\mathrm{c}}(\mathbb{R} \times M) \text{ and } \sum_k \|\tau_k^* u\|_{H^s} < \infty$$

provided we choose a fixed norm on $H^s_{\mathrm{c}}(\mathbb{R} \times M)$ giving the usual topology for functions supported in a fixed compact set, for example by embedding $[-T, T]$ in a torus $\mathbb{T}$ and then taking the norm on $H^s(\mathbb{T} \times M)$.

LEMMA 1.2. *With* $\mathrm{Diff}^m_{ti}(\mathbb{R}\times M)$ *defined by* (1.10) *and the translation-invariant Sobolev spaces by* (1.13),

$$P \in \mathrm{Diff}^m_{ti}(\mathbb{R}\times M) \Longrightarrow \\ P : H^{s+m}_{ti}(\mathbb{R}\times M) \longrightarrow H^s_{ti}(\mathbb{R}\times M)\ \forall\, s \in \mathbb{R}. \tag{1.14}$$

PROOF. This is basically an exercise. Really we also need to check a little more carefully that the two definitions of $H^(_{\mathrm{ti}}\mathbb{R}\times M)$ for k a positive integer, are the same. In fact this is similar to the proof of (1.14) so is omitted. So, to prove (1.14) we will proceed by induction over m. For $m = 0$ there is nothing to prove. Now observe that the translation-invariant of P means that $P\tau_k^*u = \tau_k^*(Pu)$ so

$$u \in H^{s+m}_{\mathrm{ti}}(\mathbb{R}\times M) \Longrightarrow \\ P(\tau_k^* u\mu) = \tau_k^*(Pu) + \sum_{m'<m} \tau_k^*(P_{m'}u)D_t^{m-m'}\mu,\ P_{m'} \in \mathrm{Diff}^{m'}_{\mathrm{ti}}(\mathbb{R}\times M). \tag{1.15}$$

The left side is in $H^s_{\mathrm{ti}}(\mathbb{R}\times M)$, with the sum over k of the squares of the norms bounded, by the regularity of u. The same is easily seen to be true for the sum on the right by the inductive hypothesis, and hence for the first term on the right. This proves the mapping property (1.14) and continuity follows by the same argument or the closed graph theorem. □

We can, and shall, extend this in various ways. If $\mathbb{E} = (E_1, E_2)$ is a pair of vector bundles over M then it lifts to a pair of vector bundles over $\mathbb{R}\times M$, which we can again denote by $\mathbb{E}$. It is then straightforward to define $\mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M);\mathbb{E})$ and the Sobolev spaces $H^s_{\mathrm{ti}}(\mathbb{R}\times M; E_i)$ and to check that (1.14) extends in the obvious way.

Then main question we want to understand is the *invertibility* of an operator such as P in (1.14). However, let me look first at these Sobolev spaces a little more carefully. As already noted we really have two definitions in the case of positive integral order. Thinking about these we can also make the following provisional definitions in terms of the 1-dimensional Fourier transform discussed above – where the '$\tilde{H}$' notation is only temporary since these will turn out to be the same as the spaces just considered.

For any compact manifold define

$$\tilde{H}^s_{\mathrm{ti}}(\mathbb{R}\times M)=\Big\{u\in L^2(\mathbb{R}\times M);\ \|u\|_s^2=\int_{\mathbb{R}}\left(\langle\tau\rangle^s|\hat{u}(\tau,\cdot)|^2_{L^2(M)}+\int_{\mathbb{R}}|\hat{u}(\tau,\cdot)|^2_{H^s(M)}\right)d\tau<\infty\Big\},\ s\ge 0 \tag{1.16}$$

$$\tilde{H}^s_{\mathrm{ti}}(\mathbb{R}\times M)=\Big\{u\in\mathcal{S}'(\mathbb{R}\times M); u=u_1+u_2,\ u_1\in L^2(\mathbb{R};H^s(M)),\ u_2\in L^2(M;H^s(\mathbb{R}))\Big\},\ \|u\|_s^2=\inf\|u_1\|^2+\|u_2\|^2,\ s<0. \tag{1.17}$$

The following interpolation result for Sobolev norms on M should be back in Chapter 5.

Lemma 1.3. *If M is a compact manifold or $\mathbb{R}^n$ then for any $m_1\ge m_2\ge m_3$ and any R, the Sobolev norms are related by*

$$\|u\|_{m_2}\le C\left((1+R)^{m_2-m_1}\|u\|_{m_1}+(1+R)^{m_2-m_3}\|u\|_{m_3}\right). \tag{1.18}$$

Proof. On $\mathbb{R}^n$ this follows directly by dividing Fourier space in two pieces

$$\begin{aligned}\|u\|^2_{m_2}&=\int_{|\zeta|>R}\langle\zeta\rangle^{2m_2}|\hat{u}|d\zeta+\int_{|\zeta|\le R}\langle\zeta\rangle^{2m_2}|\hat{u}|d\zeta\\&\le\langle R\rangle^{2(m_1-m_2)}\int_{|\zeta|>R}\langle\zeta\rangle^{2m_1}|\hat{u}|d\zeta+\langle R\rangle^{2(m_2-m_3)}\int_{|\zeta|\le R}\langle\zeta\rangle^{2m_3}|\hat{u}|d\zeta\\&\le\langle R\rangle^{2(m_1-m_2)}\|u\|^2_{m_1}+\langle R\rangle^{2(m_2-m_3)}\|u\|^2_{m_3}.\end{aligned} \tag{1.19}$$

On a compact manifold we have defined the norms by using a partition ϕ_i of unity subordinate to a covering by coordinate patches $F_i:Y_i\longrightarrow U_i'$:

$$\|u\|^2_m=\sum_i\|(F_i)^*(\phi_i u)\|^2_m \tag{1.20}$$

where on the right we are using the Sobolev norms on $\mathbb{R}^n$. Thus, applying the estimates for Euclidean space to each term on the right we get the same estimate on any compact manifold. □

Corollary 1.4. *If $u\in\tilde{H}^s_{ti}(\mathbb{R}\times M)$, for $s>0$, then for any $0<t<s$*

$$\int_{\mathbb{R}}\langle\tau\rangle^{2t}\|\hat{u}(\tau,\cdot)\|^2_{H^{s-t}(M)}d\tau<\infty \tag{1.21}$$

which we can interpret as meaning '$u\in H^t(\mathbb{R};H^{s-t}(M))$ or $u\in H^{s-t}(M;H^s(\mathbb{R}))$.'

Proof. Apply the estimate to $\hat{u}(\tau,\cdot) \in H^s(M)$, with $R = |\tau|$, $m_1 = s$ and $m_3 = 0$ and integrate over τ. $\square$

Lemma 1.5. *The Sobolev spaces* $\tilde{H}^s_{ti}(\mathbb{R} \times M)$ *and* $H^s_{ti}(\mathbb{R} \times M)$ *are the same.*

Proof. $\square$

Lemma 1.6. *For* $0 < s < 1$ $u \in H^s_{ti}(\mathbb{R} \times M)$ *if and only if* $u \in L^2(\mathbb{R} \times M)$ *and*

(1.22)
$$\int_{\mathbb{R}^2 \times M} \frac{|u(t,z) - u(t',z)|^2}{|t-t'|^{2s+1}} dt dt' \nu + \int_{\mathbb{R} \times M^2} \frac{|u(t,z') - u(t,z)|^2}{\rho(z,z')^{s+\frac{n}{2}}} dt \nu(z) \nu(z') < \infty,$$
$$n = \dim M,$$

where $0 \le \rho \in \mathcal{C}^\infty(M^2)$ *vanishes exactly quadratically at* $\mathrm{Diag} \subset M^2$.

Proof. This follows as in the cases of $\mathbb{R}^n$ and a compact manifold discussed earlier since the second term in (1.22) gives (with the L^2 norm) a norm on $L^2(\mathbb{R}; H^s(M))$ and the first term gives a norm on $L^2(M; H^s(\mathbb{R}))$. $\square$

Using these results we can see directly that the Sobolev spaces in (1.16) have the following 'obvious' property as in the cases of $\mathbb{R}^n$ and M.

Lemma 1.7. *Schwartz space* $\mathcal{S}(\mathbb{R} \times M) = \mathcal{C}^\infty(M; \mathcal{S}(\mathbb{R}))$ *is dense in each* $H^s_{ti}(\mathbb{R} \times M)$ *and the* L^2 *pairing extends by continuity to a jointly continuous non-degenerate pairing*

$$H^s_{ti}(\mathbb{R} \times M) \times H^{-s}_{ti}(\mathbb{R} \times M) \longrightarrow \mathbb{C} \tag{1.23}$$

which identifies $H^{-s}_{ti}(\mathbb{R} \times M)$ *with the dual of* $H^s_{ti}(\mathbb{R} \times M)$ *for any* $s \in \mathbb{R}$.

Proof. I leave the density as an exercise – use convolution in $\mathbb{R}$ and the density of $\mathcal{C}^\infty(M)$ in $H^s(M)$ (explicity, using a partition of unity on M and convolution on $\mathbb{R}^n$ to get density in each coordinate patch).

Then the existence and continuity of the pairing follows from the definitions and the corresponding pairings on $\mathbb{R}$ and M. We can assume that $s > 0$ in (1.23) (otherwise reverse the factors). Then if $u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M)$ and $v = v_1 + v_2 \in H^{-s}_{\mathrm{ti}}(\mathbb{R} \times M)$ as in (1.17),

$$(u,v) = \int_{\mathbb{R}} (u(t,\cdot), u_1(t,\cdot))\, dt + \int_M (u(\cdot,z), v_2(\cdot,z))\, \nu_z \tag{1.24}$$

where the first pairing is the extension of the L^2 pairing to $H^s(M) \times H^{-s}(M)$ and in the second case to $H^s(\mathbb{R}) \times H^{-s}(\mathbb{R})$. The continuity of the pairing follows directly from (1.24).

So, it remains only to show that the pairing is non-degenerate – so that

$$H^{-s}_{\mathrm{ti}}(\mathbb{R} \times M) \ni v \longmapsto \sup_{\|u\|_{H^s_{\mathrm{ti}}(\mathbb{R}\times M)}=1} |(u,v)| \tag{1.25}$$

is equivalent to the norm on $H^{-s}_{\mathrm{ti}}(\mathbb{R} \times M)$. We already know that this is bounded above by a multiple of the norm on H^{-s}_{ti} so we need the estimate the other way. To see this we just need to go back to Euclidean space. Take a partition of unity ψ_i with our usual ϕ_i on M subordinate to a coordinate cover and consider with $\phi_i = 1$ in a neighbourhood of the support of ψ_i. Then

$$(u, \psi_i v) = (\phi_i u, \psi_i v) \tag{1.26}$$

allows us to extend $\psi_i v$ to a continuous linear functional on $H^s(\mathbb{R}^n)$ by reference to the local coordinates and using the fact that for $s > 0$ $(F_i^{-1})^*(\phi_i u) \in H^s(\mathbb{R}^{n+1})$. This shows that the coordinate representative of $\psi_i v$ is a sum as desired and summing over i gives the desired bound. □

2. Translation-invariant Operators

Some corrections from Fang Wang added, 25 July, 2007.

Next I will characterize those operators $P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M;\mathbb{E})$ which give invertible maps (1.14), or rather in the case of a pair of vector bundles $\mathbb{E} = (E_1, E_2)$ over M :

$$P : H^{s+m}_{\mathrm{ti}}(\mathbb{R}\times M; E_1) \longrightarrow H^s_{\mathrm{ti}}(\mathbb{R}\times M; E_2),\ P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M;\mathbb{E}). \tag{2.1}$$

This is a generalization of the 1-dimensional case, $M = \{\mathrm{pt}\}$ which we have already discussed. In fact it will become clear how to generalize some parts of the discussion below to products $\mathbb{R}^n \times M$ as well, but the case of a 1-dimensional Euclidean factor is both easier and more fundamental.

As with the constant coefficient case, there is a basic dichotomy here. A t-translation-invariant differential operator as in (2.1) is Fredholm if and only if it is invertible. To find necessary and sufficient conditons for invertibility we will we use the 1-dimensional Fourier transform as in (1.6).

If

$$P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R} \times M);\mathbb{E}) \Longleftrightarrow P = \sum_{i=0}^{m} D_t^i P_i,\ P_i \in \mathrm{Diff}^{m-i}(M;\mathbb{E}) \tag{2.2}$$

then

$$P : \mathcal{S}(\mathbb{R} \times M; E_1) \longrightarrow \mathcal{S}(\mathbb{R} \times M; E_2)$$

and

$$\widehat{Pu}(\tau, \cdot) = \sum_{i=0}^{m} \tau^i P_i \widehat{u}(\tau, \cdot) \tag{2.3}$$

where $\widehat{u}(\tau, \cdot)$ is the 1-dimensional Fourier transform from (1.6). So we clearly need to examine the "suspended" family of operators

$$P(\tau) = \sum_{i=0}^{m} \tau^i P_i \in \mathcal{C}^\infty \left(\mathbb{C}; \mathrm{Diff}^m(M; \mathbb{E})\right). \tag{2.4}$$

I use the term "suspended" to denote the addition of a parameter to $\mathrm{Diff}^m(M; \mathbb{E})$ to get such a family—in this case polynomial. They are sometimes called "operator pencils" for reasons that escape me. Anyway, the main result we want is

Theorem 2.1. *If $P \in \mathrm{Diff}^m_{ti}(M; \mathbb{E})$ is elliptic then the suspended family $P(\tau)$ is invertible for all $\tau \in \mathbb{C} \setminus D$ with inverse*

$$P(\tau)^{-1} : H^s(M; E_2) \longrightarrow H^{s+m}(M; E_1) \tag{2.5}$$

where

$$D \subset \mathbb{C} \text{ is discrete and } D \subset \{\tau \in \mathbb{C};\ |\operatorname{Re}\tau| \le c|\operatorname{Im}\tau| + 1/c\} \tag{2.6}$$

for some $c > 0$ (see Fig. **??** *– still not quite right).*

In fact we need some more information on $P(\tau)^{-1}$ which we will pick up during the proof of this result. The translation-invariance of P can be written in operator form as

$$Pu(t + s, \cdot) = (Pu)(t + s, \cdot)\ \forall\ s \in \mathbb{R} \tag{2.7}$$

Lemma 2.2. *If $P \in \mathrm{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ is elliptic then it has a parametrix*

$$Q : \mathcal{S}(\mathbb{R} \times M; E_2) \longrightarrow \mathcal{S}(\mathbb{R} \times M; E_1) \tag{2.8}$$

which is translation-invariant in the sense of (2.7) *and preserves the compactness of supports in* $\mathbb{R}$,

$$Q : \mathcal{C}^\infty_c(\mathbb{R} \times M; E_2) \longrightarrow \mathcal{C}^\infty_c(\mathbb{R} \times M; E_1) \tag{2.9}$$

Proof. In the case of a compact manifold we contructed a global parametrix by patching local parametricies with a partition of unity. Here we do the same thing, treating the variable $t \in \mathbb{R}$ globally throughout. Thus if $F_a : \Omega_a \to \Omega'_a$ is a coordinate patch in M over which E_1

and (hence) E_2 are trivial, P becomes a square matrix of differential operators

$$P_a = \begin{bmatrix} P_{11}(z, D_t, D_z) & \cdots & P_{l1}(z, D_t, D_z) \\ \vdots & & \vdots \\ P_{1l}(z, D_t, D_z) & \cdots & P_{ll}(z, D_t, D_z) \end{bmatrix} \tag{2.10}$$

in which the coefficients do *not* depend on t. As discussed in Sections 2 and 3 above, we can construct a local parametrix in Ω'_a using a properly supported cutoff χ. In the t variable the parametrix is global anyway, so we use a fixed cutoff $\tilde{\chi} \in \mathcal{C}^\infty_c(\mathbb{R})$, $\tilde{\chi} = 1$ in $|t| < 1$, and so construct a parametrix

$$Q_a f(t, z) = \int_{\Omega'_a} q(t - t', z, z')\tilde{\chi}(t - t')\chi(z, z')f(t', z')\, dt'\, dz'. \tag{2.11}$$

This satisfies

$$P_a Q_a = \operatorname{Id} - R_a, \quad Q_a P_a = \operatorname{Id} - R'_a \tag{2.12}$$

where R_a and R'_a are smoothing operators on Ω'_a with kernels of the form

$$\begin{aligned} R_a f(t, z) = &\int_{\Omega'_a} R_a(t - t', z, z')f(t', z')\, dt'\, dz' \\ R_a \in &\mathcal{C}^\infty(\mathbb{R} \times \Omega'^2_a),\ R_a(t, z, z') = 0 \text{ if } |t| \geq 2 \end{aligned} \tag{2.13}$$

with the support proper in Ω'_a.

Now, we can sum these local parametricies, which are all t-translation-invariant to get a global parametrix with the same properties

$$Qf = \sum_a \chi_a (F_a^{-1})^*(T_a^{-1})^* Q_a T_a^* F_a^* f \tag{2.14}$$

where T_a denotes the trivialization of bundles E_1 and E_2. It follows that Q satisfies (2.9) and since it is translation-invariant, also (2.8). The global version of (2.12) becomes

$$\begin{aligned} &PQ = \operatorname{Id} - R_2, \quad QP = \operatorname{Id} - R_1, \\ &R_i : \mathcal{C}^\infty_c(\mathbb{R} \times M; E_i) \longrightarrow \mathcal{C}^\infty_c(\mathbb{R} \times M; E_i), \\ &R_i f = \int_{\mathbb{R}\times M} R_i(t - t', z, z')f(t', z')\, dt'\, \nu_{z'} \end{aligned} \tag{2.15}$$

where the kernels

$$R_i \in \mathcal{C}^\infty_c\left(\mathbb{R} \times M^2; \operatorname{Hom}(E_i)\right),\ i = 1, 2. \tag{2.16}$$

□

In fact we can deduce directly from (2.11) the boundedness of Q.

LEMMA 2.3. *The properly-supported parametrix Q constructed above extends by continuity to a bounded operator*

$$
\begin{aligned}
Q :& H^s_{ti}(\mathbb{R}\times M;E_2) \longrightarrow H^{s+m}_{ti}(\mathbb{R}\times M;E_1)\ \forall\, s\in\mathbb{R} \\
Q :& \mathcal{S}(\mathbb{R}\times M;E_2) \longrightarrow \mathcal{S}(\mathbb{R}\times M;E_1).
\end{aligned}
\tag{2.17}
$$

PROOF. This follows directly from the earlier discussion of elliptic regularity for each term in (2.14) to show that

$$
\begin{aligned}
Q : \{f\in H^s_{\mathrm{ti}}(\mathbb{R}\times M;E_2;\ \mathrm{supp}(f)\subset[-2,2]\times M\} \\
\longrightarrow \left\{u\in H^{s+m}_{\mathrm{ti}}(\mathbb{R}\times M;E_1;\ \mathrm{supp}(u)\subset[-2-R,2+R]\times M\right\}
\end{aligned}
\tag{2.18}
$$

for some R (which can in fact be taken to be small and positive). Indeed on compact sets the translation-invariant Sobolev spaces reduce to the usual ones. Then (2.17) follows from (2.18) and the translation-invariance of Q. Using a $\mu\in\mathcal{C}^\infty_c(\mathbb{R})$ generating a ti-paritition of unity on $\mathbb{R}$ we can decompose

$$
H^s_{\mathrm{ti}}(\mathbb{R}\times M;E_2)\ni f=\sum_{k\in\mathbb{Z}}\tau_k^*(\mu\tau_{-k}^*f).
\tag{2.19}
$$

Then

$$
Qf=\sum_{k\in\mathbb{Z}}\tau_k^*\left(Q(\mu\tau_{-k}^*f)\right).
\tag{2.20}
$$

The estimates corresponding to (2.18) give

$$
\|Qf\|_{H^{s+m}_{\mathrm{ti}}}\le C\|f\|_{H^s_{\mathrm{ti}}}
$$

if f has support in $[-2,2]\times M$. The decomposition (2.19) then gives

$$
\sum\|\mu\tau_{-k}^*f\|^2_{H^s}=\|f\|^2_{H_s}<\infty\implies\|Qf\|^2\le C'\|f\|^2_{H^s}.
$$

This proves Lemma 2.3. □

Going back to the remainder term in (2.15), we can apply the 1-dimensional Fourier transform and find the following uniform results.

LEMMA 2.4. *If R is a compactly supported, t-translation-invariant smoothing operator as in* (2.15) *then*

$$
\widehat{Rf}(\tau,\cdot)=\widehat{R}(\tau)\hat{f}(\tau,\cdot)
\tag{2.21}
$$

where $\widehat{R}(\tau)\in\mathcal{C}^\infty\left(\mathbb{C}\times M^2;\mathrm{Hom}(E)\right)$ is entire in $\tau\in\mathbb{C}$ and satisfies the estimates

$$
\forall\, k,p\ \exists\, C_{p,k}\ \text{such that}\ \|\tau^k\widehat{R}(\tau)\|_{\mathcal{C}^p}\le C_{p,k}\exp(A|\operatorname{Im}\tau|).
\tag{2.22}
$$

Here A is a constant such that

$$
\operatorname{supp}R(t,\cdot)\subset[-A,A]\times M^2.
\tag{2.23}
$$

PROOF. This is a parameter-dependent version of the usual estimates for the Fourier-Laplace transform. That is,

$$\widehat{R}(\tau,\cdot)=\int e^{-i\tau t}R(t,\cdot)\,dt \tag{2.24}$$

from which all the statements follow just as in the standard case when $R\in\mathcal{C}^\infty_c(\mathbb{R})$ has support in $[-A,A]$. $\square$

PROPOSITION 2.5. *If R is as in Lemma 2.4 then there exists a discrete subset $D\subset\mathbb{C}$ such that $\left(\operatorname{Id}-\widehat{R}(\tau)\right)^{-1}$ exists for all $\tau\in\mathbb{C}\setminus D$ and*

$$\left(\operatorname{Id}-\widehat{R}(\tau)\right)^{-1}=\operatorname{Id}-\widehat{S}(\tau) \tag{2.25}$$

where $\widehat{S}:\mathbb{C}\longrightarrow\mathcal{C}^\infty(M^2;\operatorname{Hom}(E))$ is a family of smoothing operators which is meromorphic in the complex plane with poles of finite order and residues of finite rank at D. Furthermore,

$$D\subset\{\tau\in\mathbb{C};\ \log(|\operatorname{Re}\tau|)<c|\operatorname{Im}\tau|+1/c\} \tag{2.26}$$

for some $c>0$ and for any $C>0$, there exists C' such that

$$|\operatorname{Im}\tau|<C,\ |\operatorname{Re}\tau|>C'\implies\|\tau^k\widehat{S}(\tau)\|_{\mathcal{C}^p}\le C_{p,k}. \tag{2.27}$$

PROOF. This is part of "Analytic Fredholm Theory" (although usually done with compact operators on a Hilbert space). The estimates (2.22) on $\widehat{R}(\tau)$ show that, in some region as on the right in (2.26),

$$\|\widehat{R}(\tau)\|_{L^2}\le 1/2. \tag{2.28}$$

Thus, by Neumann series,

$$\widehat{S}(\tau)=\sum_{k=1}^{\infty}\left(\widehat{R}(\tau)\right)^k \tag{2.29}$$

exists as a bounded operator on $L^2(M;E)$. In fact it follows that $\widehat{S}(\tau)$ is itself a family of smoothing operators in the region in which the Neumann series converges. Indeed, the series can be rewritten

$$\widehat{S}(\tau)=\widehat{R}(\tau)+\widehat{R}(\tau)^2+\widehat{R}(\tau)\widehat{S}(\tau)\widehat{R}(\tau) \tag{2.30}$$

The smoothing operators form a "corner" in the bounded operators in the sense that products like the third here are smoothing if the outer two factors are. This follows from the formula for the kernel of the product

$$\int_{M\times M}\widehat{R}_1(\tau;z,z')\widehat{S}(\tau;z',z'')\widehat{R}_2(\tau;z'',\tilde{z})\,\nu_{z'}\,\nu_{z''}.$$

Thus $\widehat{S}(\tau) \in \mathcal{C}^\infty(M^2; \operatorname{Hom}(E))$ exists in a region as on the right in (2.26). To see that it extends to be meromorphic in $\mathbb{C} \setminus D$ for a discrete divisor D we can use a finite-dimensional approximation to $\widehat{R}(\tau)$.

Recall — if neccessary from local coordinates — that given any $p \in \mathbb{N}, R > 0, q > 0$ there are finitely many sections $f_i^{(\tau)} \in \mathcal{C}^\infty(M; E'), g_i^{(\tau)} \in \mathcal{C}^\infty(M; E)$ and such that

$$\|\widehat{R}(\tau) - \sum_i g_i(\tau, z) \cdot f_i(\tau, z')\|_{\mathcal{C}^p} < \epsilon, \quad |\tau| < R. \tag{2.31}$$

Writing this difference as $M(\tau)$,

$$\operatorname{Id} - \widehat{R}(\tau) = \operatorname{Id} - M(\tau) + F(\tau)$$

where $F(\tau)$ is a finite rank operator. In view of (2.31), $\operatorname{Id} - M(\tau)$ is invertible and, as seen above, of the form $\operatorname{Id} - \widehat{M}(\tau)$ where $\widehat{M}(\tau)$ is holomorphic in $|\tau| < R$ as a smoothing operator.

Thus

$$\operatorname{Id} - \widehat{R}(\tau) = (\operatorname{Id} - M(\tau))(\operatorname{Id} + F(\tau) - \widehat{M}(\tau)F(\tau))$$

is invertible if and only if the finite rank perturbation of the identity by $(\operatorname{Id} - \widehat{M}(\tau))F(\tau)$ is invertible. For R large, by the previous result, this finite rank perturbation must be invertible in an open set in $\{|\tau| < R\}$. Then, by standard results for finite dimensional matrices, it has a meromorphic inverse with finite rank (generalized) residues. The same is therefore true of $\operatorname{Id} - \widehat{R}(\tau)$ itself.

Since $R > 0$ is arbitrary this proves the result. □

Proof. Proof of Theorem 2.1 We have proved (2.15) and the corresponding form for the Fourier transformed kernels follows:

$$\widehat{P}(\tau)\widehat{Q}'(\tau) = \operatorname{Id} - \widehat{R}_2(\tau), \ \widehat{Q}'(\tau)\widehat{P}(\tau) = \operatorname{Id} - \widehat{R}_1(\tau) \tag{2.32}$$

where $\widehat{R}_1(\tau), \widehat{R}_2(\tau)$ are families of smoothing operators as in Proposition 2.5. Applying that result to the first equation gives a new meromorphic right inverse

$$\widehat{Q}(\tau) = \widehat{Q}'(\tau)(\operatorname{Id} - \widehat{R}_2(\tau))^{-1} = \widehat{Q}'(\tau) - \widehat{Q}'(\tau)M(\tau)$$

where the first term is entire and the second is a meromorphic family of smoothing operators with finite rank residues. The same argument on the second term gives a left inverse, but his shows that $\widehat{Q}(\tau)$ must be a two-sided inverse.

This we have proved everything except the locations of the poles of $\widehat{Q}(\tau)$ — which are only constrained by (2.26) instead of (2.6). However, we can apply the same argument to $P_\theta(z, D_t, D_z) = P(z, e^{i\theta}D_t, D_z)$ for

$|\theta| < \delta$, $\delta > 0$ small, since P_θ stays elliptic. This shows that the poles of $\widehat{Q}(\tau)$ lie in a set of the form (2.6). □

3. Invertibility

We are now in a position to characterize those t-translation-invariant differential operators which give isomorphisms on the translation-invariant Sobolev spaces.

THEOREM 3.1. *An element $P \in \mathrm{Diff}^m_{ti}(\mathbb{R} \times M; E)$ gives an isomorphism* (2.1) *(or equivalently is Fredholm) if and only if it is elliptic and $D \cap \mathbb{R} = \emptyset$, i.e. $\hat{P}(\tau)$ is invertible for all $\tau \in \mathbb{R}$.*

PROOF. We have already done most of the work for the important direction for applications, which is that the ellipticity of P and the invertibility at $\hat{P}(\tau)$ for all $\tau \in \mathbb{R}$ together imply that (2.1) is an isomorphism for any $s \in \mathbb{R}$.

Recall that the ellipticity of P leads to a parameterix Q which is translation-invariant and has the mapping property we want, namely (2.17).

To prove the same estimate for the true inverse (and its existence) consider the difference

$$\hat{P}(\tau)^{-1} - \hat{Q}(\tau) = \hat{\mathbb{R}}(\tau),\ \tau \in \mathbb{R}. \tag{3.1}$$

Since $\hat{P}(\tau) \in \mathrm{Diff}^m(M; \mathbb{E})$ depends smoothly on $\tau \in \mathbb{R}$ and $\hat{Q}(\tau)$ is a paramaterix for it, we know that

$$\hat{R}(\tau) \in \mathcal{C}^\infty(\mathbb{R}; \Psi^{-\infty}(M; \mathbb{E})) \tag{3.2}$$

is a smoothing operator on M which depends smoothly on $\tau \in \mathbb{R}$ as a parameter. On the other hand, from (2.32) we also know that for large real τ,

$$\hat{P}(\tau)^{-1} - \hat{Q}(\tau) = \hat{Q}(\tau)M(\tau)$$

where $M(\tau)$ satisfies the estimates (2.27). It follows that $\hat{Q}(\tau)M(\tau)$ also satisfies these estimates and (3.2) can be strengthened to

$$\sup_{\tau \in \mathbb{R}} \|\tau^k \hat{R}(\tau, \cdot, \cdot)\|_{\mathcal{C}^p} < \infty\ \forall\ p, k. \tag{3.3}$$

That is, the kernel $\hat{R}(\tau) \in \mathcal{S}(\mathbb{R}; \mathcal{C}^\infty(M^2; \mathrm{Hom}(\mathbb{E})))$. So if we define the t-translation-invariant operator

$$Rf(t, z) = (2\pi)^{-1} \int e^{it\tau} \hat{R}(\tau) \hat{f}(\tau, \cdot) d\tau \tag{3.4}$$

by inverse Fourier transform then

$$R : H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_2) \longrightarrow H^\infty_{\mathrm{ti}}(\mathbb{R} \times M; E_1)\ \forall\ s \in \mathbb{R}. \tag{3.5}$$

It certainly suffices to show this for $s < 0$ and then we know that the Fourier transform gives a map

$$\mathcal{F} : H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_2) \longrightarrow \langle\tau\rangle^{|s|} L^2(\mathbb{R}; H^{-|s|}(M; E_2)). \tag{3.6}$$

Since the kernel $\hat{R}(\tau)$ is rapidly decreasing in τ, as well as being smooth, for every $N > 0$,

$$\hat{R}(\tau) : \langle\tau\rangle^{|s|} L^2(\mathbb{R}; H^{-|s|}M; E_2) \longrightarrow \langle\tau\rangle^{-N} L^2(\mathbb{R}; H^N(M; E_2)) \tag{3.7}$$

and inverse Fourier transform maps

$$\mathcal{F}^{-1} : \langle\tau\rangle^{-N} H^N(M; E_2) \longrightarrow H^N_{\mathrm{ti}}(\mathbb{R} \times M; E_2)$$

which gives (3.5).

Thus $Q + R$ has the same property as Q in (2.17). So it only remains to check that $Q + R$ is the two-sided version of P and it is enough to do this on $\mathcal{S}(\mathbb{R} \times M; E_i)$ since these subspaces are dense in the Sobolev spaces. This in turn follows from (3.1) by taking the Fourier transform. Thus we have shown that the invertibility of P follows from its ellipticity and the invertibility of $\hat{P}(\tau)$ for $\tau \in \mathbb{R}$.

The converse statement is less important but certainly worth knowing! If P is an isomorphism as in (2.1), even for one value of s, then it must be elliptic — this follows as in the compact case since it is everywhere a local statement. Then if $\hat{P}(\tau)$ is not invertible for some $\tau \in \mathbb{R}$ we know, by ellipticity, that it is Fredholm and, by the stability of the index, of index zero (since $\hat{P}(\tau)$ *is* invertible for a dense set of $\tau \in \mathbb{C}$). There is therefore some $\tau_0 \in \mathbb{R}$ and $f_0 \in \mathcal{C}^\infty(M; E_2)$, $f_0 \neq 0$, such that

$$\hat{P}(\tau_0)^* f_0 = 0. \tag{3.8}$$

It follows that f_0 is *not* in the range of $\hat{P}(\tau_0)$. Then, choose a cut off function, $\rho \in \mathcal{C}^\infty_c(\mathbb{R})$ with $\rho(\tau_0) = 1$ (and supported sufficiently close to τ_0) and define $f \in \mathcal{S}(\mathbb{R} \times M; E_2)$ by

$$\hat{f}(\tau, \cdot) = \rho(\tau) f_0(\cdot). \tag{3.9}$$

Then $f \notin P \cdot H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_1)$ for any $s \in \mathbb{R}$. To see this, suppose $u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_1)$ has

$$Pu = f \Rightarrow \hat{P}(\tau)\hat{u}(\tau) = \hat{f}(\tau) \tag{3.10}$$

where $\hat{u}(\tau) \in \langle\tau\rangle^{|s|} L^2(\mathbb{R}; H^{-|s|}(M; E_1))$. The invertibility of $P(\tau)$ for $\tau \neq \tau_0$ on $\mathrm{supp}(\rho)$ (chosen with support close enough to τ_0) shows that

$$\hat{u}(\tau) = \hat{P}(\tau)^{-1} \hat{f}(\tau) \in \mathcal{C}^\infty((\mathbb{R} \backslash \{\tau_0\}) \times M; E_1).$$

Since we know that $\hat{P}(\tau)^{-1} - \hat{Q}(\tau) = \hat{R}(\tau)$ is a meromorphic family of smoothing operators it actually follows that $\hat{u}(\iota)$ is meromorphic in τ near τ_0 in the sense that

$$\hat{u}(\tau) = \sum_{j=1}^{k} (\tau - \tau_0)^{-j} u_j + v(\tau) \tag{3.11}$$

where the $u_j \in \mathcal{C}^\infty(M; E_1)$ and $v \in \mathcal{C}^\infty((\tau - \epsilon, \tau + \epsilon) \times M; E_1)$. Now, one of the u_j is not identically zero, since otherwise $\hat{P}(\tau_0)v(\tau_0) = f_0$, contradicting the choice of f_0. However, a function such as (3.11) is *not* locally in L^2 with values in any Sobolev space on M, which contradicts the existence of $u \in H^s_{\text{ti}}(\mathbb{R} \times M; E_1)$.

This completes the proof for invertibility of P. To get the Fredholm version it suffices to prove that if P is Fredholm then it is invertible. Since the arguments above easily show that the null space of P is empty on any of the $H^s_{\text{ti}}(\mathbb{R}\times M; E_1)$ spaces and the same applies to the adjoint, we easily conclude that P is an isomorphism if it is Fredholm. $\square$

This result allows us to deduce similar invertibility conditions on exponentially-weighted Sobolev spaces. Set

$$e^{at} H^s_{\text{ti}}(\mathbb{R} \times M; E) = \{u \in H^s_{\text{loc}}(\mathbb{R} \times M; E);\ e^{-at} u \in H^s_{\text{ti}}(\mathbb{R} \times M; E)\} \tag{3.12}$$

for any $\mathcal{C}^\infty$ vector bundle E over M. The translation-invariant differential operators also act on these spaces.

LEMMA 3.2. *For any $a \in \mathbb{R}$, $P \in \text{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ defines a continuous linear operator*

$$P : e^{at} H^{s+m}_{ti}(\mathbb{R} \times M; E_1) \longrightarrow e^{at} H^{s+m}_{ti}(\mathbb{R} \times M; E_2). \tag{3.13}$$

PROOF. We already know this for $a = 0$. To reduce the general case to this one, observe that (3.13) just means that

$$P \cdot e^{at} u \in e^{at} H^s_{\text{ti}}(\mathbb{R} \times M; E_2)\ \forall\ u \in H^s_{\text{ti}}(\mathbb{R} \times M; E_1) \tag{3.14}$$

with continuity meaning just continuous dependence on u. However, (3.14) in turn means that the conjugate operator

$$P_a = e^{-at} \cdot P \cdot e^{at} : H^{s+m}_{\text{ti}}(\mathbb{R} \times M; E_1) \longrightarrow H^s_{\text{ti}}(\mathbb{R} \times M; E_2). \tag{3.15}$$

Conjugation by an exponential is actually an isomorphism

$$\text{Diff}^m_{\text{ti}}(\mathbb{R} \times M; \mathbb{E}) \ni P \longmapsto e^{-at} P e^{at} \in \text{Diff}^m_{\text{ti}}(\mathbb{R} \times M; \mathbb{E}). \tag{3.16}$$

To see this, note that elements of $\text{Diff}^j(M; \mathbb{E})$ commute with multiplication by e^{at} and

$$e^{-at} D_t e^{at} = D_t - ia \tag{3.17}$$

which gives (3.16)).

The result now follows. ☐

PROPOSITION 3.3. *If* $P \in \mathrm{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ *is elliptic then as a map* (3.13) *it is invertible precisely for*

$$a \notin -\operatorname{Im}(D),\ D = D(P) \subset \mathbb{C}, \tag{3.18}$$

that is, a is not the negative of the imaginary part of an element of D.

Note that the set $-\operatorname{Im}(D) \subset \mathbb{R}$, for which invertibility fails, is discrete. This follows from the discreteness of D and the estimate (2.6). Thus in Fig **??** invertibility on the space with weight e^{at} correspond exactly to the horizonatal line with $\operatorname{Im}\tau = -a$ missing D.

PROOF. This is direct consequence of (**??**) and the discussion around (3.15). Namely, P is invertible as a map (3.13) if and only if P_a is invertible as a map (2.1) so, by Theorem 3.1, if

and only if

$$D(P_a) \cap \mathbb{R} = \emptyset.$$

From (3.17), $D(P_a) = D(P) + ia$ so this condition is just $D(P) \cap (\mathbb{R} - ia) = \emptyset$ as claimed. ☐

Although this is a characterization of the Fredholm properties on the standard Sobolev spaces, it is not the end of the story, as we shall see below.

One important thing to note is that $\mathbb{R}$ has *two* ends. The exponential weight e^{at} treats these differently – since if it is big at one end it is small at the other – and in fact we (or rather you) can easily define doubly-exponentially weighted spaces and get similar results for those. Since this is rather an informative extended exercise, I will offer some guidance.

DEFINITION 3.4. *Set*

$$\begin{aligned} H^{s,a,b}_{ti,\exp}(\mathbb{R} \times M; E) = \{&u \in H^s_{\mathrm{loc}}(\mathbb{R} \times M; E);\\ &\chi(t)e^{-at}u \in H^s_{ti}(\mathbb{R} \times M; E)(1-\chi(t))e^{bt}u \in H^s_{ti}(\mathbb{R} \times M; E)\} \end{aligned} \tag{3.19}$$

where $\chi \in \mathcal{C}^\infty(\mathbb{R})$, $\chi = 1$ *in* $t > 1$, $\chi = 0$ *in* $t < -1$.

Exercises.

(1) Show that the spaces in (3.19) are independent of the choice of χ, are all Hilbertable (are complete with respect to a Hilbert norm) and show that if $a + b \geq 0$

$$H^{s,a,b}_{\mathrm{ti},\exp}(\mathbb{R} \times M; E) = e^{at}H^s_{\mathrm{ti}}(\mathbb{R} \times M; E) + e^{-bt}H^s_{\mathrm{ti}}(\mathbb{R} \times M; E) \tag{3.20}$$

whereas if $a+b\le 0$ then

$$H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E)=e^{at}H^s_{\mathrm{ti}}(\mathbb{R}\times M;E)\cap e^{-bt}H^s_{\mathrm{ti}}(\mathbb{R}\times M;E). \tag{3.21}$$

(2) Show that any $P\in\mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M;\mathbb{E})$ defines a continuous linear map for any $s,\ a,\ b\in\mathbb{R}$

$$P:H^{s+m,a,b}_{\mathrm{ti\text{-}exp}}(\mathbb{R}\times M;E_1)\longrightarrow H^{s,a,b}_{\mathrm{ti\text{-}exp}}(\mathbb{R}\times M;E_2). \tag{3.22}$$

(3) Show that the standard L^2 pairing, with respect to dt, a smooth positive density on M and an inner product on E extends to a non-degenerate bilinear pairing

$$H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E)\times H^{-s,-a,-b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E)\longrightarrow\mathbb{C} \tag{3.23}$$

for any s, a and b. Show that the adjoint of P with respect to this pairing is P^* on the 'negative' spaces – you can use this to halve the work below.

(4) Show that if P is elliptic then (3.22) is Fredholm precisely when

$$a\notin-\mathrm{Im}(D)\text{ and }b\notin\mathrm{Im}(D). \tag{3.24}$$

Hint:- Assume for instance that $a+b\ge 0$ and use (3.20). Given (3.24) a parametrix for P can be constructed by combining the inverses on the single exponential spaces

$$Q_{a,b}=\chi'P_a^{-1}\chi+(1-\chi'')P^{-1}_{-b}(1-\chi) \tag{3.25}$$

where χ is as in (3.19) and χ' and χ'' are similar but such that $\chi'\chi=1$, $(1-\chi'')(1-\chi)=1-\chi$.

(5) Show that P is an isomorphism if and only if

$a+b\le 0$ and $[a,-b]\cap-\mathrm{Im}(D)=\emptyset$ or $a+b\ge 0$ and $[-b,a]\cap-\mathrm{Im}(D)=\emptyset$.

(6) Show that if $a+b\le 0$ and (3.24) holds then

$$\mathrm{ind}(P)=\dim\mathrm{null}(P)=\sum_{\tau_i\in D\cap(\mathbb{R}\times[b,-a])}\mathrm{Mult}(P,\tau_i)$$

where $\mathrm{Mult}(P,\tau_i)$ is the *algebraic* multiplicity of τ as a 'zero' of $\hat{P}(\tau)$, namely the dimension of the generalized null space

$$\mathrm{Mult}(P,\tau_i)=\dim\left\{u=\sum_{p=0}^N u_p(z)D^p_\tau\delta(\tau-\tau_i);P(\tau)u(\tau)\equiv 0\right\}.$$

(7) Characterize these multiplicities in a more algebraic way. Namely, if τ' is a zero of $P(\tau)$ set $E_0=\mathrm{null}\,P(\tau')$ and $F_0=\mathcal{C}^\infty(M;E_2)/P(\tau')\mathcal{C}^\infty(M;E_1)$. Since $P(\tau)$ is Fredholm of index zero, these are finite dimensional vector spaces of the same dimension. Let the derivatives of P be $T_i=\partial^iP/\partial\tau^i$ at $\tau=\tau'$ Then define $R_1:E_0\longrightarrow F_0$

as T_1 restricted to E_0 and projected to F_0. Let E_1 be the null space of R_1 and $F_1 = F_0/R_1E_0$. Now proceed inductively and define for each i the space E_i as the null space of R_i, $F_i = F_{i-1}/R_iE_{i-1}$ and $R_{i+1} : E_i \longrightarrow F_i$ as T_i restricted to E_i and projected to F_i. Clearly E_i and F_i have the same, finite, dimension which is non-increasing as i increases. The properties of $P(\tau)$ can be used to show that for large enough i, $E_i = F_i = \{0\}$ and

$$\mathrm{Mult}(P,\tau') = \sum_{i=0}^{\infty} \dim(E_i) \tag{3.26}$$

where the sum is in fact finite.

(8) Derive, by duality, a similar formula for the index of P when $a + b \geq 0$ and (3.24) holds, showing in particular that it is injective.

4. Resolvent operator

Addenda to Chapter 7

More?

- Why – manifold with boundary later for Euclidean space, but also resolvent (Photo-C5-01)
- Hölder type estimates – Photo-C5-03. Gives interpolation.

As already noted even a result such as Proposition 3.3 and the results in the exercises above by no means exhausts the possibile realizations of an element $P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R} \times M; \mathbb{E})$ as a Fredholm operator. Necessarily these other realization cannot simply be between spaces like those in (3.19). To see what else one can do, suppose that the condition in Theorem 3.1 is violated, so

$$D(P) \cap \mathbb{R} = \{\tau_1, \dots, \tau_N\} \neq \emptyset. \tag{4.1}$$

To get a Fredholm operator we need to change either the domain or the range space. Suppose we want the range to be $L^2(\mathbb{R}\times M; E_2)$. Now, the condition (3.24) guarantees that P is Fredholm as an operator (3.22). So in particular

$$P : H^{m,\epsilon,\epsilon}_{\mathrm{ti-exp}}(\mathbb{R} \times M; E_1) \longrightarrow H^{0,\epsilon,\epsilon}_{\mathrm{ti\text{-}exp}}(\mathbb{R} \times M; E_2) \tag{4.2}$$

is Fredholm for all $\epsilon > 0$ sufficiently small (becuase D is discrete). The image space (which is necessarily the range in this case) just consists of the sections of the form $\exp(a|t|)f$ with f in L^2. So, in this case the range certainly contains L^2 so we can define

(4.3)

$$\mathrm{Dom}_{\mathrm{AS}}(P)=\{u\in H^{m,\epsilon,\epsilon}_{\text{ti-exp}}(\mathbb{R}\times M;E_1);Pu\in L^2(\mathbb{R}\times M;E_2)\},\ \epsilon>0 \text{ sufficiently small.}$$

This space is independent of $\epsilon > 0$ if it is taken smalle enough, so the same space arises by taking the intersection over $\epsilon > 0$.

Proposition 4.1. *For any elliptic element $P \in \mathrm{Diff}^m_{ti}(\mathbb{R}\times M;\mathbb{E})$ the space in* (4.3) *is Hilbertable space and*

$$P:\mathrm{Dom}_{AS}(P)\longrightarrow L^2(\mathbb{R}\times M;E_2) \text{ is Fredholm.} \tag{4.4}$$

I have not made the assumption (4.1) since it is relatively easy to see that if $D\cap\mathbb{R}=\emptyset$ then the domain in (4.3) reduces again to $H^m_{\mathrm{ti}}(\mathbb{R}\times M;E_1)$ and (4.4) is just the standard realization. Conversely of course under the assumption (4.1) the domain in (4.4) is strictly larger than the standard Sobolev space. To see what it actually is requires a bit of work but if you did the exercises above you are in a position to work this out! Here is the result when there is only one pole of $\hat{P}(\tau)$ on the real line and it has order one.

Proposition 4.2. *Suppose $P\in\mathrm{Diff}^m_{ti}(\mathbb{R}\times M;\mathbb{E})$ is elliptic, $\hat{P}(\tau)$ is invertible for $\tau\in\mathbb{R}\setminus\{0\}$ and in addition $\tau\hat{P}(\tau)^{-1}$ is holomorphic near* 0*. Then the Atiyah-Singer domain in* (4.4) *is*

$$\begin{aligned}(4.5)\quad \mathrm{Dom}_{AS}(P)=\big\{u=u_1+u_2;u_1\in H^m_{ti}(\mathbb{R}\times M;E_1),\\ u_2=f(t)v,\ v\in\mathcal{C}^\infty(M;E_1),\ \hat{P}(0)v=0,\ f(t)=\int_0^t g(t)dt,\ g\in H^{m-1}(\mathbb{R})\big\}.\end{aligned}$$

Notice that the 'anomalous' term here, u_2, need not be square-integrable. In fact for any $\delta>0$ the power $\langle t\rangle^{\frac12-\delta}v\in\langle t\rangle^{1-\delta}L^2(\mathbb{R}\times M;E_1)$ is included and conversely

$$f\in\bigcap_{\delta>0}\langle t\rangle^{1+\delta}H^{m-1}(\mathbb{R}). \tag{4.6}$$

One can say a lot more about the growth of f if desired but it is generally quite close to $\langle t\rangle L^2(\mathbb{R})$.

Domains of this sort are sometimes called 'extended L^2 domains' – see if you can work out what happens more generally.

CHAPTER 8

Manifolds with boundary

- Dirac operators – Photos-C5-16, C5-17.
- Homogeneity etc Photos-C5-18, C5-19, C5-20, C5-21, C5-23, C5-24.

1. Compactifications of $\mathbb{R}$.

As I will try to show by example later in the course, there are I believe considerable advantages to looking at compactifications of non-compact spaces. These advantages show up last in geometric and analytic considerations. Let me start with the simplest possible case, namely the real line. There are two standard compactifications which one can think of as 'exponential' and 'projective'. Since there is only one connected compact manifold with boundary compactification corresponds to the choice of a diffeomorphism onto the interior of $[0, 1]$:

$$\begin{gathered}\gamma : \mathbb{R} \longrightarrow [0, 1],\ \gamma(\mathbb{R}) = (0, 1),\\ \gamma^{-1} : (0, 1) \longrightarrow \mathbb{R},\ \gamma, \gamma^{-1}\mathcal{C}^\infty.\end{gathered} \tag{1.1}$$

In fact it is not particularly pleasant to have to think of the global maps γ, although we can. Rather we can think of separate maps

$$\begin{gathered}\gamma_+ : (T_+, \infty) \longrightarrow [0, 1]\\ \gamma_- : (T_-, -\infty) \longrightarrow [0, 1]\end{gathered} \tag{1.2}$$

which *both* have images $(0, x_\pm)$ and as diffeomorphism other than signs. In fact if we want the two ends to be the 'same' then we can take $\gamma_-(t) = \gamma_+(-t)$. I leave it as an exercise to show that γ then exists with

$$\begin{cases}\gamma(t) = \gamma_+(t) & t \gg 0\\ \gamma(t) = 1 - \gamma_-(t) & t \ll 0.\end{cases} \tag{1.3}$$

So, all we are really doing here is identifying a 'global coordinate' $\gamma_+^* x$ near ∞ and another near $-\infty$. Then two choices I refer to above are

$$\begin{aligned} x &= e^{-t} && \text{exponential compactification} \\ x &= 1/t && \text{projective compactification}\,. \end{aligned} \tag{CR.4}$$

Note that these are *alternatives*!

Rather than just consider $\mathbb{R}$, I want to consider $\mathbb{R} \times M$, with M compact, as discussed above.

LEMMA 1.1. *If $R : H \longrightarrow H$ is a compact operator on a Hilbert space then* $\operatorname{Id} - R$ *is Fredholm.*

PROOF. A compact operator is one which maps the unit ball (and hence any bounded subset) of H onto a precompact set, a set with compact closure. The unit ball in the null space of $\operatorname{Id} - R$ is

$$\{u \in H; \|u\| = 1\,,\ u = Ru\} \subset R\{u \in H; \|u\| = 1\}$$

and is therefore precompact. Since it is closed, it is compact and any Hilbert space with a compact unit ball is finite dimensional. Thus the null space of $(\operatorname{Id} - R)$ is finite dimensional.

Consider a sequence $u_n = v_n - Rv_n$ in the range of $\operatorname{Id} - R$ and suppose $u_n \to u$ in H. We may assume $u \neq 0$, since 0 is in the range, and by passing to a subsequence suppose that of γ on ?? fields. Clearly

$$\begin{aligned} \gamma(t) = e^{-t} &\ \Rightarrow\ \gamma_*(\partial_t) = -x(\partial_x) \\ \tilde{\gamma}(t) = 1/t &\ \Rightarrow\ \tilde{\gamma}_*(\partial_t) = -s^2\partial_s \end{aligned} \tag{CR.5}$$

where I use 's' for the variable in the second case to try to reduce confusion, it is just a variable in $[0,1]$. Dually

$$\begin{aligned} \gamma^*\left(\frac{dx}{x}\right) &= -dt \\ \tilde{\gamma}^*\left(\frac{ds}{s^2}\right) &= -dt \end{aligned} \tag{CR.6}$$

in the two cases. The minus signs just come from the fact that both γ's reverse orientation.

PROPOSITION 1.2. *Under exponential compactification the translation-invariant Sobolev spaces on $\mathbb{R} \times M$ are identified with*

$$\begin{aligned} H_b^k([0,1] \times M) = \Big\{ u \in L^2\left([0,1] \times M; \frac{dx}{x} V_M\right)\ ;\ \forall\ \ell, p \le k \\ P_p \in \operatorname{Diff}^p(M)\,,\ (xD_x)^\ell P_p u \in L^2\big([0,1] \times M; \frac{dx}{x} V_M\big) \Big\} \end{aligned} \tag{1.4}$$

for k a positive integer, $\dim M = n$,

$$(1.5)\quad H^s_b([0,1]\times M) = \Big\{u \in L^2\left([0,1]\times M; \frac{dx}{x}V_M\right);$$
$$\iint \frac{|u(x,z)-u(x',z')|^2}{\left(|\log\frac{x}{x'}|^2+\rho(z,z')\right)^{\frac{n+s+1}{2}}}\frac{dx}{x}\frac{dx'}{x'}\nu\nu' < \infty\Big\} 0 < s < 1$$

and for $s<0$, $k\in\mathbb{N}$ s.t., $0\le s+k<1$,

$$(1.6)\quad H^s_b([0,1]\times M) = \Big\{u = \sum_{0\le j+p\le k}(Xd^J_X)p_P u_{j,p},$$
$$P_p \in \mathrm{Diff}^p(M)\,,\ u_{j,p}\in H^{s+k}_b([0,1]\times M)\Big\}.$$

Moreover the L^2 pairing with respect to the measure $\frac{dx}{x}\nu$ extends by continuity from the dense subspaces $\mathcal{C}^\infty_c((0,1)\times M)$ to a non-degenerate pairing

$$(1.7)\quad H^s_b([0,1]\times M)\times H^{-s}_b([0,1]\times M)\ni(n,u)\longmapsto\int u\cdot v\frac{dx}{x}\nu\in\mathbb{C}.$$

□

PROOF. This is all just translation of the properties of the space $H^s_{\mathrm{ti}}(\mathbb{R}\times M)$ to the new coordinates. □

Note that there are other properties I have not translated into this new setting. There is one additional fact which it is easy to check. Namely $\mathcal{C}^\infty([0,1]\times M)$ acts as multipliers on all the spaces $H^s_{\mathrm{b}}([0,1]\times M)$. This follows directly from Proposition 1.2;

(CR.12)

$$\mathcal{C}^\infty([0,1]\times M)\times H^s_{\mathrm{b}}([0,1]\times M)\ni(\varphi,u)\mapsto\varphi u\in H^s_{\mathrm{b}}([0,1]\times M)\,.$$

What about the 'b' notation? Notice that $(1-x)x\partial_x$ and the smooth vector fields on M span, over $\mathcal{C}^\infty(X)$, for $X=[0,1]\times M$, *all* the vector fields tangent to $\{x=0|u|x=1\}$. Thus we can define the 'boundary differential operators' as

(CR.13)

$$\mathrm{Diff}^m_{\mathrm{b}}([0,1]\times M_i)^E = \left\{P = \sum_{0\le j+p\le m}a_{j,p}(x_j)((1-x)xD_x)^jP_p\ ,\right.$$
$$P_p\in\mathrm{Diff}^p(M_i)^E\Big\}$$

and conclude from (CR.12) and the earlier properties that

$$\text{(CR.14)} \qquad \begin{aligned} &P \in \mathrm{Diff}^m_{\mathrm{b}}(X; E) \Rightarrow \\ &P : H^{s+m}_{\mathrm{b}}(X; E) \to H^s_{\mathrm{b}}(X; E) \, \forall s \in \mathbb{R} \, . \end{aligned}$$

THEOREM 1.3. *A differential operator as in* (1.3) *is Fredholm if and only if it is elliptic in the interior and the two "normal operators'*

$$\text{(CR.16)} \qquad I_\pm(P) = \sum_{0 \le j+p \le m} a_{j,p}(x_{\pm 1})(\pm D_k)^i P_p \quad x_+ = 0 \, , \, x_- = 1$$

derived from (CR.13)*, are elliptic and invertible on the translation-invariant Sobolev spaces.*

PROOF. As usual we are more interested in the sufficiency of these conditions than the necessity. To prove this result by using the present (slightly low-tech) methods requires going back to the beginning and redoing most of the proof of the Fredholm property for elliptic operators on a compact manifold.

The first step then is *a priori* bounds. What we want to show is that if the conditions of the theorem hold then for $u \in H^{s+m}_{\mathrm{b}}(X; E)$, $x = \mathbb{R} \times M$, $\exists C > 0$ s.t.

$$\text{(CR.17)} \qquad \|u\|_{m+s} \le C_s \|Pu\|_s + C_s \|x(1-x)u\|_{s-1+m} \, .$$

Notice that the norm on the right has a factor, $x(1-x)$, which vanishes at the boundary. Of course this is supposed to come from the invertibility of $I_\pm(P)$ in $\mathbb{R}(0)$ and the ellipticity of P.

By comparison $I_\pm(P) : H^{s+m}_{\hbar}(\mathbb{R} \times M) \to H^s_{\hbar}(\mathbb{R} \times M)$ are isomorphisms — necessary and sufficient conditions for this are given in Theorem ???. We can use the compactifying map γ to convert this to a statement as in (CR.17) for the operators

$$\text{(CR.18)} \qquad P_\pm \in \mathrm{Diff}^m_{\mathrm{b}}(X) \, , \, P_\pm = I_\pm(P)(\gamma_* D_t, \cdot) \, .$$

Namely

$$\text{(CR.19)} \qquad \|u\|_{m+s} \le C_s \|P_\pm u\|_s$$

where these norms, as in (CR.17) are in the H^s_{b} spaces. Note that near $x = 0$ or $x = 1$, $P_\pm$ are obtained by substituting $D_t \mapsto x D_x$ or $(1-x)D_x$ in (CR.17). Thus

$$\text{(CR.20)} \qquad P - P_\pm \in (x - x_\pm) \, \mathrm{Diff}^m_{\mathrm{b}}(X) \, , \quad x_\pm = 0, 1$$

have coefficients which *vanish* at the appropriate boundary. This is precisely how (CR.16) is derived from (CR.13). Now choose $\varphi \in$

$\mathcal{C}^\infty, (0,1)\times M$ which is equal to 1 on a sufficiently large set (and has $0 \le \varphi \le 1$) so that

$$1-\varphi = \varphi_+ + \varphi_- \,,\ \varphi_\pm \in \mathcal{C}^\infty([0,1]\times M) \tag{CR.21}$$

have $\operatorname{supp}(\varphi_\pm) \subset \{|x-x_\pm| \le \epsilon)$, $0 \le \varphi_+ 1$.

By the interim elliptic estimate,

$$\|\varphi u\|_{s+m} \le C_s\|\varphi P u\|_s + C_s'\|\psi u\|_{s-1+m} \tag{CR.22}$$

where $\psi \in \mathcal{C}_c^\infty((0,1)\times M)$. On the other hand, because of (CR.20)

(CR.23)

$$\begin{aligned}\|\varphi_\pm u\|_{m+s} &\le C_s\|\varphi_\pm P_\pm u\|_s + C_s\|[\varphi_\pm, P_\pm u]\|_s \\ &\le C_s\|\varphi_\pm P u\|_s + C_s\varphi_\pm(P-P_\pm)u\|_s + C_s\|[\varphi_\pm, P_\pm]u\|_s\,.\end{aligned}$$

Now, if we can choose the support at $\varphi_\pm$ small enough — recalling that C_s truly depends on $I_\pm(P_t)$ and s — then the second term on the right in (CR.23) is bounded by $\frac{1}{4}\|u\|_{m+s}$, since *all* the coefficients of $P-P_\pm$ are small on the support off $\varphi_\pm$. Then (CR.24) ensures that the final term in (CR.17), since the coefficients vanish at $x = x_\pm$.

The last term in (CR.22) has a similar bound since ψ has compact support in the interim. This combining (CR.2) and (CR.23) gives the desired bound (CR.17).

To complete the proof that P is Fredholm, we need another property of these Sobolev spaces.

Lemma 1.4. *The map*

$$Xx(1-x): H_b^s(X) \longrightarrow H_b^{s-1}(X) \tag{1.8}$$

is compact.

Proof. Follow it back to $\mathbb{R}\times M$!

$\square$

Now, it follows from the *a priori* estimate (CR.17) that, as a map (CR.14), P has finite dimensional null space and closed range. This is really the proof of Proposition ?? again. Moreover the adjoint of P with respect to $\frac{dx}{x}V, P^*$, is again elliptic and satisfies the condition of the theorem, so it too has finite-dimensional null space. Thus the range of P has finite codimension so it is Fredholm.

$\square$

A corresponding theorem, with similar proof follows for the cusp compactification. I will formulate it later.

2. Basic properties

A discussion of manifolds with boundary goes here.

3. Boundary Sobolev spaces

Generalize results of Section 1 to arbitrary compact manifolds with boundary.

4. Dirac operators

Euclidean and then general Dirac operators

5. Homogeneous translation-invariant operators

One application of the results of Section 3 is to homogeneous constant-coefficient operators on $\mathbb{R}^n$, including the Euclidean Dirac operators introduced in Section 4. Recall from Chapter 4 that an elliptic constant-coefficient operator is Fredholm, on the standard Sobolev spaces, if and only if its characteristic polynomial has no real zeros. If P is homogeneous

$$P_{ij}(t\zeta) = t^m P_{ij}(\zeta) \ \forall \ \zeta \in \mathbb{C}^n \, , \ t \in \mathbb{R} \, , \tag{5.1}$$

and elliptic, then the only real zero (of the determinant) is at $\zeta = 0$. We will proceed to discuss the radial compactification of Euclidean space to a ball, or more conveniently a half-sphere

$$\gamma_R : \mathbb{R}^n \hookrightarrow \mathbb{S}^{n,1} = \{Z \in \mathbb{R}^{n+1} \, ; \, |Z| = 1 \, , \, Z_0 \geq 0\} \, . \tag{5.2}$$

Transferring P to $\mathbb{S}^{n,1}$ gives

$$P_R \in Z_0^m \operatorname{Diff}_{\mathrm{b}}^m(\mathbb{S}^{n,1} \, ; \, \mathbb{C}^N) \tag{5.3}$$

which is elliptic and to which the discussion in Section 3 applies.

In the 1-dimensional case, the map (5.2) reduces to the second 'projective' compactification of $\mathbb{R}$ discussed above. It can be realized globally by

$$\gamma_R(z) = \left(\frac{1}{\sqrt{1+|z|^2}} \, , \, \frac{z}{\sqrt{1+|z|^2}} \right) \in \mathbb{S}^{n,1} \, . \tag{5.4}$$

Geometrically this corresponds to a form of stereographic projection. Namely, if $\mathbb{R}^n \ni z \mapsto (1, z) \in \mathbb{R}^{n+1}$ is embedded as a 'horizontal plane' which is then projected radially onto the sphere (of radius one around the origin) one arrives at (5.4). It follows easily that γ_R is a diffeomorphism onto the open half-sphere with inverse

$$z = Z'/Z_0 \, , \ Z' = (Z_1, \dots, Z_n) \, . \tag{5.5}$$

Whilst (5.4) is concise it is not a convenient form of the compactification as far as computation is concerned. Observe that

$$x \mapsto \frac{x}{\sqrt{1+x^2}}$$

is a diffeomorphism of neighborhoods of $0 \in \mathbb{R}$. It follows that Z_0, the first variable in (5.4) can be replaced, near $Z_0 = 0$, by $1/|z| = x$. That is, there is a diffeomorphism

$$\{0 \le Z_0 \le \epsilon\} \cap \mathbb{S}^{n,1} \leftrightarrow [0, \delta]_x \times \mathbb{S}^{n-1}_\theta \tag{5.6}$$

which composed with (5.4) gives $x = 1/|z|$ and $\theta = z/|z|$. In other words the compactification (5.4) is equivalent to the introduction of polar coordinates near infinity on $\mathbb{R}^n$ followed by inversion of the radial variable.

LEMMA 5.1. *If $P = (P_{ij}(D_z))$ is an $N \times N$ matrix of constant coefficient operators in $\mathbb{R}^n$ which is homogeneous of degree $-m$ then* (5.3) *holds after radial compactification. If P is elliptic then P_R is elliptic.*

PROOF. This is a bit tedious if one tries to do it by direct computation. However, it is really only the homogeneity that is involved. Thus if we use the coordinates $x = 1/|z|$ and $\theta = z/|z|$ valid near the boundary of the compactification (i.e., near ∞ on $\mathbb{R}^n$) then

$$P_{ij} = \sum_{0 \le \ell \le m} D_x^\ell P_{\ell,i,j}(x, \theta, D_\theta)\,,\ P_{\ell,i,j} \in \mathcal{C}^\infty(0, \delta)_x\,;\, \mathrm{Diff}^{m-\ell}(\mathbb{S}^{n-1}). \tag{5.7}$$

Notice that we *do* know that the coefficients are smooth in $0 < x < \delta$, since we are applying a diffeomorphism there. Moreover, the operators $P_{\ell,i,j}$ are uniquely determined by (5.7).

So we can exploit the assumed homogeneity of P_{ij}. This means that for any $t > 0$, the transformation $z \mapsto tz$ gives

$$P_{ij} f(tz) = t^m (P_{ij} f)(tz)\,. \tag{5.8}$$

Since $|tz| = t|z|$, this means that the transformed operator must satisfy (5.9)

$$\sum_\ell D_x^\ell P_{\ell,i,j}(x, \theta, D_\theta) f(x/t, \theta) = t^m (\sum_\ell D^\ell P_{\ell,i,j}(\cdot, \theta, D_\theta) f(\cdot, \theta))(x/t)\,.$$

Expanding this out we conclude that

$$x^{-m-\ell} P_{\ell,i,j}(x, \theta, D_\theta) = P_{\ell,i,j}(\theta, D_\theta) \tag{5.10}$$

is independent of x. Thus in fact (5.7) becomes

$$P_{ij} = x^m \sum_{0 \le j \le \ell} x^\ell D_x^\ell P_{\ell,j,i}(\theta, D_\theta)\,. \tag{5.11}$$

Since we can rewrite

$$x^\ell D_x = \sum_{0 \le j \le \ell} C_{\ell,j} (x D_x)^j \tag{5.12}$$

(with explicit coefficients if you want) this gives (5.3). Ellipticity in this sense, meaning that

$$x^{-m}P_R \in \mathrm{Diff}^m_{\mathrm{b}}(\mathbb{S}^{n,1};\mathbb{C}^N) \tag{5.13}$$

(5.11) and the original ellipticity at P. Namely, when expressed in terms of xD_x the coefficients of 5.13 are independent of x (this of course just reflects the homogeneity), ellipticity in $x > 0$ follows by the coordinate independence of ellipticity, and hence extends down to $x = 0$. □

Now the coefficient function Z_0^{w+m} in (5.3) always gives an isomorphism

$$\times Z_0^m : Z_0^w H^s_{\mathrm{b}}(\mathbb{S}^{n,1}) \longrightarrow Z_0^{w+m} H^s_{\mathrm{b}}(\mathbb{S}^{n,1})\,. \tag{5.14}$$

Combining this with the results of Section 3 we find most of

Theorem 5.2. *If P is an $N \times N$ matrix of constant coefficient differential operators on $\mathbb{R}^n$ which is elliptic and homogeneous of degree $-m$ then there is a discrete set $-\operatorname{Im}(D(P)) \subset \mathbb{R}$ such that*

(5.15)

$$P : Z_0^w H_b^{m+s}(\mathbb{S}^{n,1}) \longrightarrow Z_0^{w+m} H_b^s(\mathbb{S}^{n,1}) \text{ is Fredholm } \forall\ w \notin -\operatorname{Im}(D(P))$$

where (5.4) *is used to pull these spaces back to $\mathbb{R}^n$. Moreover,*

$$\begin{aligned} &P \text{ is injective for } w \in [0,\infty) \text{ and} \\ &P \text{ is surjective for } w \in (-\infty, n-m] \cap (-\operatorname{Im}(D)(P))\,. \end{aligned} \tag{5.16}$$

Proof. The conclusion (5.15) is exactly what we get by applying Theorem X knowing (5.3).

To see the specific restriction (5.16) on the null space and range, observe that the domain spaces in (5.15) are tempered. Thus the null space is contained in the null space on $\mathcal{S}'(\mathbb{R}^n)$. Fourier transform shows that $P(\zeta)\hat{u}(\zeta) = 0$. From the assumed ellipticity of P and homogeneity it follows that $\operatorname{supp}(\hat{u}(\zeta)) \subset \{0\}$ and hence $\hat{u}$ is a sum of derivatives of delta functions and finally that u itself is a polynomial. If $w \geq 0$ the domain in (5.15) contains no polynomials and the first part of (5.16) follows.

The second part of (5.16) follows by a duality argument. Namely, the adjoint of P with respect to $L^2(\mathbb{R}^n)$, the usual Lebesgue space, is P^* which is another elliptic homogeneous differential operator with constant coefficients. Thus the first part of (5.16) applies to P^*. Using the homogeneity of Lebesgue measure,

$$|dz| = \frac{dx}{x^{n+1}} \cdot \nu_\theta \text{ near } \infty \tag{5.17}$$

and the shift in weight in (5.15), the second part of (5.16) follows. □

One important consequence of this is a result going back to Nirenberg and Walker (although expressed in different language).

COROLLARY 5.3. *If P is an elliptic $N \times N$ matrix constant coefficient differential operator which is homogeneous of degree m, with $n > m$, the the map* (5.15) *is an isomorphism for $w \in (0, n-m)$.*

In particular this applies to the Laplacian in dimensions $n > 2$ and to the constant coefficient Dirac operators discussed above in dimensions $n > 1$. In these cases it is also straightforward to compute the index and to identify the surjective set. Namely, for a constant coefficient Dirac operator

$$D(P) = i\mathbb{N}_0 \cup i(n - m + \mathbb{N}_0)\,. \tag{5.18}$$

Figure goes here.

6. Scattering structure

Let me briefly review how the main result of Section 5 was arrived at. To deal with a constant coefficient Dirac operator we first radially compactified $\mathbb{R}^n$ to a ball, then peeled off a multiplicative factor Z_0 from the operator showed that the remaining operator was Fredholm by identifing a neighbourhood of the boundary with part of $\mathbb{R} \times \mathbb{S}^{n-1}$ using the exponential map to exploit the results of Section 1 near infinity. Here we will use a similar, but different, procedure to treat a different class of operators which are Fredholm on the *standard* Sobolev spaces.

Although we will only apply this in the case of a ball, coming from $\mathbb{R}^n$, I cannot resist carrying out the discussed for a general compact manifolds — since I think the generality clarifies what is going on. Starting from a compact manifold with boundary, M, the first step is essentially the reverse of the radial compactification of $\mathbb{R}^n$.

Near any point on the boundary, $p \in \partial M$, we can introduce 'admissible' coordinates, $x, y_1, \dots, y_{n-1}$ where $\{x = 0\|$ is the local form of the boundary and $y_1, \dots, y_{n-1}$ are tangential coordinates; we normalize $y_1 = \cdots = y_{n-1} = 0$ at p. By reversing the radial compactification of $\mathbb{R}^n$ I mean we can introduce a diffeomorphism of a neighbourhood of p to a conic set in $\mathbb{R}^n$:

$$z_n = 1/x\,,\ z_j = y_j/x\,,\ j = 1, \dots, n-1\,. \tag{6.1}$$

Clearly the 'square' $|y| < \epsilon$, $0 < x < \epsilon$ is mapped onto the truncated conic set

$$z_n \geq 1/\epsilon\,,\ |z'| < \epsilon|z_n|\,, z' = (z_1, \dots, z_{n-1})\,. \tag{6.2}$$

Definition 6.1. *We define spaces $H^s_{sc}(M)$ for any compact manifold with boundary M by the requirements*

$$u \in H^s_{sc}(M) \Longleftrightarrow u \in H^s_{\mathrm{loc}}(M \setminus \partial M) \text{ and } R^*_j(\varphi_j u) \in H^s(\mathbb{R}^n) \tag{6.3}$$

for $\varphi_j \in \mathcal{C}^\infty(M)$, $0 \le \varphi_i \le 1$, $\sum \varphi_i = 1$ in a neighbourhood of the boundary and where each φ_j is supported in a coordinate patch (**??**), (6.2) *with R given by* (6.1).

Of course such a definition would not make much sense if it depended on the choice of the partition of unity near the boundary $\{\varphi_i\|$ or the choice of coordinate. So really (6.1) should be preceded by such an invariance statement. The key to this is the following observation.

Proposition 6.2. *If we set $\mathcal{V}_{\mathrm{sc}}(M) = x\mathcal{V}_{\mathrm{b}}(M)$ for any compact manifold with boundary then for any $\psi \in \mathcal{C}^\infty(M)$ supported in a coordinate patch* (**??**)*, and any $\mathcal{C}^\infty$ vector field V on M*

$$\psi V \in \mathcal{V}_{\mathrm{sc}}(M) \Longleftrightarrow \psi V = \sum_{j=1}^n \mu_j (R^{-1})_*(D_{z_j}) \,, \ \mu_j \in \mathcal{C}^\infty(M) \,. \tag{6.4}$$

Proof. The main step is to compute the form of D_{z_j} in terms of the coordinate obtained by inverting (6.1). Clearly

$$D_{z_n} = x^2 D_x \,, \, D_{z_j} = x D_{y_j} - y_i x^2 D_x \,, \ j < n \,. \tag{6.5}$$

Now, as discussed in Section 3, xD_x and D_{y_j} locally span $\mathcal{V}_{\mathrm{b}}(M)$, so x^2D_x, xD_{y_j} locally span $\mathcal{V}_{\mathrm{sc}}(M)$. Thus (6.5) shows that in the singular coordinates (6.1), $\mathcal{V}_{\mathrm{sc}}(M)$ is spanned by the D_{z_ℓ}, which is exactly what (6.4) claims. □

Next let's check what happens to Euclidean measure under R, actually we did this before:

$$|dz| = \frac{|dx|}{x^{n+1}} \nu_y \,. \tag{SS.9}$$

Thus we can first identify what (6.3) means in the case of $s = 0$.

Lemma 6.3. *For $s = 0$, Definition* (6.1) *unambiguously defines*

$$H^0_{sc}(M) = \left\{ u \in L^2_{\mathrm{loc}}(M) \,; \int |u|^2 \frac{\nu_M}{x^{n+1}} < \infty \right\} \tag{6.6}$$

where ν_M is a positive smooth density on M (smooth up to the boundary of course) and $x \in \mathcal{C}^\infty(M)$ is a boundary defining function.

Proof. This is just what (6.3) and (SS.9) mean. □

Combining this with Proposition 6.2 we can see directly what (6.3) means for $kin\mathbb{N}$.

LEMMA 6.4. *If* (6.3) *holds for* $s = k \in \mathbb{N}$ *for any one such partition of unity then* $u \in H^0_{sc}(M)$ *in the sense of* (6.6) *and*

$$V_1 \dots V_j u \in H^0_{sc}(M) \ \forall \ V_i \in \mathcal{V}_{\mathrm{sc}}(M) \ \textit{if} \ j \le k \,, \tag{6.7}$$

and conversely.

PROOF. For clarity we can proceed by induction on k and replace (6.7) by the statements that $u \in H^{k-1}_{\mathrm{sc}}(M)$ and $Vu \in H^{k-1}_{\mathrm{sc}}(M)$ $\forall V \in \mathcal{V}_{\mathrm{sc}}(M)$. In the interior this is clear and follows immediately from Proposition 6.2 provided we carry along the inductive statement that

$$\mathcal{C}^\infty(M) \text{ acts by multiplication on } H^k_{\mathrm{sc}}(M)\,. \tag{6.8}$$

□

As usual we can pass to general $s \in \mathbb{R}$ by treating the cases $0 < s < 1$ first and then using the action of the vector fields.

PROPOSITION 6.5. *For* $0 < s < 1$ *the condition* (6.3) *(for any one partition of unity) is equivalent to requiring* $u \in H^0_{sc}(M)$ *and*

$$\iint_{M\times M} \frac{|u(p) - u(p')|^2}{\rho_{sc}^{n+2s}} \frac{\nu_M}{x^{n+1}} \frac{\nu'_M}{(x')^{n+1}} < \infty \tag{6.9}$$

where $\rho_{sc}(p, p') = \chi\chi' p(p, p') + \sum_j \varphi_j \varphi'_j \langle z - z'\rangle$.

PROOF. Use local coordinates. □

Then for $s \ge 1$ if k is the integral part of s, so $0 \le s - k < 1$, $k \in \mathbb{N}$,

$$u \in H^s_{\mathrm{sc}}(M) \Longleftrightarrow V_1, \dots, V_j u \in H^{s-k}_{\mathrm{sc}}(M)\,, V_i \in \mathcal{V}_{\mathrm{sc}}(M)\,,\ j \le k \tag{6.10}$$

and for $s < 0$ if $k \in \mathbb{N}$ is chosen so that $0 \le k + s < 1$, then

$$\begin{gathered} u \in H^s_{\mathrm{sc}}(M) \Leftrightarrow \exists\ V_j \in H^{s+k}_{\mathrm{sc}}(M)\,,\ j = 1, \dots, \mathbb{N}\,, \\ u_j \in H^{s-k}_{\mathrm{sc}}(M)\,, V_{j,i}(M)\,,\ 1 \le i \le \ell_j \le k \text{ s.t.} \\ u = u_0 + \sum_{j=1}^N V_{j,i} \cdots V_{j,\ell_j} u_j. \end{gathered} \tag{6.11}$$

All this complexity is just because we are preceding in such a 'low-tech' fashion. The important point is that these Sobolev spaces are determined by the choice of 'structure vector fields', $V \in \mathcal{V}_{\mathrm{sc}}(M)$. I leave it as an important exercise to check that

LEMMA 6.6. *For the ball, or half-sphere,*

$$\gamma_R^* H^s_{sc}(\mathbb{S}^{n,1}) = H^s(\mathbb{R}^n)\,.$$

Thus on Euclidean space we have done *nothing*. However, my claim is that we understand things better by doing this! The idea is that we should Fourier analysis on $\mathbb{R}^n$ to analyse differential operators which are made up out of $\mathcal{V}_{\mathrm{sc}}(M)$ on any compact manifold with boundary M, and this includes $\mathbb{S}^{n,1}$ as the radial compactification of $\mathbb{R}^n$. Thus set

$$\begin{aligned}\operatorname{Diff}^m_{\mathrm{sc}}(M) = \Big\{P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M);\ \exists\ f \in \mathcal{C}^\infty(M) \text{ and}\\ V_{i,j} \in \mathcal{V}_{\mathrm{sc}}(M) \text{ s.t. } P = f + \sum_{i,1\le j\le m} V_{i,1}\dots V_{i,j}\Big\}.\end{aligned} \tag{6.12}$$

In local coordinates this is just a differential operator and it is smooth up to the boundary. Since only scattering vector fields are allowed in the definition such an operator is quite degenerate at the boundary. It always looks like

$$P = \sum_{k+|\alpha|\le m} a_{k,\alpha}(x,y)(x^2D_x)^k(xD_y)^\alpha, \tag{6.13}$$

with smooth coefficients in terms of local coordinates (**??**).

Now, if we *freeze* the coefficients at a point, p, on the boundary of M we get a polynomial

$$\sigma_{\mathrm{sc}}(P)(p) = \sum_{k+|\alpha|\le m} a_{k,\alpha}(p)\tau^k\eta^\alpha. \tag{6.14}$$

Note that this is *not* in general homogeneous since the lower order terms are retained. Despite this one gets essentially the same polynomial at each point, independent of the admissible coordinates chosen, as will be shown below. Let's just assume this for the moment so that the condition in the following result makes sense.

Theorem 6.7. *If $P \in \operatorname{Diff}^m_{sc}(M;\mathbb{E})$ acts between vector bundles over M, is elliptic in the interior and each of the polynomials (matrices) (6.14) is elliptic and has no real zeros then*

$$P : H^{s+m}_{sc}(M,E_1) \longrightarrow H^s_{sc}(M;E_2) \text{ is Fredholm} \tag{6.15}$$

for each $s \in \mathbb{R}$ and conversely.

Last time at the end I gave the following definition and theorem.

Definition 6.8. *We define weighted (non-standard) Sobolev spaces for $(m,w) \in \mathbb{R}^2$ on $\mathbb{R}^n$ by*

$$\tilde{H}^{m,w}(\mathbb{R}^n) = \{u \in M^m_{\mathrm{loc}}(\mathbb{R}^n); F^*\big((1-\chi)r^{-w}u\big) \in H^m_{\mathrm{ti}}(\mathbb{R}\times\mathbb{S}^{n-1})\} \tag{6.16}$$

where $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, $\chi(y) = 1$ in $|y| < 1$ and

$$F : \mathbb{R}\times\mathbb{S}^{n-1} \ni (t,\theta) \longrightarrow (e^t, e^t\theta) \in \mathbb{R}^n\setminus\{0\}. \tag{6.17}$$

THEOREM 6.9. *If* $P = \sum_{i=1}^{n} \Gamma_i D_i$, $\Gamma_i \in M(N, \mathbb{C})$, *is an elliptic, constant coefficient, homogeneous differential operator of first order then*

$$P : \tilde{H}^{m,w}(\mathbb{R}^n) \longrightarrow \tilde{H}^{m-1,w+1}(\mathbb{R}^n) \ \forall \ (m, w) \in \mathbb{R}^2 \tag{6.18}$$

is continuous and is Fredholm for $w \in \mathbb{R} \setminus \tilde{D}$ *where* $\tilde{D}$ *is discrete.*

If P *is a Dirac operators, which is to say explicitly here that the coefficients are 'Pauli matrices' in the sense that*

$$\Gamma_i^* = \Gamma_i, \ \Gamma_i^2 = \mathrm{Id}_{N\times N}, \ \forall \ i, \ \Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 0, \ i \neq j, \tag{6.19}$$

then

$$\tilde{D} = -\mathbb{N}_0 \cup (n - 2 + \mathbb{N}_0) \tag{6.20}$$

and if $n > 2$ *then for* $w \in (0, n-2)$ *the operator* P *in* (6.18) *is an isomorphism.*

I also proved the following result from which this is derived

LEMMA 6.10. *In polar coordinates on* $\mathbb{R}^n$ *in which* $\mathbb{R}^n \setminus \{0\} \simeq (0, \infty) \times \mathbb{S}^{n-1}$, $y = r\theta$,

$$D_{y_j} = \tag{6.21}$$

CHAPTER 9

Electromagnetism

1. Maxwell's equations

Maxwell's equations in a vacuum take the standard form

$$\begin{aligned} \operatorname{div} \mathbf{E} &= \rho & \operatorname{div} \mathbf{B} &= 0 \\ \operatorname{curl} \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \operatorname{curl} \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \end{aligned} \tag{1.1}$$

where $\mathbf{E}$ is the electric and $\mathbf{B}$ the magnetic field strength, both are 3-vectors depending on position $z \in \mathbb{R}^3$ and time $t \in \mathbb{R}$. The external quantities are ρ, the charge density which is a scalar, and $\mathbf{J}$, the current density which is a vector.

We will be interested here in stationary solutions for which $\mathbf{E}$ and $\mathbf{B}$ are independent of time and with $\mathbf{J} = 0$, since this also represents motion in the standard description. Thus we arrive at

$$\begin{aligned} \operatorname{div} \mathbf{E} &= \rho & \operatorname{div} \mathbf{B} &= 0 \\ \operatorname{curl} \mathbf{E} &= 0 & \operatorname{curl} \mathbf{B} &= 0. \end{aligned} \tag{1.2}$$

The simplest interesting solutions represent charged particles, say with the charge at the origin, $\rho = c\delta_0(z)$, and with no magnetic field, $\mathbf{B} = 0$. By identifying $\mathbf{E}$ with a 1-form, instead of a vector field on $\mathbb{R}^3$,

$$\mathbf{E} = (E_1, E_2, E_3) \Longrightarrow e = E_1 dz_1 + E_2 dz_2 + E_3 dz_3 \tag{1.3}$$

we may identify $\operatorname{curl} \mathbf{E}$ with the 2-form de,

$$de = \left(\frac{\partial E_2}{\partial z_1} - \frac{\partial E_1}{\partial z_2}\right) dz_1 \wedge dz_2 + \left(\frac{\partial E_3}{\partial z_2} - \frac{\partial E_2}{\partial z_3}\right) dz_2 \wedge dz_3 + \left(\frac{\partial E_1}{\partial z_3} - \frac{\partial E_3}{\partial z_1}\right) dz_3 \wedge dz_1. \tag{1.4}$$

Thus (1.2) implies that e is a closed 1-form, satisfying

$$\frac{\partial E_1}{\partial z_1} + \frac{\partial E_2}{\partial z_2} + \frac{\partial E_3}{\partial z_3} = c\delta_0(z). \tag{1.5}$$

By the Poincaré Lemma, a closed 1-form on $\mathbb{R}^3$ is exact, $e = dp$, with p determined up to an additive constant. If e is smooth (which it cannot be, because of (1.5)), then

(1.6)
$$p(z) - p(z') = \int_0^1 \gamma^* e \qquad \text{along } \gamma : [0,1] \longrightarrow \mathbb{R}^3, \gamma(0) = z', \gamma(1) = z.$$

It is reasonable to look for a particular p and 1-form e which satisfy (1.5) and are smooth outside the origin. Then (1.6) gives a potential which is well defined, up to an additive constant, outside 0, once z' is fixed, since $de = 0$ implies that the integral of $\gamma^* e$ along a closed curve vanishes. This depends on the fact that $\mathbb{R}^3 \backslash \{0\}$ is simply connected.

So, modulo confirmation of these simple statements, it suffices to look for $p \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash \{0\})$ satisfying $e = dp$ and (1.5), so

$$\Delta p = -\left(\frac{\partial^2 p}{\partial z_1^2} + \frac{\partial^2 p}{\partial z_2^2} + \frac{\partial^2 p}{\partial z_3^2} \right) = -c\delta_0(z). \tag{1.7}$$

Then $\mathbf{E}$ is recovered from $e = dp$.

The operator 'div' can also be understood in terms of de Rham d together with the Hodge star $*$. If we take $\mathbb{R}^3$ to have the standard orientation and Euclidean metric $dz_1^2 + dz_2^2 + dz_3^2$, the Hodge star operator is given on 1-forms by

$$*dz_1 = dz_2 \wedge dz_3, \quad *dz_2 = dz_3 \wedge dz_1, \quad *dz_3 = dz_1 \wedge dz_2. \tag{1.8}$$

Thus $*e$ is a 2-form,

$$\begin{aligned} &(1.9) \quad * e = E_1\, dz_2 \wedge dz_3 + E_2\, dz_3 \wedge dz_1 + E_3\, dz_1 \wedge dz_2 \\ &\Longrightarrow d{*}e = \left(\frac{\partial E_1}{\partial z_1} + \frac{\partial E_2}{\partial z_2} + \frac{\partial E_3}{\partial z_3} \right) dz_1 \wedge dz_2 \wedge dz_3 = (\operatorname{div} \mathbf{E})\, dz_1 \wedge dz_2 \wedge dz_3. \end{aligned}$$

The stationary Maxwell's equations on e become

$$d * e = \rho\, dz_1 \wedge dz_2 \wedge dz_3, \ de = 0. \tag{1.10}$$

There is essential symmetry in (1.1) except for the appearance of the "source" terms, ρ and $\mathbf{J}$. To reduce (1.1) to two equations, analogous to (1.10) but in 4-dimensional (Minkowski) space requires $\mathbf{B}$ to be identified with a 2-form on $\mathbb{R}^3$, rather than a 1-form. Thus, set

$$\beta = B_1\, dz_2 \wedge dz_3 + B_2\, dz_3 \wedge dz_1 + B_3\, dz_1 \wedge dz_2. \tag{1.11}$$

Then

$$d\beta = \operatorname{div} \mathbf{B}\, dz_1 \wedge dz_2 \wedge dz_3 \tag{1.12}$$

as follows from (1.9) and the second equation in (1.1) implies β is closed.

Thus e and β are respectively a closed 1-form and a closed 2-form on $\mathbb{R}^3$. If we return to the general time-dependent setting then we may define a 2-form on $\mathbb{R}^4$ by

$$\lambda = e \wedge dt + \beta \tag{1.13}$$

where e and β are pulled back by the projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$. Computing directly,

$$d\lambda = d'e \wedge dt + d'\beta + \frac{\partial \beta}{\partial t} \wedge dt \tag{1.14}$$

where d' is now the differential on $\mathbb{R}^3$. Thus

$$d\lambda = 0 \Leftrightarrow d'e + \frac{\partial \beta}{\partial t} = 0, \quad d'\beta = 0 \tag{1.15}$$

recovers two of Maxwell's equations. On the other hand we can define a 4-dimensional analogue of the Hodge star but corresponding to the Minkowski metric, not the Euclidean one. Using the natural analogue of the 3-dimensional Euclidean Hodge by formally inserting an i into the t-component, gives

$$\begin{cases} *_4 dz_1 \wedge dz_2 = i dz_3 \wedge dt \\ *_4 dz_1 \wedge dz_3 = i dt \wedge dz_2 \\ *_4 dz_1 \wedge dt = -i dz_2 \wedge dz_3 \\ *_4 dz_2 \wedge dz_3 = i dz_1 \wedge dt \\ *_4 dz_2 \wedge dt = -i dz_3 \wedge dz_1 \\ *_4 dz_3 \wedge dt = -i dz_1 \wedge dz_2. \end{cases} \tag{1.16}$$

The other two of Maxwell's equations then become

$$d *_4 \lambda = d(-i * e + i(*\beta) \wedge dt) = -i(\rho \, dz_1 \wedge dz_2 \wedge dz_3 + j \wedge dt) \tag{1.17}$$

where j is the 1-form associated to $\mathbf{J}$ as in (1.3). For our purposes this is really just to confirm that it is best to think of $\mathbf{B}$ as the 2-form β rather than try to make it into a 1-form. There are other good reasons for this, related to behaviour under linear coodinate changes.

Returning to the stationary setting, note that (1.7) has a 'preferred' solution

$$p = \frac{1}{4\pi|z|}. \tag{1.18}$$

This is in fact the only solution which vanishes at infinity.

Proposition 1.1. *The only tempered solutions of* (1.7) *are of the form*

$$p = \frac{1}{4\pi|z|} + q, \ \Delta q = 0, \ q \text{ a polynomial.} \tag{1.19}$$

PROOF. The only solutions are of the form (1.19) where $q \in \mathcal{S}'(\mathbb{R}^3)$ is harmonic. Thus $\widehat{q} \in \mathcal{S}'(\mathbb{R}^3)$ satisfies $|\xi|^2\widehat{q} = 0$, which implies that q is a polynomial. □

2. Hodge Theory

The Hodge $*$ operator discussed briefly above in the case of $\mathbb{R}^3$ (and Minkowski 4-space) makes sense in any oriented real vector space, V, with a Euclidean inner product—that is, on a finite dimensional real Hilbert space. Namely, if $e_1, \dots, e_n$ is an oriented orthonormal basis then

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \operatorname{sgn}(i_*) e_{i_{k+1}} \wedge \cdots e_{i_n} \tag{2.1}$$

extends by linearity to

$$* : \bigwedge^k V \longrightarrow \bigwedge^{n-k} V. \tag{2.2}$$

PROPOSITION 2.1. *The linear map* (2.2) *is independent of the oriented orthonormal basis used to define it and so depends only on the choice of inner product and orientation of V. Moreover,*

$$*^2 = (-1)^{k(n-k)}, \ \text{on} \bigwedge^k V. \tag{2.3}$$

PROOF. Note that $\operatorname{sgn}(i_*)$, the sign of the permutation defined by $\{i_1, \dots, i_n\}$ is fixed by

$$e_{i_1} \wedge \cdots \wedge e_{i_n} = \operatorname{sgn}(i_*) e_1 \wedge \cdots \wedge e_n. \tag{2.4}$$

Thus, on the basis $e_{i_1} \wedge \ldots \wedge e_{i_n}$ of $\bigwedge^k V$ given by strictly increasing sequences $i_1 < i_2 < \cdots < i_k$ in $\{1, \dots, n\}$,

$$e_* \wedge *e_* = \operatorname{sgn}(i_*)^2 e_1 \wedge \cdots \wedge e_n = e_1 \wedge \cdots \wedge e_n. \tag{2.5}$$

The standard inner product on $\bigwedge^k V$ is chosen so that this basis is orthonormal. Then (2.5) can be rewritten

$$e_I \wedge *e_J = \langle e_I, e_J \rangle e_1 \wedge \cdots \wedge e_n. \tag{2.6}$$

This in turn fixes $*$ uniquely since the pairing given by

$$\bigwedge^k V \times \bigwedge^{k-1} V \ni (u, v) \mapsto (u \wedge v)/_{e_1 \wedge \cdots \wedge e_n} \tag{2.7}$$

is non-degenerate, as can be checked on these bases.

Thus it follows from (2.6) that $*$ depends only on the choice of inner product and orientation as claimed, provided it is shown that the inner product on $\bigwedge^k V$ only depends on that of V. This is a standard fact following from the embedding

$$\bigwedge^k V \hookrightarrow V^{\otimes k} \tag{2.8}$$

as the totally antisymmetric part, the fact that $V^{\otimes k}$ has a natural inner product and the fact that this induces one on $\bigwedge^k V$ after normalization (depending on the convention used in (2.8). These details are omitted. □

Since $*$ is uniquely determined in this way, it necessarily depends smoothly on the data, in particular the inner product. On an oriented Riemannian manifold the induced inner product on T_p^*M varies smoothly with p (by assumption) so

$$*: \bigwedge_p^k M \longrightarrow \bigwedge_p^{n-k} M,\ \bigwedge_p^k M = \bigwedge_p^k (T_p^* M) \tag{2.9}$$

varies smoothly and so defines a smooth bundle map

$$* \in \mathcal{C}^\infty(M; \bigwedge^k M, \bigwedge^{n-k} M). \tag{2.10}$$

An oriented Riemannian manifold carries a natural volume form $\nu \in \mathcal{C}^\infty(M, \bigwedge^n M)$, and this allows (2.6) to be written in integral form:

$$\int_M \langle \alpha, \beta \rangle \, \nu = \int_M \alpha \wedge *\beta \ \forall\, \alpha, \beta \in \mathcal{C}^\infty(M, \bigwedge^k M). \tag{2.11}$$

Lemma 2.2. *On an oriented, (compact) Riemannian manifold the adjoint of d with respect to the Riemannian inner product and volume form is*

$$d^* = \delta = (-1)^{k+n(n-k+1)} * d * \ \text{ on } \bigwedge^k M. \tag{2.12}$$

Proof. By definition,

$$\begin{gathered} d: \mathcal{C}^\infty(M, \bigwedge^k M) \longrightarrow \mathcal{C}^\infty(M, \bigwedge^{k+1} M) \\ \Longrightarrow \delta: \mathcal{C}^\infty(M, \bigwedge^{k+1} M) \longrightarrow \mathcal{C}^\infty(M, \bigwedge^k M), \end{gathered} \tag{2.13}$$
$$\int_M \langle d\alpha, \alpha' \rangle \, \nu = \int_M \langle \alpha, \delta\alpha' \rangle \, \nu \ \forall\, \alpha \in \mathcal{C}^\infty(M, \bigwedge^k M), \alpha' \in \mathcal{C}^\infty(M, \bigwedge^{k+1} M).$$

Applying (2.11) and using Stokes' theorem, (and compactness of either M or the support of at least one of α, α'),

$$\int_M \langle \delta\alpha, \alpha' \rangle \, \nu = \int_M d\alpha \wedge *\alpha'$$
$$= \int_M d(\alpha \wedge *\alpha') + (-1)^{k+1} \int_M \alpha \wedge d*\alpha' = 0 + (-1)^{k+1} \int_M \langle \alpha, *^{-1} d*\alpha' \rangle \, \nu.$$

Taking into account (2.3) to compute $*^{-1}$ on $n-k$ forms shows that

$$\delta\alpha' = (-1)^{k+1+n(n-k)} * d * \quad \text{on } (k+1)\text{-forms} \tag{2.14}$$

which is just (2.12) on k-forms. □

Notice that changing the orientation simply changes the sign of $*$ on all forms. Thus (2.12) does not depend on the orientation and as a local formula is valid even if M is not orientable — since the existence of $\delta = d^*$ does *not* require M to be orientable.

THEOREM 2.3 (Hodge/Weyl). *On any compact Riemannian manifold there is a canonical isomorphism*

$$H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{Ho}}(M) = \left\{u \in L^2(M; {\textstyle\bigwedge}^k M);\ (d+\delta)u = 0\right\} \tag{2.15}$$

where the left-hand side is either the $\mathcal{C}^\infty$ or the distributional de Rham cohomology

$$\begin{aligned}\left\{u \in \mathcal{C}^\infty(M; {\textstyle\bigwedge}^k M);\ du = 0\right\} \Big/ d\,\mathcal{C}^\infty(M; {\textstyle\bigwedge}^k M) \\ \cong \left\{u \in \mathcal{C}^{-\infty}(M; {\textstyle\bigwedge}^k M);\ du = 0\right\} \Big/ d\,\mathcal{C}^{-\infty}(M; {\textstyle\bigwedge}^k M).\end{aligned} \tag{2.16}$$

PROOF. The critical point of course is that

$$d + \delta \in \mathrm{Diff}^1(M; {\textstyle\bigwedge}^* M) \text{ is elliptic.} \tag{2.17}$$

We know that the symbol of d at a point $\zeta \in T^*_p M$ is the map

$${\textstyle\bigwedge}^k M \ni \alpha \mapsto i\zeta \wedge \alpha. \tag{2.18}$$

We are only interested in $\zeta \neq 0$ and by homogeneity it is enough to consider $|\zeta| = 1$. Let $e_1 = \zeta$, $e_2, \ldots, e_n$ be an orthonormal basis of $T^*_p M$, then from (2.12) with a fixed sign throughout:

$$\sigma(\delta, \zeta)\alpha = \pm * (i\zeta \wedge \cdot) * \alpha. \tag{2.19}$$

Take $\alpha = e_I$, $*\alpha = \pm e_{I'}$ where $I \cup I' = \{1, \ldots, n\}$. Thus

$$\sigma(\delta, \zeta)\alpha = \left\{ \begin{array}{ll} 0 & 1 \notin I \\ \pm i\alpha_{I\setminus\{1\}} & 1 \in I \end{array} \right. . \tag{2.20}$$

In particular, $\sigma(d + \delta)$ is an isomorphism since it satisfies

$$\sigma(d + \delta)^2 = |\zeta|^2 \tag{2.21}$$

as follows from (2.18) and (2.20) or directly from the fact that

$$(d + \delta)^2 = d^2 + d\delta + \delta d + \delta^2 = d\delta + \delta d \tag{2.22}$$

again using (2.18) and (2.20).

Once we know that $d + \delta$ is elliptic we conclude from the discussion of Fredholm properties above that the distributional null space

$$\left\{u \in \mathcal{C}^{-\infty}(M, {\textstyle\bigwedge}^* M);\ (d + \delta)u = 0\right\} \subset \mathcal{C}^\infty(M, {\textstyle\bigwedge}^* M) \tag{2.23}$$

is finite dimensional. From this it follows that

$$
\begin{aligned}
H^k_{\mathrm{Ho}} =& \{u \in \mathcal{C}^{-\infty}(M, \textstyle\bigwedge^k M);\ (d+\delta)u = 0\} \\
=& \{u \in \mathcal{C}^{\infty}(M, \textstyle\bigwedge^k M);\ du = \delta u = 0\}
\end{aligned}
\tag{2.24}
$$

and that the null space in (2.23) is simply the direct sum of these spaces over k. Indeed, from (2.23) the integration by parts in

$$
0 = \int \langle du, (d+\delta)u\rangle\,\nu = \|du\|^2_{L^2} + \int \langle u, \delta^2 u\rangle\,\nu = \|du\|^2_{L^2}
$$

is justified.

Thus we can consider $d+\delta$ as a Fredholm operator in three forms

$$
\begin{aligned}
d+\delta :& \mathcal{C}^{-\infty}(M, \textstyle\bigwedge^* M) \longrightarrow \mathcal{C}^{-\infty}(M, \textstyle\bigwedge^* M), \\
d+\delta :& H^1(M, \textstyle\bigwedge^* M) \longrightarrow H^1(M, \textstyle\bigwedge^* M), \\
d+\delta :& \mathcal{C}^{\infty}(M, \textstyle\bigwedge^* M) \longrightarrow \mathcal{C}^{\infty}(M, \textstyle\bigwedge^* M)
\end{aligned}
\tag{2.25}
$$

and obtain the three direct sum decompositions

$$
\begin{aligned}
\mathcal{C}^{-\infty}(M, \textstyle\bigwedge^* M) &= H^*_{\mathrm{Ho}} \oplus (d+\delta)\mathcal{C}^{-\infty}(M, \textstyle\bigwedge^* M), \\
L^2(M, \textstyle\bigwedge^* M) &= H^*_{\mathrm{Ho}} \oplus (d+\delta)L^2(M, \textstyle\bigwedge^* M), \\
\mathcal{C}^{\infty}(M, \textstyle\bigwedge^* M) &= H^*_{\mathrm{Ho}} \oplus (d+\delta)\mathcal{C}^{\infty}(M, \textstyle\bigwedge^* M).
\end{aligned}
\tag{2.26}
$$

The same complement occurs in all three cases in view of (2.24).

From (2.24) directly, all the "harmonic" forms in $H^k_{\mathrm{Ho}}(M)$ are closed and so there is a natural map

$$
H^k_{\mathrm{Ho}}(M) \longrightarrow H^k_{\mathrm{dR}}(M) \longrightarrow H^k_{\mathrm{dR},\mathcal{C}^{-\infty}}(M)
\tag{2.27}
$$

where the two de Rham spaces are those in (2.16), not yet shown to be equal.

We proceed to show that the maps in (2.27) are isomorphisms. First to show injectivity, suppose $u \in H^k_{\mathrm{Ho}}(M)$ is mapped to zero in either space. This means $u = dv$ where v is either $\mathcal{C}^\infty$ or distributional, so it suffices to suppose $v \in \mathcal{C}^{-\infty}(M, \bigwedge^{k-1} M)$. Since u is smooth the integration by parts in the distributional pairing

$$
\|u\|^2_{L^2} = \int_M \langle u, dv\rangle\,\nu = \int_M \langle \delta u, v\rangle\,\nu = 0
$$

is justified, so $u = 0$ and the maps are injective.

To see surjectivity, use the Hodge decomposition (2.26). If $u' \in \mathcal{C}^{-\infty}(M, \bigwedge^k M)$ or $\mathcal{C}^{\infty}(M, \bigwedge^k M)$, we find

$$
u' = u_0 + (d+\delta)v
\tag{2.28}
$$

where correspondingly, $v \in \mathcal{C}^{-\infty}(M, \bigwedge^* M)$ or $\mathcal{C}^{\infty}(M, \bigwedge^* M)$ and $u_0 \in H^k_{\mathrm{Ho}}(M)$. If u' is closed, $du' = 0$, then $d\delta v = 0$ follows from applying

d to (2.28) and hence $(d+\delta)\delta v = 0$, since $\delta^2 = 0$. Thus $\delta v \in H^*_{\mathrm{Ho}}(M)$ and in particular, $\delta v \in \mathcal{C}^\infty(M, \bigwedge^* M)$. Then the integration by parts in

$$\|\delta v\|^2_{L^2} = \int \langle \delta v, \delta v\rangle\, \nu = \int \langle v, (d+\delta)\delta v\rangle\, \nu = 0$$

is justified, so $\delta v = 0$. Then (2.28) shows that any closed form, smooth or distributional, is cohomologous in the same sense to $u_0 \in H^k_{\mathrm{Ho}}(M)$. Thus the natural maps (2.27) are isomorphisms and the Theorem is proved. □

Thus, on a compact Riemannian manifold (whether orientable or not), each de Rham class has a unique harmonic representative.

3. Coulomb potential

4. Dirac strings

Addenda to Chapter 9

CHAPTER 10

Monopoles

1. Gauge theory

2. Bogomolny equations

(1) Compact operators, spectral theorem
(2) Families of Fredholm operators(*)
(3) Non-compact self-adjoint operators, spectral theorem
(4) Spectral theory of the Laplacian on a compact manifold
(5) Pseudodifferential operators(*)
(6) Invertibility of the Laplacian on Euclidean space
(7) Lie groups(‡), bundles and gauge invariance
(8) Bogomolny equations on $\mathbb{R}^3$
(9) Gauge fixing
(10) Charge and monopoles
(11) Monopole moduli spaces

* I will drop these if it looks as though time will become an issue.
†,‡ I will provide a brief and elementary discussion of manifolds and Lie groups if that is found to be necessary.

3. Problems

Problem 1. Prove that u_+, defined by (15.10) is linear.

Problem 2. Prove Lemma 15.7.

Hint(s). All functions here are supposed to be continuous, I just don't bother to keep on saying it.

(1) Recall, or check, that the local compactness of a metric space X means that for each point $x \in X$ there is an $\epsilon > 0$ such that the ball $\{y \in X; d(x, y) \le \delta\}$ is compact for $\delta \le \epsilon$.
(2) First do the case $n = 1$, so $K \Subset U$ is a compact set in an open subset.
   (a) Given $\delta > 0$, use the local compactness of X, to cover K with a finite number of compact closed balls of radius at most δ.

(b) Deduce that if $\epsilon > 0$ is small enough then the set $\{x \in X; d(x, K) \le \epsilon\}$, where

$$d(x, K) = \inf_{y \in K} d(x, y),$$

is compact.

(c) Show that $d(x, K)$, for K compact, is continuous.

(d) Given $\epsilon > 0$ show that there is a continuous function $g_\epsilon : \mathbb{R} \longrightarrow [0, 1]$ such that $g_\epsilon(t) = 1$ for $t \le \epsilon/2$ and $g_\epsilon(t) = 0$ for $t > 3\epsilon/4$.

(e) Show that $f = g_\epsilon \circ d(\cdot, K)$ satisfies the conditions for $n = 1$ if $\epsilon > 0$ is small enough.

(3) Prove the general case by induction over n.

(a) In the general case, set $K' = K \cap U_1^{\complement}$ and show that the inductive hypothesis applies to K' and the U_j for $j > 1$; let f'_j, $j = 2, \dots, n$ be the functions supplied by the inductive assumption and put $f' = \sum_{j \ge 2} f'_j$.

(b) Show that $K_1 = K \cap \{f' \le \frac{1}{2}\}$ is a compact subset of U_1.

(c) Using the case $n = 1$ construct a function F for K_1 and U_1.

(d) Use the case $n = 1$ again to find G such that $G = 1$ on K and $\operatorname{supp}(G) \Subset \{f' + F > \frac{1}{2}\}$.

(e) Make sense of the functions

$$f_1 = F \frac{G}{f' + F},\ f_j = f'_j \frac{G}{f' + F},\ j \ge 2$$

and show that they satisfies the inductive assumptions.

Problem 3. Show that σ-algebras are closed under countable intersections.

Problem 4. (Easy) Show that if μ is a complete measure and $E \subset F$ where F is measurable and has measure 0 then $\mu(E) = 0$.

Problem 5. Show that compact subsets are measurable for any Borel measure. (This just means that compact sets are Borel sets if you follow through the tortuous terminology.)

Problem 6. Show that the smallest σ-algebra containing the sets

$$(a, \infty] \subset [-\infty, \infty]$$

for all $a \in \mathbb{R}$, generates what is called above the 'Borel' σ-algebra on $[-\infty, \infty]$.

Problem 7. Write down a careful proof of Proposition 1.1.

Problem 8. Write down a careful proof of Proposition 1.2.

Problem 9. Let X be the metric space

$$X = \{0\} \cup \{1/n; n \in \mathbb{N} = \{1, 2, \dots\}\} \subset \mathbb{R}$$

with the induced metric (i.e. the same distance as on $\mathbb{R}$). Recall why X is compact. Show that the space $\mathcal{C}_0(X)$ and its dual are infinite dimensional. Try to describe the dual space in terms of sequences; at least *guess* the answer.

Problem 10. For the space $Y = \mathbb{N} = \{1, 2, \dots\} \subset \mathbb{R}$, describe $\mathcal{C}_0(Y)$ and guess a description of its dual in terms of sequences.

Problem 11. Let $(X, \mathcal{M}, \mu)$ be any measure space (so μ is a measure on the σ-algebra $\mathcal{M}$ of subsets of X). Show that the set of equivalence classes of μ-integrable functions on X, with the equivalence relation given by (4.8), is a normed linear space with the usual linear structure and the norm given by

$$\|f\| = \int_X |f| d\mu.$$

Problem 12. Let $(X, \mathcal{M})$ be a set with a σ-algebra. Let $\mu : \mathcal{M} \to \mathbb{R}$ be a finite measure in the sense that $\mu(\phi) = 0$ and for any $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$ with $E_i \cap E_j = \phi$ for $i \neq j$,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \tag{3.1}$$

with the series on the right *always* absolutely convergenct (i.e., this is part of the requirement on μ). Define

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)| \tag{3.2}$$

for $E \in \mathcal{M}$, with the supremum over *all* measurable decompositions $E = \bigcup_{i=1}^{\infty} E_i$ with the E_i disjoint. Show that $|\mu|$ is a finite, positive measure.

Hint 1. You must show that $|\mu|(E) = \sum_{i=1}^{\infty} |\mu|(A_i)$ if $\bigcup_i A_i = E$, $A_i \in \mathcal{M}$ being disjoint. Observe that if $A_j = \bigcup_l A_{jl}$ is a measurable decomposition of A_j then together the A_{jl} give a decomposition of E. Similarly, if $E = \bigcup_j E_j$ is any such decomposition of E then $A_{jl} = A_j \cap E_l$ gives such a decomposition of A_j.

Hint 2. See [**6**] p. 117!

Problem 13. (Hahn Decomposition)
With assumptions as in Problem 12:

(1) Show that $\mu_+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu_- = \frac{1}{2}(|\mu| - \mu)$ are positive measures, $\mu = \mu_+ - \mu_-$. Conclude that the definition of a measure based on (4.16) is the *same* as that in Problem 12.
(2) Show that $\mu_\pm$ so constructed are orthogonal in the sense that there is a set $E \in \mathcal{M}$ such that $\mu_-(E) = 0$, $\mu_+(X \setminus E) = 0$.

Hint. Use the definition of $|\mu|$ to show that for any $F \in \mathcal{M}$ and any $\epsilon > 0$ there is a subset $F' \in \mathcal{M}$, $F' \subset F$ such that $\mu_+(F') \geq \mu_+(F) - \epsilon$ and $\mu_-(F') \leq \epsilon$. Given $\delta > 0$ apply this result repeatedly (say with $\epsilon = 2^{-n}\delta$) to find a decreasing sequence of sets $F_1 = X$, $F_n \in \mathcal{M}$, $F_{n+1} \subset F_n$ such that $\mu_+(F_n) \geq \mu_+(F_{n-1}) - 2^{-n}\delta$ and $\mu_-(F_n) \leq 2^{-n}\delta$. Conclude that $G = \bigcap_n F_n$ has $\mu_+(G) \geq \mu_+(X) - \delta$ and $\mu_-(G) = 0$. Now let G_m be chosen this way with $\delta = 1/m$. Show that $E = \bigcup_m G_m$ is as required.

Problem 14. Now suppose that μ is a finite, positive Radon measure on a locally compact metric space X (meaning a finite positive Borel measure outer regular on Borel sets and inner regular on open sets). Show that μ is inner regular on all Borel sets and hence, given $\epsilon > 0$ and $E \in \mathcal{B}(X)$ there exist sets $K \subset E \subset U$ with K compact and U open such that $\mu(K) \geq \mu(E) - \epsilon$, $\mu(E) \geq \mu(U) - \epsilon$.

Hint. First take U open, then use *its* inner regularity to find K with $K' \Subset U$ and $\mu(K') \geq \mu(U) - \epsilon/2$. How big is $\mu(E \backslash K')$? Find $V \supset K' \backslash E$ with V open and look at $K = K' \backslash V$.

Problem 15. Using Problem 14 show that if μ is a finite Borel measure on a locally compact metric space X then the following three conditions are equivalent

(1) $\mu = \mu_1 - \mu_2$ with μ_1 and μ_2 both positive finite Radon measures.
(2) $|\mu|$ is a finite positive Radon measure.
(3) μ_+ and μ_- are finite positive Radon measures.

Problem 16. Let $\| \ \|$ be a norm on a vector space V. Show that $\|u\| = (u, u)^{1/2}$ for an inner product satisfying (1.1) - (1.4) if and only if the parallelogram law holds for every pair $u, v \in V$.

Hint (From Dimitri Kountourogiannis)

If $\| \cdot \|$ comes from an inner product, then it must satisfy the polarisation identity:

$$(x, y) = 1/4(\|x + y\|^2 - \|x - y\|^2 - i\|x + iy\|^2 - i\|x - iy\|^2)$$

i.e, the inner product is recoverable from the norm, so use the RHS (right hand side) to define an inner product on the vector space. You will need the paralellogram law to verify the additivity of the RHS. Note the polarization identity is a bit more transparent for real vector spaces. There we have

$$(x, y) = 1/2(\|x + y\|^2 - \|x - y\|^2)$$

both are easy to prove using $\|a\|^2 = (a, a)$.

PROBLEM 17. Show (Rudin does it) that if $u : \mathbb{R}^n \to \mathbb{C}$ has continuous partial derivatives then it is differentiable at each point in the sense of (6.19).

PROBLEM 18. Consider the function $f(x) = \langle x \rangle^{-1} = (1 + |x|^2)^{-1/2}$. Show that

$$\frac{\partial f}{\partial x_j} = l_j(x) \cdot \langle x \rangle^{-3}$$

with $l_j(x)$ a linear function. Conclude by *induction* that $\langle x \rangle^{-1} \in \mathcal{C}_0^k(\mathbb{R}^n)$ for *all* k.

PROBLEM 19. Show that $\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$.

PROBLEM 20. Prove (2.8), probably by induction over k.

PROBLEM 21. Prove Lemma 2.4.

Hint. Show that a set $U \ni 0$ in $\mathcal{S}(\mathbb{R}^n)$ is a neighbourhood of 0 if and only if for some k and $\epsilon > 0$ it contains a set of the form

$$\left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) \, ; \sum_{\substack{|\alpha| \le k, \\ |\beta| \le k}} \sup \left| x^\alpha D^\beta \varphi \right| < \epsilon \right\} .$$

PROBLEM 22. Prove (3.7), by estimating the integrals.

PROBLEM 23. Prove (3.9) where

$$\psi_j(z; x') = \int_0^{\prime} \frac{\partial \psi}{\partial z_j}(z + tx') \, dt \, .$$

PROBLEM 24. Prove (3.20). You will probably have to go back to first principles to do this. Show that it is enough to assume $u \ge 0$ has compact support. Then show it is enough to assume that u is a simple, and integrable, function. Finally look at the definition of Lebesgue measure and show that if $E \subset \mathbb{R}^n$ is Borel and has finite Lebesgue measure then

$$\lim_{|t| \to \infty} \mu(E \backslash (E + t)) = 0$$

where $\mu =$ Lebesgue measure and

$$E + t = \{ p \in \mathbb{R}^n \, ; \, p' + t \, , \, p' \in E \} \, .$$

PROBLEM 25. Prove Leibniz' formula

$$D^\alpha{}_x(\varphi\psi) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\alpha{}_x\varphi \cdot d_x^{\alpha-\beta}\psi$$

for any $\mathcal{C}^\infty$ functions and φ and ψ. Here α and β are multiindices, $\beta \le \alpha$ means $\beta_j \le \alpha_j$ for each j? and

$$\binom{\alpha}{\beta} = \prod_j \binom{\alpha_j}{\beta_j}.$$

I suggest induction!

PROBLEM 26. Prove the generalization of Proposition 3.10 that $u \in \mathcal{S}'(\mathbb{R}^n)$, $\operatorname{supp}(w) \subset \{0\}$ implies there are constants $c\alpha$, $|\alpha| \le m$, for some m, such that

$$u = \sum_{|\alpha|\le m} c_\alpha D^\alpha \delta\,.$$

Hint This is not so easy! I would be happy if you can show that $u \in M(\mathbb{R}^n)$, $\operatorname{supp} u \subset \{0\}$ implies $u = c\delta$. To see this, you can show that

$$\begin{aligned}&\varphi \in \mathcal{S}(\mathbb{R}^n),\ \varphi(0) = 0\\ &\qquad\Rightarrow \exists\varphi_j \in \mathcal{S}(\mathbb{R}^n)\,,\ \varphi_j(x) = 0 \text{ in } |x| \le \epsilon_j > 0(\downarrow 0)\,,\\ &\qquad\qquad\qquad \sup|\varphi_j - \varphi| \to 0 \text{ as } j \to \infty\,.\end{aligned}$$

To prove the general case you need something similar — that given m, if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $D^\alpha{}_x\varphi(0) = 0$ for $|\alpha| \le m$ then $\exists\,\varphi_j \in \mathcal{S}(\mathbb{R}^n)$, $\varphi_j = 0$ in $|x| \le \epsilon_j$, $\epsilon_j \downarrow 0$ such that $\varphi_j \to \varphi$ in the $\mathcal{C}^m$ norm.

PROBLEM 27. If $m \in \mathbb{N}$, $m' > 0$ show that $u \in H^m(\mathbb{R}^n)$ and $D^\alpha u \in H^{m'}(\mathbb{R}^n)$ for all $|\alpha| \le m$ implies $u \in H^{m+m'}(\mathbb{R}^n)$. Is the converse true?

PROBLEM 28. Show that every element $u \in L^2(\mathbb{R}^n)$ can be written as a sum

$$u = u_0 + \sum_{j=1}^n D_j u_j\,,\ u_j \in H^1(\mathbb{R}^n)\,,\ j = 0, \dots, n\,.$$

PROBLEM 29. Consider for $n = 1$, the locally integrable function (the Heaviside function),

$$H(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 1\,. \end{cases}$$

Show that $D_x H(x) = c\delta$; what is the constant c?

PROBLEM 30. For what range of orders m is it true that $\delta \in H^m(\mathbb{R}^n)$, $\delta(\varphi) = \varphi(0)$?

PROBLEM 31. Try to write the Dirac measure explicitly (as possible) in the form (5.8). How many derivatives do you think are necessary?

PROBLEM 32. Go through the computation of $\overline{\partial} E$ again, but cutting out a disk $\{x^2 + y^2 \le \epsilon^2\}$ instead.

PROBLEM 33. Consider the Laplacian, (6.4), for $n = 3$. Show that $E = c(x^2 + y^2)^{-1/2}$ is a fundamental solution for some value of c.

PROBLEM 34. Recall that a topology on a set X is a collection $\mathcal{F}$ of subsets (called the *open* sets) with the properties, $\phi \in \mathcal{F}$, $X \in \mathcal{F}$ and $\mathcal{F}$ is closed under finite intersections and arbitrary unions. Show that the following definition of an open set $U \subset \mathcal{S}'(\mathbb{R}^n)$ defines a topology:

$$\begin{aligned} &\forall\ u \in U \text{ and all } \varphi \in \mathcal{S}(\mathbb{R}^n)\ \exists \epsilon > 0 \text{ st.} \\ &\qquad |(u' - u)(\varphi)| < \epsilon \Rightarrow u' \in U\,. \end{aligned}$$

This is called the weak topology (because there are very few open sets). Show that $u_j \to u$ weakly in $\mathcal{S}'(\mathbb{R}^n)$ means that for every open set $U \ni u$ $\exists N$ st. $u_j \in U$ $\forall$ $j \ge N$.

PROBLEM 35. Prove (6.18) where $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$.

PROBLEM 36. Show that for fixed $v \in \mathcal{S}'(\mathbb{R}^n)$ with compact support

$$\mathcal{S}(\mathbb{R}^n) \ni \varphi \mapsto v * \varphi \in \mathcal{S}(\mathbb{R}^n)$$

is a continuous linear map.

PROBLEM 37. Prove the ?? to properties in Theorem 6.6 for $u * v$ where $u \in \mathcal{S}'(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^n)$ with at least one of them having compact support.

PROBLEM 38. Use Theorem 6.9 to show that if $P(D)$ is hypoelliptic then *every* parametrix $F \in \mathcal{S}(\mathbb{R}^n)$ has $\operatorname{sing\,supp}(F) = \{0\}$.

PROBLEM 39. Show that if $P(D)$ is an ellipitic differential operator of order m, $u \in L^2(\mathbb{R}^n)$ and $P(D)u \in L^2(\mathbb{R}^n)$ then $u \in H^m(\mathbb{R}^n)$.

PROBLEM 40 (Taylor's theorem). . Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued function which is k times continuously differentiable. Prove that there is a polynomial p and a continuous function v such that

$$u(x) = p(x) + v(x) \text{ where } \lim_{|x| \downarrow 0} \frac{|v(x)|}{|x|^k} = 0.$$

Problem 41. Let $\mathcal{C}(\mathbb{B}^n)$ be the space of continuous functions on the (closed) unit ball, $\mathbb{B}^n = \{x \in \mathbb{R}^n; |x| \le 1\}$. Let $\mathcal{C}_0(\mathbb{B}^n) \subset \mathcal{C}(\mathbb{B}^n)$ be the subspace of functions which vanish at each point of the boundary and let $\mathcal{C}(\mathbb{S}^{n-1})$ be the space of continuous functions on the unit sphere. Show that inclusion and restriction to the boundary gives a short exact sequence

$$\mathcal{C}_0(\mathbb{B}^n) \hookrightarrow \mathcal{C}(\mathbb{B}^n) \longrightarrow \mathcal{C}(\mathbb{S}^{n-1})$$

(meaning the first map is injective, the second is surjective and the image of the first is the null space of the second.)

Problem 42 (Measures). A measure on the ball is a continuous linear functional $\mu : \mathcal{C}(\mathbb{B}^n) \longrightarrow \mathbb{R}$ where continuity is with respect to the supremum norm, i.e. there must be a constant C such that

$$|\mu(f)| \le C \sup_{x \in \mathbb{R}^n} |f(x)| \; \forall \; f \in \mathcal{C}(\mathbb{B}^n).$$

Let $M(\mathbb{B}^n)$ be the linear space of such measures. The space $M(\mathbb{S}^{n-1})$ of measures on the sphere is defined similarly. Describe an injective map

$$M(\mathbb{S}^{n-1}) \longrightarrow M(\mathbb{B}^n).$$

Can you define another space so that this can be extended to a short exact sequence?

Problem 43. Show that the Riemann integral defines a measure

$$\mathcal{C}(\mathbb{B}^n) \ni f \longmapsto \int_{\mathbb{B}^n} f(x) dx. \tag{3.3}$$

Problem 44. If $g \in \mathcal{C}(\mathbb{B}^n)$ and $\mu \in M(\mathbb{B}^n)$ show that $g\mu \in M(\mathbb{B}^n)$ where $(g\mu)(f) = \mu(fg)$ for all $f \in \mathcal{C}(\mathbb{B}^n)$. Describe all the measures with the property that

$$x_j \mu = 0 \text{ in } M(\mathbb{B}^n) \text{ for } j = 1, \ldots, n.$$

Problem 45 (Hörmander, Theorem 3.1.4). Let $I \subset \mathbb{R}$ be an open, non-empty interval.

i) Show (you may use results from class) that there exists $\psi \in \mathcal{C}_c^\infty(I)$ with $\int_{\mathbb{R}} \psi(x) ds = 1$.
ii) Show that any $\phi \in \mathcal{C}_c^\infty(I)$ may be written in the form

$$\phi = \tilde{\phi} + c\psi, \; c \in \mathbb{C}, \; \tilde{\phi} \in \mathcal{C}_c^\infty(I) \text{ with } \int_{\mathbb{R}} \tilde{\phi} = 0.$$

iii) Show that if $\tilde{\phi} \in \mathcal{C}_c^\infty(I)$ and $\int_{\mathbb{R}} \tilde{\phi} = 0$ then there exists $\mu \in \mathcal{C}_c^\infty(I)$ such that $\frac{d\mu}{dx} = \tilde{\phi}$ in I.

iv) Suppose $u \in \mathcal{C}^{-\infty}(I)$ satisfies $\frac{du}{dx} = 0$, i.e.

$$u(-\frac{d\phi}{dx}) = 0 \ \forall \ \phi \in \mathcal{C}_c^\infty(I),$$

show that $u = c$ for some constant c.

v) Suppose that $u \in \mathcal{C}^{-\infty}(I)$ satisfies $\frac{du}{dx} = c$, for some constant c, show that $u = cx + d$ for some $d \in \mathbb{C}$.

Problem 46. [Hörmander Theorem 3.1.16]

i) Use Taylor's formula to show that there is a fixed $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ can be written in the form

$$\phi = c\psi + \sum_{j=1}^{n} x_j \psi_j$$

where $c \in \mathbb{C}$ and the $\psi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ depend on ϕ.

ii) Recall that δ_0 is the distribution defined by

$$\delta_0(\phi) = \phi(0) \ \forall \ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n);$$

explain why $\delta_0 \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$.

iii) Show that if $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and $u(x_j\phi) = 0$ for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $j = 1, \dots, n$ then $u = c\delta_0$ for some $c \in \mathbb{C}$.

iv) Define the 'Heaviside function'

$$H(\phi) = \int_0^\infty \phi(x)dx \ \forall \ \phi \in \mathcal{C}_c^\infty(\mathbb{R});$$

show that $H \in \mathcal{C}^{-\infty}(\mathbb{R})$.

v) Compute $\frac{d}{dx}H \in \mathcal{C}^{-\infty}(\mathbb{R})$.

Problem 47. Using Problems 45 and 46, find all $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ satisfying the differential equation

$$x\frac{du}{dx} = 0 \text{ in } \mathbb{R}.$$

These three problems are all about homogeneous distributions on the line, extending various things using the fact that

$$x_+^z = \begin{cases} \exp(z \log x) & x > 0 \\ 0 & x \le 0 \end{cases}$$

is a continuous function on $\mathbb{R}$ if $\operatorname{Re} z > 0$ and is differentiable if $\operatorname{Re} z > 1$ and then satisfies

$$\frac{d}{dx}x_+^z = zx_+^{z-1}.$$

We used this to define

$$x_+^z = \frac{1}{z+k}\frac{1}{z+k-1}\cdots\frac{1}{z+1}\frac{d^k}{dx^k}x_+^{z+k} \text{ if } z\in\mathbb{C}\setminus-\mathbb{N}. \tag{3.4}$$

PROBLEM 48. [Hadamard regularization]

i) Show that (3.4) just means that for each $\phi\in\mathcal{C}_c^\infty(\mathbb{R})$

$$x_+^z(\phi) = \frac{(-1)^k}{(z+k)\cdots(z+1)}\int_0^\infty \frac{d^k\phi}{dx^k}(x)x^{z+k}dx,\ \operatorname{Re} z > -k,\ z\notin-\mathbb{N}.$$

ii) Use integration by parts to show that

$$x_+^z(\phi) = \lim_{\epsilon\downarrow 0}\left[\int_\epsilon^\infty \phi(x)x^z dx - \sum_{j=1}^k C_j(\phi)\epsilon^{z+j}\right],\ \operatorname{Re} z > -k,\ z\notin-\mathbb{N} \tag{3.5}$$

for certain constants $C_j(\phi)$ which you should give explicitly. [This is called Hadamard regularization after Jacques Hadamard, feel free to look at his classic book [**3**].]

iii) Assuming that $-k+1\ge \operatorname{Re} z > -k$, $z\neq -k+1$, show that there can only be one set of the constants with $j<k$ (for each choice of $\phi\in\mathcal{C}_c^\infty(\mathbb{R})$) such that the limit in (3.5) exists.

iiv) Use ii), and maybe iii), to show that

$$\frac{d}{dx}x_+^z = zx_+^{z-1} \text{ in } \mathcal{C}^{-\infty}(\mathbb{R})\ \forall\ z\notin -\mathbb{N}_0 = \{0,1,\dots\}.$$

v) Similarly show that $xx_+^z = x_+^{z+1}$ for all $z\notin-\mathbb{N}$.

vi) Show that $x_+^z = 0$ in $x<0$ for all $z\notin-\mathbb{N}$. (Duh.)

PROBLEM 49. [Null space of $x\frac{d}{dx}-z$]

i) Show that if $u\in\mathcal{C}^{-\infty}(\mathbb{R})$ then $\tilde u(\phi) = u(\tilde\phi)$, where $\tilde\phi(x) = \phi(-x)\ \forall\ \phi\in\mathcal{C}_c^\infty(\mathbb{R})$, defines an element of $\mathcal{C}^{-\infty}(\mathbb{R})$. What is $\tilde u$ if $u\in\mathcal{C}^0(\mathbb{R})$? Compute $\widetilde{\delta_0}$.

ii) Show that $\frac{d}{dx}\tilde u = -\widetilde{\frac{d}{dx}u}$.

iii) Define $x_-^z = \widetilde{x_+^z}$ for $z\notin-\mathbb{N}$ and show that $\frac{d}{dx}x_-^z = -zx_-^{z-1}$ and $xx_-^z = -x_-^{z+1}$.

iv) Suppose that $u\in\mathcal{C}^{-\infty}(\mathbb{R})$ satisfies the distributional equation $(x\frac{d}{dx}-z)u = 0$ (meaning of course, $x\frac{du}{dx} = zu$ where z is a constant). Show that

$$u\big|_{x>0} = c_+x_-^z\big|_{x>0} \text{ and } u\big|_{x<0} = c_-x_-^z\big|_{x<0}$$

for some constants $c_\pm$. Deduce that $v = u - c_+x_+^z - c_-x_-^z$ satisfies

$$(x\frac{d}{dx} - z)v = 0 \text{ and } \operatorname{supp}(v) \subset \{0\}. \tag{3.6}$$

v) Show that for each $k \in \mathbb{N}$, $(x\frac{d}{dx} + k + 1)\frac{d^k}{dx^k}\delta_0 = 0$.

vi) Using the *fact* that any $v \in \mathcal{C}^{-\infty}(\mathbb{R})$ with $\operatorname{supp}(v) \subset \{0\}$ is a finite sum of constant multiples of the $\frac{d^k}{dx^k}\delta_0$, show that, for $z \notin -\mathbb{N}$, the only solution of (3.6) is $v = 0$.

vii) Conclude that for $z \notin -\mathbb{N}$

$$\left\{u \in \mathcal{C}^{-\infty}(\mathbb{R}); (x\frac{d}{dx} - z)u = 0\right\} \tag{3.7}$$

is a two-dimensional vector space.

Problem 50. [Negative integral order] To do the same thing for negative integral order we need to work a little differently. Fix $k \in \mathbb{N}$.

i) We define *weak convergence* of distributions by saying $u_n \to u$ in $\mathcal{C}_c^\infty(X)$, where $u_n, u \in \mathcal{C}^{-\infty}(X)$, $X \subset \mathbb{R}^n$ being open, if $u_n(\phi) \to u(\phi)$ for each $\phi \in \mathcal{C}_c^\infty(X)$. Show that $u_n \to u$ implies that $\frac{\partial u_n}{\partial x_j} \to \frac{\partial u}{\partial x_j}$ for each $j = 1, \dots, n$ and $fu_n \to fu$ if $f \in \mathcal{C}^\infty(X)$.

ii) Show that $(z+k)x_+^z$ is weakly continuous as $z \to -k$ in the sense that for any sequence $z_n \to -k$, $z_n \notin -\mathbb{N}$, $(z_n+k)x_+^{z_n} \to v_k$ where

$$v_k = \frac{1}{-1}\cdots\frac{1}{-k+1}\frac{d^{k+1}}{dx^{k+1}}x_+,\ x_+ = x_+^1.$$

iii) Compute v_k, including the constant factor.

iv) Do the same thing for $(z+k)x_-^z$ as $z \to -k$.

v) Show that there is a linear combination $(k+z)(x_+^z + c(k)x_-^z)$ such that as $z \to -k$ the limit is zero.

vi) If you get this far, show that in fact $x_+^z + c(k)x_-^z$ also has a weak limit, u_k, as $z \to -k$. [This may be the hardest part.]

vii) Show that this limit distribution satisfies $(x\frac{d}{dx} + k)u_k = 0$.

viii) Conclude that (3.7) does in fact hold for $z \in -\mathbb{N}$ as well. [There are still some things to prove to get this.]

Problem 51. Show that for any set $G \subset \mathbb{R}^n$

$$v^*(G) = \inf \sum_{i=1}^\infty v(A_i)$$

where the infimum is taken over coverings of G by rectangular sets (products of intervals).

Problem 52. Show that a σ-algebra is closed under countable intersections.

Problem 53. Show that compact sets are Lebesgue measurable and have finite volume and also show the inner regularity of the Lebesgue measure on open sets, that is if E is open then

$$v(E) = \sup\{v(K); K \subset E,\ K \text{ compact}\}. \tag{3.8}$$

Problem 54. Show that a set $B \subset \mathbb{R}^n$ is Lebesgue measurable if and only if

$$v^*(E) = v^*(E \cap B) + v^*(E \cap B^{\complement})\ \forall \text{ open } E \subset \mathbb{R}^n.$$

[The definition is this for all $E \subset \mathbb{R}^n$.]

Problem 55. Show that a real-valued continuous function $f : U \longrightarrow \mathbb{R}$ on an open set, is Lebesgue measurable, in the sense that $f^{-1}(I) \subset U \subset \mathbb{R}^n$ is measurable for each interval I.

Problem 56. Hilbert space and the Riesz representation theorem. If you need help with this, it can be found in lots of places – for instance [**7**] has a nice treatment.

i) A pre-Hilbert space is a vector space V (over $\mathbb{C}$) with a 'positive definite sesquilinear inner product' i.e. a function

$$V \times V \ni (v, w) \mapsto \langle v, w\rangle \in \mathbb{C}$$

satisfying
- $\langle w, v\rangle = \overline{\langle v, w\rangle}$
- $\langle a_1v_1 + a_2v_2, w\rangle = a_1\langle v_1, w\rangle + a_2\langle v_2, w\rangle$
- $\langle v, v\rangle \geq 0$
- $\langle v, v\rangle = 0 \Rightarrow v = 0.$

Prove Schwarz' inequality, that

$$|\langle u, v\rangle| \leq \langle u\rangle^{\frac{1}{2}}\langle v\rangle^{\frac{1}{2}}\ \forall\ u, v \in V.$$

Hint: Reduce to the case $\langle v, v\rangle = 1$ and then expand

$$\langle u - \langle u, v\rangle v, u - \langle u, v\rangle v\rangle \geq 0.$$

ii) Show that $\|v\| = \langle v, v\rangle^{1/2}$ is a norm and that it satisfies the parallelogram law:

$$\|v_1 + v_2\|^2 + \|v_1 - v_2\|^2 = 2\|v_1\|^2 + 2\|v_2\|^2\ \forall\ v_1, v_2 \in V. \tag{3.9}$$

iii) Conversely, suppose that V is a linear space over $\mathbb{C}$ with a norm which satisfies (3.9). Show that

$$4\langle v, w\rangle = \|v + w\|^2 - \|v - w\|^2 + i\|v + iw\|^2 - i\|v - iw\|^2$$

defines a pre-Hilbert inner product which gives the original norm.

iv) Let V be a Hilbert space, so as in (i) but complete as well. Let $C \subset V$ be a closed non-empty convex subset, meaning $v, w \in C \Rightarrow (v+w)/2 \in C$. Show that there exists a unique $v \in C$ minimizing the norm, i.e. such that

$$\|v\| = \inf_{w \in C} \|w\|.$$

Hint: Use the parallelogram law to show that a norm minimizing sequence is Cauchy.

v) Let $u : H \to \mathbb{C}$ be a continuous linear functional on a Hilbert space, so $|u(\varphi)| \le C\|\varphi\| \ \forall \ \varphi \in H$. Show that $N = \{\varphi \in H; u(\varphi) = 0\}$ is closed and that if $v_0 \in H$ has $u(v_0) \neq 0$ then each $v \in H$ can be written uniquely in the form

$$v = cv_0 + w, \ c \in \mathbb{C}, \ w \in N.$$

vi) With u as in v), not the zero functional, show that there exists a unique $f \in H$ with $u(f) = 1$ and $\langle w, f \rangle = 0$ for all $w \in N$.
Hint: Apply iv) to $C = \{g \in V; u(g) = 1\}$.

vii) Prove the Riesz Representation theorem, that every continuous linear functional on a Hilbert space is of the form

$$u_f : H \ni \varphi \mapsto \langle \varphi, f \rangle \text{ for a unique } f \in H.$$

Problem 57. Density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$.

i) Recall in a few words why simple integrable functions are dense in $L^1(\mathbb{R}^n)$ with respect to the norm $\|f\|_{L^1} = \int_{\mathbb{R}^n} |f(x)|dx$.

ii) Show that simple functions $\sum_{j=1}^N c_j \chi(U_j)$ where the U_j are open and bounded are also dense in $L^1(\mathbb{R}^n)$.

iii) Show that if U is open and bounded then $F(y) = v(U \cap U_y)$, where $U_y = \{z \in \mathbb{R}^n; z = y + y', y' \in U\}$ is continuous in $y \in \mathbb{R}^n$ and that

$$v(U \cap U_y^{\complement}) + v(U^{\complement} \cap U_y) \to 0 \text{ as } y \to 0.$$

iv) If U is open and bounded and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ show that

$$f(x) = \int_U \varphi(x-y)dy \in \mathcal{C}_c^\infty(\mathbb{R}^n).$$

v) Show that if U is open and bounded then

$$\sup_{|y| \le \delta} \int |\chi_U(x) - \chi_U(x-y)|dx \to 0 \text{ as } \delta \downarrow 0.$$

vi) If U is open and bounded and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\varphi \geq 0$, $\int \varphi = 1$ then

$$f_\delta \to \chi_U \text{ in } L^1(\mathbb{R}^n) \text{ as } \delta \downarrow 0$$

where

$$f_\delta(x) = \delta^{-n} \int \varphi\left(\frac{y}{\delta}\right) \chi_U(x-y)dy.$$

Hint: Write $\chi_U(x) = \delta^{-n} \int \varphi\left(\frac{y}{\delta}\right) \chi_U(x)$ and use v).

vii) Conclude that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.

viii) Show that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for any $1 \leq p < \infty$.

PROBLEM 58. Schwartz representation theorem. Here we (well you) come to grips with the general structure of a tempered distribution.

i) Recall briefly the proof of the Sobolev embedding theorem and the corresponding estimate

$$\sup_{x\in\mathbb{R}^n} |\phi(x)| \leq C\|\phi\|_{H^m}, \ \frac{n}{2} < m \in \mathbb{R}.$$

ii) For $m = n+1$ write down a(n equivalent) norm on the right in a form that does not involve the Fourier transform.

iii) Show that for any $\alpha \in \mathbb{N}_0$

$$|D^\alpha\left((1+|x|^2)^N\phi\right)| \leq C_{\alpha,N} \sum_{\beta\leq\alpha} (1+|x|^2)^N |D^\beta\phi|.$$

iv) Deduce the general estimates

$$\sup_{\substack{|\alpha|\leq N\\ x\in\mathbb{R}^n}} (1+|x|^2)^N |D^\alpha\phi(x)| \leq C_N \|(1+|x|^2)^N\phi\|_{H^{N+n+1}}.$$

v) Conclude that for each tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ there is an integer N and a constant C such that

$$|u(\phi)| \leq C\|(1+|x|^2)^N\phi\|_{H^{2N}} \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n).$$

vi) Show that $v = (1+|x|^2)^{-N}u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies

$$|v(\phi)| \leq C\|(1+|D|^2)^N\phi\|_{L^2} \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n).$$

vi) Recall (from class or just show it) that if v is a tempered distribution then there is a unique $w \in \mathcal{S}'(\mathbb{R}^n)$ such that $(1+|D|^2)^N w = v$.

vii) Use the Riesz Representation Theorem to conclude that for each tempered distribution u there exists N and $w \in L^2(\mathbb{R}^n)$ such that

$$u = (1+|D|^2)^N(1+|x|^2)^N w. \tag{3.10}$$

viii) Use the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ (and the fact that it is an isomorphism on $L^2(\mathbb{R}^n)$) to show that any tempered distribution can be written in the form

$$u = (1+|x|^2)^N (1+|D|^2)^N w \text{ for some } N \text{ and some } w \in L^2(\mathbb{R}^n).$$

ix) Show that any tempered distribution can be written in the form

$$u = (1+|x|^2)^N (1+|D|^2)^{N+n+1} \tilde{w} \text{ for some } N \text{ and some } \tilde{w} \in H^{2(n+1)}(\mathbb{R}^n).$$

x) Conclude that any tempered distribution can be written in the form

$$u = (1+|x|^2)^N (1+|D|^2)^M U \text{ for some } N, M \text{ and a bounded continuous function } U$$

Problem 59. Distributions of compact support.

i) Recall the definition of the support of a distribution, defined in terms of its complement

$$\mathbb{R}^n \backslash \operatorname{supp}(u) = \left\{ p \in \mathbb{R}^n; \exists\, U \subset \mathbb{R}^n, \text{ open, with } p \in U \text{ such that } u\big|_U = 0 \right\}$$

ii) Show that if $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}(u) \cap \operatorname{supp}(\phi) = \emptyset$$

then $u(\phi) = 0$.

iii) Consider the space $\mathcal{C}^\infty(\mathbb{R}^n)$ of all smooth functions on $\mathbb{R}^n$, without restriction on supports. Show that for each N

$$\|f\|_{(N)} = \sup_{|\alpha| \le N,\ |x| \le N} |D^\alpha f(x)|$$

is a seminorn on $\mathcal{C}^\infty(\mathbb{R}^n)$ (meaning it satisfies $\|f\| \ge 0$, $\|cf\| = |c|\|f\|$ for $c \in \mathbb{C}$ and the triangle inequality but that $\|f\| = 0$ does not necessarily imply that $f = 0$.)

iv) Show that $\mathcal{C}^\infty_c(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$ is dense in the sense that for each $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ there is a sequence f_n in $\mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $\|f - f_n\|_{(N)} \to 0$ for each N.

v) Let $\mathcal{E}'(\mathbb{R}^n)$ temporarily (or permanantly if you prefer) denote the dual space of $\mathcal{C}^\infty(\mathbb{R}^n)$ (which is also written $\mathcal{E}(\mathbb{R}^n)$), that is, $v \in \mathcal{E}'(\mathbb{R}^n)$ is a linear map $v : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathbb{C}$ which is continuous in the sense that for some N

$$|v(f)| \le C\|f\|_{(N)}\ \forall\ f \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{3.11}$$

Show that such a v 'is' a distribution and that the map $\mathcal{E}'(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}^n)$ is injective.

vi) Show that if $v \in \mathcal{E}'(\mathbb{R}^n)$ satisfies (3.11) and $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ has $f = 0$ in $|x| < N + \epsilon$ for some $\epsilon > 0$ then $v(f) = 0$.

vii) Conclude that each element of $\mathcal{E}'(\mathbb{R}^n)$ has compact support when considered as an element of $\mathcal{C}^{-\infty}(\mathbb{R}^n)$.

viii) Show the converse, that each element of $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ with compact support is an element of $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and hence conclude that $\mathcal{E}'(\mathbb{R}^n)$ 'is' the space of distributions of compact support.

I will denote the space of distributions of compact support by $\mathcal{C}_c^{-\infty}(\mathbb{R})$.

Problem 60. Hypoellipticity of the heat operator $H = iD_t + \Delta = iD_t + \sum_{j=1}^{n} D_{x_j}^2$ on $\mathbb{R}^{n+1}$.

(1) Using τ to denote the 'dual variable' to t and $\xi \in \mathbb{R}^n$ to denote the dual variables to $x \in \mathbb{R}^n$ observe that $H = p(D_t, D_x)$ where $p = i\tau + |\xi|^2$.

(2) Show that $|p(\tau,\xi)| > \frac{1}{2}\left(|\tau| + |\xi|^2\right)$.

(3) Use an inductive argument to show that, in $(\tau,\xi) \neq 0$ where it makes sense,

$$D_\tau^k D_\xi^\alpha \frac{1}{p(\tau,\xi)} = \sum_{j=1}^{|\alpha|} \frac{q_{k,\alpha,j}(\xi)}{p(\tau,\xi)^{k+j+1}} \tag{3.12}$$

where $q_{k,\alpha,j}(\xi)$ is a polynomial of degree (at most) $2j - |\alpha|$.

(4) Conclude that if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ is identically equal to 1 in a neighbourhood of 0 then the function

$$g(\tau,\xi) = \frac{1 - \phi(\tau,\xi)}{i\tau + |\xi|^2}$$

is the Fourier transform of a distribution $F \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{sing\,supp}(F) \subset \{0\}$. [Remember that $\operatorname{sing\,supp}(F)$ is the complement of the largest open subset of $\mathbb{R}^n$ the restriction of F to which is smooth].

(5) Show that F is a parametrix for the heat operator.

(6) Deduce that $iD_t + \Delta$ is *hypoelliptic* – that is, if $U \subset \mathbb{R}^n$ is an open set and $u \in \mathcal{C}^{-\infty}(U)$ satisfies $(iD_t + \Delta)u \in \mathcal{C}^\infty(U)$ then $u \in \mathcal{C}^\infty(U)$.

(7) Show that $iD_t - \Delta$ is also hypoelliptic.

Problem 61. Wavefront set computations and more – all pretty easy, especially if you use results from class.

i) Compute $\mathrm{WF}(\delta)$ where $\delta \in \mathcal{S}'(\mathbb{R}^n)$ is the Dirac delta function at the origin.

ii) Compute $\mathrm{WF}(H(x))$ where $H(x) \in \mathcal{S}'(\mathbb{R})$ is the Heaviside function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}.$$

Hint: D_x is elliptic in one dimension, hit H with it.

iii) Compute $\mathrm{WF}(E)$, $E = iH(x_1)\delta(x')$ which is the Heaviside in the first variable on $\mathbb{R}^n$, $n > 1$, and delta in the others.

iv) Show that $D_{x_1}E = \delta$, so E is a fundamental solution of D_{x_1}.

v) If $f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ show that $u = E \star f$ solves $D_{x_1}u = f$.

vi) What does our estimate on $\mathrm{WF}(E \star f)$ tell us about $\mathrm{WF}(u)$ in terms of $\mathrm{WF}(f)$?

PROBLEM 62. The wave equation in two variables (or one spatial variable).

i) Recall that the Riemann function

$$E(t, x) = \begin{cases} -\frac{1}{4} & \text{if } t > x \text{ and } t > -x \\ 0 & \text{otherwise} \end{cases}$$

is a fundamental solution of $D_t^2 - D_x^2$ (check my constant).

ii) Find the singular support of E.

iii) Write the Fourier transform (dual) variables as τ, ξ and show that

$$\mathrm{WF}(E) \subset \{0\} \times \mathbb{S}^1 \cup \{(t, x, \tau, \xi); x = t > 0 \text{ and } \xi + \tau = 0\} \\ \cup \{(t, x, \tau, \xi); -x = t > 0 \text{ and } \xi = \tau\}.$$

iv) Show that if $f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^2)$ then $u = E \star f$ satisfies $(D_t^2 - D_x^2)u = f$.

v) With u defined as in iv) show that

$$\mathrm{supp}(u) \subset \{(t, x);\ \exists \\ (t', x') \in \mathrm{supp}(f) \text{ with } t' + x' \le t + x \text{ and } t' - x' \le t - x\}.$$

vi) Sketch an illustrative example of v).

vii) Show that, still with u given by iv),

$$\mathrm{sing\,supp}(u) \subset \{(t, x);\ \exists\ (t', x') \in \mathrm{sing\,supp}(f) \text{ with} \\ t \ge t' \text{ and } t + x = t' + x' \text{ or } t - x = t' - x'\}.$$

viii) Bound $\mathrm{WF}(u)$ in terms of $\mathrm{WF}(f)$.

PROBLEM 63. A little uniqueness theorems. Suppose $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ recall that the Fourier transform $\hat{u} \in \mathcal{C}^\infty(\mathbb{R}^n)$. Now, suppose $u \in$

$\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ satisfies $P(D)u = 0$ for some non-trivial polynomial P, show that $u = 0$.

PROBLEM 64. Work out the elementary behavior of the heat equation.

i) Show that the function on $\mathbb{R} \times \mathbb{R}^n$, for $n \geq 1$,

$$F(t,x) = \begin{cases} t^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is measurable, bounded on the any set $\{|(t,x)| \geq R\}$ and is integrable on $\{|(t,x)| \leq R\}$ for any $R > 0$.

ii) Conclude that F defines a tempered distribution on $\mathbb{R}^{n+1}$.

iii) Show that F is $\mathcal{C}^\infty$ outside the origin.

iv) Show that F satisfies the heat equation

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)F(t,x) = 0 \text{ in } (t,x) \neq 0.$$

v) Show that F satisfies

$$F(s^2t, sx) = s^{-n}F(t,x) \text{ in } \mathcal{S}'(\mathbb{R}^{n+1}) \tag{3.13}$$

where the left hand side is defined by duality "$F(s^2t, sx) = F_s$" where

$$F_s(\phi) = s^{-n-2}F(\phi_{1/s}),\ \phi_{1/s}(t,x) = \phi(\frac{t}{s^2}, \frac{x}{s}).$$

vi) Conclude that

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)F(t,x) = G(t,x)$$

where $G(t,x)$ satisfies

$$G(s^2t, sx) = s^{-n-2}G(t,x) \text{ in } \mathcal{S}'(\mathbb{R}^{n+1}) \tag{3.14}$$

in the same sense as above and has support at most $\{0\}$.

vii) Hence deduce that

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)F(t,x) = c\delta(t)\delta(x) \tag{3.15}$$

for some real constant c.

Hint: Check which distributions with support at $(0,0)$ satisfy (3.14).

viii) If $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ show that $u = F \star \psi$ satisfies

$$(3.16) \quad u \in \mathcal{C}^\infty(\mathbb{R}^{n+1}) \text{ and } \sup_{x\in\mathbb{R}^n,\ t\in[-S,S]} (1+|x|)^N |D^\alpha u(t,x)| < \infty \ \forall\ S > 0, \alpha \in \mathbb{N}^{n+1},\ N.$$

ix) Supposing that u satisfies (3.16) and is a real-valued solution of

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2) u(t,x) = 0$$

in $\mathbb{R}^{n+1}$, show that

$$v(t) = \int_{\mathbb{R}^n} u^2(t,x)$$

is a non-increasing function of t.

Hint: Multiply the equation by u and integrate over a slab $[t_1,t_2]\times\mathbb{R}^n$.

x) Show that c in (3.15) is non-zero by arriving at a contradiction from the assumption that it is zero. Namely, show that if $c=0$ then u in viii) satisfies the conditions of ix) and also vanishes in $t<T$ for some T (depending on ψ). Conclude that $u=0$ for all ψ. Using properties of convolution show that this in turn implies that $F=0$ which is a contradiction.

xi) So, finally, we know that $E = \frac{1}{c}F$ is a fundamental solution of the heat operator which vanishes in $t<0$. Explain why this allows us to show that for any $\psi \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^n)$ there is a solution of

$$(3.17) \qquad (\partial_t - \sum_{j=1}^n \partial_{x_j}^2) u = \psi,\ \ u=0 \text{ in } t<T \text{ for some } T.$$

What is the largest value of T for which this holds?

xii) Can you give a heuristic, or indeed a rigorous, explanation of why

$$c = \int_{\mathbb{R}^n} \exp(-\frac{|x|^2}{4}) dx?$$

xiii) Explain why the argument we used for the wave equation to show that there is *only one* solution, $u \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$, of (3.17) does not apply here. (Indeed such uniqueness does not hold without some growth assumption on u.)

PROBLEM 65. (Poisson summation formula) As in class, let $L \subset \mathbb{R}^n$ be an integral lattice of the form

$$L = \left\{ v = \sum_{j=1}^{n} k_j v_j,\ k_j \in \mathbb{Z} \right\}$$

where the v_j form a basis of $\mathbb{R}^n$ and using the dual basis w_j (so $w_j \cdot v_i = \delta_{ij}$ is 0 or 1 as $i \neq j$ or $i = j$) set

$$L^\circ = \left\{ w = 2\pi \sum_{j=1}^{n} k_j w_j,\ k_j \in \mathbb{Z} \right\}.$$

Recall that we defined

$$\mathcal{C}^\infty(\mathbb{T}_L) = \{u \in \mathcal{C}^\infty(\mathbb{R}^n); u(z+v) = u(z)\ \forall\ z \in \mathbb{R}^n,\ v \in L\}. \tag{3.18}$$

i) Show that summation over shifts by lattice points:

$$A_L : \mathcal{S}(\mathbb{R}^n) \ni f \longmapsto A_L f(z) = \sum_{v \in L} f(z - v) \in \mathcal{C}^\infty(\mathbb{T}_L). \tag{3.19}$$

defines a map into smooth periodic functions.

ii) Show that there exists $f \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $A_L f \equiv 1$ is the costant function on $\mathbb{R}^n$.

iii) Show that the map (3.19) is surjective. Hint: Well obviously enough use the f in part ii) and show that if u is periodic then $A_L(uf) = u$.

iv) Show that the infinite sum

$$F = \sum_{v \in L} \delta(\cdot - v) \in \mathcal{S}'(\mathbb{R}^n) \tag{3.20}$$

does indeed define a tempered distribution and that F is L-periodic and satisfies $\exp(iw \cdot z)F(z) = F(z)$ for each $w \in L^\circ$ with equality in $\mathcal{S}'(\mathbb{R}^n)$.

v) Deduce that $\hat{F}$, the Fourier transform of F, is L° periodic, conclude that it is of the form

$$\hat{F}(\xi) = c \sum_{w \in L^\circ} \delta(\xi - w) \tag{3.21}$$

vi) Compute the constant c.

vii) Show that $A_L(f) = F \star f$.

viii) Using this, or otherwise, show that $A_L(f) = 0$ in $\mathcal{C}^\infty(\mathbb{T}_L)$ if and only if $\hat{f} = 0$ on L°.

PROBLEM 66. For a measurable set $\Omega \subset \mathbb{R}^n$, with non-zero measure, set $H = L^2(\Omega)$ and let $\mathcal{B} = \mathcal{B}(H)$ be the algebra of bounded linear operators on the Hilbert space H with the norm on $\mathcal{B}$ being

$$\|B\|_{\mathcal{B}} = \sup\{\|Bf\|_H; f \in H,\ \|f\|_H = 1\}. \tag{3.22}$$

i) Show that $\mathcal{B}$ is complete with respect to this norm. Hint (probably not necessary!) For a Cauchy sequence $\{B_n\}$ observe that $B_n f$ is Cauchy for each $f \in H$.
ii) If $V \subset H$ is a finite-dimensional subspace and $W \subset H$ is a closed subspace with a finite-dimensional complement (that is $W + U = H$ for some finite-dimensional subspace U) show that there is a closed subspace $Y \subset W$ with finite-dimensional complement (in H) such that $V \perp Y$, that is $\langle v, y\rangle = 0$ for all $v \in V$ and $y \in Y$.
iii) If $A \in \mathcal{B}$ has finite rank (meaning AH is a finite-dimensional vector space) show that there is a finite-dimensional space $V \subset H$ such that $AV \subset V$ and $AV^\perp = \{0\}$ where

$$V^\perp = \{f \in H; \langle f, v\rangle = 0\ \forall\ v \in V\}.$$

Hint: Set $R = AH$, a finite dimensional subspace by hypothesis. Let N be the null space of A, show that $N^\perp$ is finite dimensional. Try $V = R + N^\perp$.
iv) If $A \in \mathcal{B}$ has finite rank, show that $(\mathrm{Id} - zA)^{-1}$ exists for all but a finite set of $\lambda \in \mathbb{C}$ (just quote some matrix theory). What might it mean to say in this case that $(\mathrm{Id} - zA)^{-1}$ is meromorphic in z? (No marks for this second part).
v) Recall that $\mathcal{K} \subset \mathcal{B}$ is the algebra of compact operators, defined as the closure of the space of finite rank operators. Show that $\mathcal{K}$ is an ideal in $\mathcal{B}$.
vi) If $A \in \mathcal{K}$ show that

$$\mathrm{Id} + A = (\mathrm{Id} + B)(\mathrm{Id} + A')$$

where $B \in \mathcal{K}$, $(\mathrm{Id} + B)^{-1}$ exists and A' has finite rank. Hint: Use the invertibility of $\mathrm{Id} + B$ when $\|B\|_{\mathcal{B}} < 1$ proved in class.
vii) Conclude that if $A \in \mathcal{K}$ then

$$\{f \in H; (\mathrm{Id} + A)f = 0\} \text{ and } \big((\mathrm{Id} + A)H\big)^\perp \text{ are finite dimensional.}$$

PROBLEM 67. [Separable Hilbert spaces]

i) (Gramm-Schmidt Lemma). Let $\{v_i\}_{i\in\mathbb{N}}$ be a sequence in a Hilbert space H. Let $V_j \subset H$ be the span of the first j elements and set $N_j = \dim V_j$. Show that there is an orthonormal sequence $e_1, \dots, e_j$ (finite if N_j is bounded above) such that V_j is the span of the first N_j elements. Hint: Proceed by induction over N such that the result is true for all j with $N_j < N$. So, consider what happens for a value of j with $N_j = N_{j-1}+1$ and add element $e_{N_j} \in V_j$ which is orthogonal to all the previous e_k's.

ii) A Hilbert space is separable if it has a countable dense subset (sometimes people say Hilbert space when they mean separable Hilbert space). Show that every separable Hilbert space has a complete orthonormal sequence, that is a sequence $\{e_j\}$ such that $\langle u, e_j\rangle = 0$ for all j implies $u = 0$.

iii) Let $\{e_j\}$ an orthonormal sequence in a Hilbert space, show that for any $a_j \in \mathbb{C}$,

$$\|\sum_{j=1}^{N} a_j e_j\|^2 = \sum_{j=1}^{N} |a_j|^2.$$

iv) (Bessel's inequality) Show that if e_j is an orthormal sequence in a Hilbert space and $u \in H$ then

$$\|\sum_{j=1}^{N} \langle u, e_j\rangle e_j\|^2 \leq \|u\|^2$$

and conclude (assuming the sequence of e_j's to be infinite) that the series

$$\sum_{j=1}^{\infty} \langle u, e_j\rangle e_j$$

converges in H.

v) Show that if e_j is a complete orthonormal basis in a separable Hilbert space then, for each $u \in H$,

$$u = \sum_{j=1}^{\infty} \langle u, e_j\rangle e_j.$$

Problem 68. [Compactness] Let's agree that a compact set in a metric space is one for which every open cover has a finite subcover. You may use the compactness of closed bounded sets in a finite dimensional vector space.

i) Show that a compact subset of a Hilbert space is closed and bounded.

ii) If e_j is a complete orthonormal subspace of a separable Hilbert space and K is compact show that given $\epsilon > 0$ there exists N

such that

$$\sum_{j\ge N}|\langle u,e_j\rangle|^2\le\epsilon\ \forall\ u\in K. \tag{3.23}$$

iii) Conversely show that any closed bounded set in a separable Hilbert space for which (3.23) holds for some orthonormal basis is indeed compact.

iv) Show directly that any sequence in a compact set in a Hilbert space has a convergent subsequence.

v) Show that a subspace of H which has a precompact unit ball must be finite dimensional.

vi) Use the existence of a complete orthonormal basis to show that any bounded sequence $\{u_j\}$, $\|u_j\|\le C$, has a weakly convergent subsequence, meaning that $\langle v,u_j\rangle$ converges in $\mathbb{C}$ along the subsequence for each $v\in H$. Show that the subsequnce can be chosen so that $\langle e_k,u_j\rangle$ converges for each k, where e_k is the complete orthonormal sequence.

PROBLEM 69. [Spectral theorem, compact case] Recall that a bounded operator A on a Hilbert space H is compact if $A\{\|u\|\le 1\}$ is precompact (has compact closure). Throughout this problem A will be a compact operator on a separable Hilbert space, H.

i) Show that if $0\ne\lambda\in\mathbb{C}$ then

$$E_\lambda=\{u\in H;Au=\lambda u\}.$$

is finite dimensional.

ii) If A is self-adjoint show that all eigenvalues (meaning $E_\lambda\ne\{0\}$) are real and that different eigenspaces are orthogonal.

iii) Show that $\alpha_A=\sup\{|\langle Au,u\rangle|^2\};\|u\|=1\}$ is attained. Hint: Choose a sequence such that $|\langle Au_j,u_j\rangle|^2$ tends to the supremum, pass to a weakly convergent sequence as discussed above and then using the compactness to a furhter subsequence such that Au_j converges.

iv) If v is such a maximum point and $f\perp v$ show that $\langle Av,f\rangle+\langle Af,v\rangle=0$.

v) If A is also self-adjoint and u is a maximum point as in iii) deduce that $Au=\lambda u$ for some $\lambda\in\mathbb{R}$ and that $\lambda=\pm\alpha$.

vi) Still assuming A to be self-adjoint, deduce that there is a finite-dimensional subspace $M\subset H$, the sum of eigenspaces with eigenvalues $\pm\alpha$, containing all the maximum points.

vii) Continuing vi) show that A restricts to a self-adjoint bounded operator on the Hilbert space $M^\perp$ and that the supremum in iii) for this new operator is smaller.

viii) Deduce that for any compact self-adjoint operator on a separable Hilbert space there is a complete orthonormal basis of eigenvectors. Hint: Be careful about the null space – it could be big.

PROBLEM 70. Show that a (complex-valued) square-integrable function $u \in L^2(\mathbb{R}^n)$ is continuous in the mean, in the sense that

$$\lim_{\epsilon\downarrow 0} \sup_{|y|<\epsilon} \int |u(x+y)-u(x)|^2 dx = 0. \tag{3.24}$$

Hint: Show that it is enough to prove this for non-negative functions and then that it suffices to prove it for non-negative simple functions and finally that it is enough to check it for the characteristic function of an open set of finite measure. Then use Problem 57 to show that it is true in this case.

PROBLEM 71. [Ascoli-Arzela] Recall the proof of the theorem of Ascoli and Arzela, that a subset of $\mathcal{C}^0_0(\mathbb{R}^n)$ is precompact (with respect to the supremum norm) if and only if it is equicontinuous and equi-small at infinity, i.e. given $\epsilon > 0$ there exists $\delta > 0$ such that for all elements $u \in B$

$$|y| < \delta \Longrightarrow \sup_{x\in\mathbb{R}^n} |u(x+y) = u(x)| < \epsilon \text{ and } |x| > 1/\delta \Longrightarrow |u(x)| < \epsilon. \tag{3.25}$$

PROBLEM 72. [Compactness of sets in $L^2(\mathbb{R}^n)$.] Show that a subset $B \subset L^2(\mathbb{R}^n)$ is *precompact* in $L^2(\mathbb{R}^n)$ if and only if it satisfies the following two conditions:

i) (Equi-continuity in the mean) For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^n} |u(x+y)-u(x)|^2 dx < \epsilon \;\forall\; |y| < \delta,\; u \in B. \tag{3.26}$$

ii) (Equi-smallness at infinity) For each $\epsilon > 0$ there exists R such that

$$\int_{|x|>R|} |u|^2 dx < \epsilon \;\forall\; u \in B. \tag{3.27}$$

Hint: Problem 70 shows that (3.26) holds for each $u \in L^2(\mathbb{R}^n)$; check that (3.27) also holds for each function. Then use a covering argument to prove that both these conditions must hold for a compact subset of $L^2(\mathbb{R})$ and hence for a precompact set. One method to prove the converse is to show that if (3.26) and (3.27) hold then B is bounded and to use this to extract a weakly convergent sequence from any given sequence in B. Next show that (3.26) is equivalent to (3.27) for the set $\mathcal{F}(B)$, the image of B under the Fourier transform. Show, possibly using Problem 71, that if χ_R is cut-off to a ball of radius R then $\chi_R \mathcal{G}(\chi_R \hat{u}_n)$ converges strongly if u_n converges weakly. Deduce from this that the weakly convergent subsequence in fact converges strongly so $\bar{B}$ is sequently compact, and hence is compact.

Problem 73. Consider the space $\mathcal{C}_{\mathrm{c}}(\mathbb{R}^n)$ of all continuous functions on $\mathbb{R}^n$ with compact support. Thus each element vanishes in $|x| > R$ for some R, depending on the function. We want to give this a toplogy in terms of which is complete. We will use the *inductive limit* topology. Thus the whole space can be written as a countable union

$$\mathcal{C}_{\mathrm{c}}(\mathbb{R}^n) = \bigcup_n \{u : \mathbb{R}^n; u \text{ is continuous and } u(x) = 0 \text{ for } |x| > R\}. \tag{3.28}$$

Each of the space on the right is a Banach space for the supremum norm.

(1) Show that the supreumum norm is not complete on the whole of this space.
(2) Define a subset $U \subset \mathcal{C}_{\mathrm{c}}(\mathbb{R}^n)$ to be open if its intersection with each of the subspaces on the right in (3.28) is open w.r.t. the supremum norm.
(3) Show that this definition does yield a topology.
(4) Show that any sequence $\{f_n\}$ which is 'Cauchy' in the sense that for any open neighbourhood U of 0 there exists N such that $f_n - f_m \in U$ for all $n, m \ge N$, is convergent (in the corresponding sense that there exists f in the space such that $f - f_n \in U$ eventually).
(5) If you are determined, discuss the corresponding issue for nets.

Problem 74. Show that the continuity of a linear functional $u : \mathcal{C}^\infty_c(\mathbb{R}^n) \longrightarrow \mathbb{C}$ with respect to the inductive limit topology defined in (1.16) means precisely that for each $n \in \mathbb{N}$ there exists $k = k(n)$ and $C = C_n$ such that

$$|u(\varphi)| \le C\|\varphi\|_{\mathcal{C}^k},\ \forall\ \varphi \in \dot{\mathcal{C}}^\infty(B(n)). \tag{3.29}$$

The point of course is that the 'order' k and the constnat C can both increase as n, measuring the size of the support, increases.

Problem 75. [Restriction from Sobolev spaces] The Sobolev embedding theorem shows that a function in $H^m(\mathbb{R}^n)$, for $m > n/2$ is continuous – and hence can be restricted to a subspace of $\mathbb{R}^n$. In fact this works more generally. Show that there is a well defined *restriction* *map*

$$H^m(\mathbb{R}^n) \longrightarrow H^{m-\frac{1}{2}}(\mathbb{R}^n) \text{ if } m > \frac{1}{2} \tag{3.30}$$

with the following properties:

(1) On $\mathcal{S}(\mathbb{R}^n)$ it is given by $u \longmapsto u(0, x')$, $x' \in \mathbb{R}^{n-1}$.
(2) It is continuous and linear.

Hint: Use the usual method of finding a weak version of the map on smooth Schwartz functions; namely show that in terms of the Fourier transforms on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$

$$\widehat{u(0,\cdot)}(\xi') = (2\pi)^{-1} \int_{\mathbb{R}} \hat{u}(\xi_1, \xi') d\xi_1, \ \forall \ \xi' \in \mathbb{R}^{n-1}. \tag{3.31}$$

Use Cauchy's inequality to show that this is continuous as a map on Sobolev spaces as indicated and then the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^m(\mathbb{R}^n)$ to conclude that the map is well-defined and unique.

Problem 76. [Restriction by WF] From class we know that the product of two distributions, one with compact support, is defined provided they have no 'opposite' directions in their wavefront set:

$$(x, \omega) \in \mathrm{WF}(u) \Longrightarrow (x, -\omega) \notin \mathrm{WF}(v) \text{ then } uv \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n). \tag{3.32}$$

Show that this product has the property that $f(uv) = (fu)v = u(fv)$ if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. Use this to define a restriction map to $x_1 = 0$ for distributions of compact support satisfying $((0, x'), (\omega_1, 0)) \notin \mathrm{WF}(u)$ as the product

$$u_0 = u\delta(x_1). \tag{3.33}$$

[Show that $u_0(f)$, $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ only depends on $f(0, \cdot) \in \mathcal{C}^\infty(\mathbb{R}^{n-1})$.

Problem 77. [Stone's theorem] For a bounded self-adjoint operator A show that the spectral measure can be obtained from the resolvent in the sense that for $\phi, \psi \in H$

$$\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \langle [(A - t - i\epsilon)^{-1} - (A + t + i\epsilon)^{-1}]\phi, \psi\rangle \longrightarrow \mu_{\phi,\psi} \tag{3.34}$$

in the sense of distributions – or measures if you are prepared to work harder!

Problem 78. If $u \in \mathcal{S}(\mathbb{R}^n)$ and $\psi' = \psi_R + \mu$ is, as in the proof of Lemma 7.5, such that

$$\mathrm{supp}(\psi') \cap \mathrm{Css}(u) = \emptyset$$

show that

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto \phi\psi' u \in \mathcal{S}(\mathbb{R}^n)$$

is continuous and hence (or otherwise) show that the functional u_1u_2 defined by (7.20) is an element of $\mathcal{S}'(\mathbb{R}^n)$.

Problem 79. Under the conditions of Lemma 7.10 show that

$$\mathrm{Css}(u{*}v)\cap\mathbb{S}^{n-1}\subset\{\frac{sx+ty}{|sx+ty|},\ |x|=|y|=1,\ x\in\mathrm{Css}(u),\ y\in\mathrm{Css}(v),\ 0\le s,t\le 1\}. \tag{3.35}$$

Notice that this make sense exactly because $sx+ty=0$ implies that $t/s=1$ but $x+y\neq 0$ under these conditions by the assumption of Lemma 7.10.

Problem 80. Show that the pairing $u(v)$ of two distributions $u,v\in {}^{\mathrm{b}}S'(\mathbb{R}^n)$ may be defined under the hypothesis (7.50).

Problem 81. Show that under the hypothesis (7.51)

$$\begin{aligned}(3.36)\quad &\mathrm{WF}_{\mathrm{sc}}(u{*}v)\subset\{(x{+}y,p);(x,p)\in\mathrm{WF}_{\mathrm{sc}}(u)\cap(\mathbb{R}^n{\times}\mathbb{S}^{n-1}),\ (y,p)\in\mathrm{WF}_{\mathrm{sc}}(v)\cap(\mathbb{R}^n{\times}\mathbb{S}^{n-1})\}\\ &\quad\cup\{(\theta,q)\in\mathbb{S}^{n-1}\times\mathbb{B}^n;\theta=\frac{s'\theta'+s''\theta''}{|s'\theta'+s''\theta''|},\ 0\le s',s''\le 1,\\ &(\theta',q)\in\mathrm{WF}_{\mathrm{sc}}(u)\cap(\mathbb{S}^{n-1}\times\mathbb{B}^n),\ (\theta'',q)\in\mathrm{WF}_{\mathrm{sc}}(v)\cap(\mathbb{S}^{n-1}\times\mathbb{B}^n)\}.\end{aligned}$$

Problem 82. Formulate and prove a bound similar to (3.36) for $\mathrm{WF}_{\mathrm{sc}}(uv)$ when $u,\ v\in\mathcal{S}'(\mathbb{R}^n)$ satisfy (7.50).

Problem 83. Show that for convolution $u*v$ defined under condition (7.51) it is still true that

$$P(D)(u*v)=(P(D)u)*v=u*(P(D)v). \tag{3.37}$$

Problem 84. Using Problem 80 (or otherwise) show that integration is defined as a functional

$$\{u\in\mathcal{S}'(\mathbb{R}^n);(\mathbb{S}^{n-1}\times\{0\})\cap\mathrm{WF}_{\mathrm{sc}}(u)=\emptyset\}\longrightarrow\mathbb{C}. \tag{3.38}$$

If u satisfies this condition, show that $\int P(D)u=c\int u$ where c is the constant term in $P(D)$, i.e. $P(D)1=c$.

Problem 85. Compute $\mathrm{WF}_{\mathrm{sc}}(E)$ where $E=C/|x-y|$ is the standard fundamental solution for the Laplacian on $\mathbb{R}^3$. Using Problem 83 give a condition on $\mathrm{WF}_{\mathrm{sc}}(f)$ under which $u=E*f$ is defined and satisfies $\Delta u=f$. Show that under this condition $\int f$ is defined using Problem 84. What can you say about $\mathrm{WF}_{\mathrm{sc}}(u)$? Why is it not the case that $\int\Delta u=0$, even though this is true if u has compact support?

4. Solutions to (some of) the problems

Solution 4.1 (To Problem 10). (by Matjaž Konvalinka).

Since the topology on $\mathbb{N}$, inherited from $\mathbb{R}$, is discrete, a set is compact if and only if it is finite. If a sequence $\{x_n\}$ (i.e. a function $\mathbb{N} \to \mathbb{C}$) is in $\mathcal{C}_0(\mathbb{N})$ if and only if for any $\epsilon > 0$ there exists a compact (hence finite) set F_ϵ so that $|x_n| < \epsilon$ for any n not in F_ϵ. We can assume that $F_\epsilon = \{1, \ldots, n_\epsilon\}$, which gives us the condition that $\{x_n\}$ is in $\mathcal{C}_0(\mathbb{N})$ if and only if it converges to 0. We denote this space by c_0, and the supremum norm by $\|\cdot\|_0$. A sequence $\{x_n\}$ will be abbreviated to x.

Let l^1 denote the space of (real or complex) sequences x with a finite 1-norm

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|.$$

We can define pointwise summation and multiplication with scalars, and $(l^1, \|\cdot\|_1)$ is a normed (in fact Banach) space. Because the functional

$$y \mapsto \sum_{n=1}^{\infty} x_n y_n$$

is linear and bounded ($|\sum_{n=1}^{\infty} x_n y_n| \leq \sum_{n=1}^{\infty} |x_n||y_n| \leq \|x\|_0 \, \|y\|_1$) by $\|x\|_0$, the mapping

$$\Phi \colon l^1 \longmapsto c_0^*$$

defined by

$$x \mapsto \left(y \mapsto \sum_{n=1}^{\infty} x_n y_n \right)$$

is a (linear) well-defined mapping with norm at most 1. In fact, Φ is an isometry because if $|x_j| = \|x\|_0$ then $|\Phi(x)(e_j)| = 1$ where e_j is the j-th unit vector. We claim that Φ is also surjective (and hence an isometric isomorphism). If φ is a functional on c_0 let us denote $\varphi(e_j)$ by x_j. Then $\Phi(x)(y) = \sum_{n=1}^{\infty} \varphi(e_n) y_n = \sum_{n=1}^{\infty} \varphi(y_n e_n) = \varphi(y)$ (the last equality holds because $\sum_{n=1}^{\infty} y_n e_n$ converges to y in c_0 and φ is continuous with respect to the topology in c_0), so $\Phi(x) = \varphi$.

Solution 4.2 (To Problem 29). (Matjaž Konvalinka) Since

$$D_x H(\varphi) = H(-D_x \varphi) = i \int_{-\infty}^{\infty} H(x) \varphi'(x)\, dx = \\ i \int_0^{\infty} \varphi'(x)\, dx = i(0 - \varphi(0)) = -i\delta(\varphi),$$

we get $D_x H = C\delta$ for $C = -i$.

SOLUTION 4.3 (To Problem 40). (Matjaž Konvalinka) Let us prove this in the case where $n = 1$. Define (for $b \neq 0$)

$$U(x) = u(b) - u(x) - (b-x)u'(x) - \ldots - \frac{(b-x)^{k-1}}{(k-1)!}u^{(k-1)}(x);$$

then

$$U'(x) = -\frac{(b-x)^{k-1}}{(k-1)!}u^{(k)}(x).$$

For the continuously differentiable function $V(x) = U(x) - (1-x/b)^k U(0)$ we have $V(0) = V(b) = 0$, so by Rolle's theorem there exists ζ between 0 and b with

$$V'(\zeta) = U'(\zeta) + \frac{k(b-\zeta)^{k-1}}{b^k}U(0) = 0$$

Then

$$U(0) = -\frac{b^k}{k(b-\zeta)^{k-1}}U'(\zeta),$$

$$u(b) = u(0) + u'(0)b + \ldots + \frac{u^{(k-1)}(0)}{(k-1)!}b^{k-1} + \frac{u^{(k)}(\zeta)}{k!}b^k.$$

The required decomposition is $u(x) = p(x) + v(x)$ for

$$p(x) = u(0) + u'(0)x + \frac{u''(0)}{2}x^2 + \ldots + \frac{u^{(k-1)}(0)}{(k-1)!}x^{k-1} + \frac{u^{(k)}(0)}{k!}x^k,$$

$$v(x) = u(x) - p(x) = \frac{u^{(k)}(\zeta) - u^{(k)}(0)}{k!}x^k$$

for ζ between 0 and x, and since $u^{(k)}$ is continuous, $(u(x) - p(x))/x^k$ tends to 0 as x tends to 0.

The proof for general n is not much more difficult. Define the function $w_x \colon I \to \mathbb{R}$ by $w_x(t) = u(tx)$. Then w_x is k-times continuously differentiable,

$$w_x'(t) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(tx)x_i,$$

$$w_x''(t) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(tx)x_i x_j,$$

$$w_x^{(l)}(t) = \sum_{l_1+l_2+\ldots+l_i=l} \frac{l!}{l_1!l_2!\cdots l_i!}\frac{\partial^l u}{\partial x_1^{l_1}\partial x_2^{l_2}\cdots\partial x_i^{l_i}}(tx)x_1^{l_1}x_2^{l_2}\cdots x_i^{l_i}$$

so by above $u(x) = w_x(1)$ is the sum of some polynomial p (od degree k), and we have

$$\frac{u(x)-p(x)}{|x|^k} = \frac{v_x(1)}{|x|^k} = \frac{w_x^{(k)}(\zeta_x) - w_x^{(k)}(0)}{k!|x|^k},$$

so it is bounded by a positive combination of terms of the form

$$\left|\frac{\partial^l u}{\partial x_1^{l_1}\partial x_2^{l_2}\cdots\partial x_i^{l_i}}(\zeta_x x) - \frac{\partial^l u}{\partial x_1^{l_1}\partial x_2^{l_2}\cdots\partial x_i^{l_i}}(0)\right|$$

with $l_1 + \ldots + l_i = k$ and $0 < \zeta_x < 1$. This tends to zero as $x \to 0$ because the derivative is continuous.

SOLUTION 4.4 (Solution to Problem 41). (Matjž Konvalinka) Obviously the map $\mathcal{C}_0(\mathbb{B}^n) \to \mathcal{C}(\mathbb{B}^n)$ is injective (since it is just the inclusion map), and $f \in \mathcal{C}(\mathbb{B}^n)$ is in $\mathcal{C}_0(\mathbb{B}^n)$ if and only if it is zero on $\partial\mathbb{B}^n$, ie. if and only if $f|_{\mathbb{S}^{n-1}} = 0$. It remains to prove that any map g on $\mathbb{S}^{n-1}$ is the restriction of a continuous function on $\mathbb{B}^n$. This is clear since

$$f(x) = \begin{cases} |x|g(x/|x|) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is well-defined, coincides with f on $\mathbb{S}^{n-1}$, and is continuous: if M is the maximum of $|g|$ on $\mathbb{S}^{n-1}$, and $\epsilon > 0$ is given, then $|f(x)| < \epsilon$ for $|x| < \epsilon/M$.

SOLUTION 4.5. (partly Matjaž Konvalinka)
For any $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned}|\int_{-\infty}^{\infty}\varphi(x)dx| \le \int_{-\infty}^{\infty}|\varphi(x)|dx \le \sup((1{+}x|^2)|\varphi(x)|)\int_{-\infty}^{\infty}(1{+}|x|^2)^{-1}dx \\ \le C\sup((1+x|^2)|\varphi(x)|).\end{aligned}$$

Thus $\mathcal{S}(\mathbb{R}) \ni \varphi \longmapsto \int_{\mathbb{R}}\varphi dx$ is continous.

Now, choose $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ with $\int_{\mathbb{R}}\phi(x)dx = 1$. Then, for $\psi \in \mathcal{S}(\mathbb{R})$, set

$$A\psi(x) = \int_{-\infty}^{x}(\psi(t) - c(\psi)\phi(t))\,dt,\ c(\psi) = \int_{-\infty}^{\infty}\psi(s)\,ds. \tag{4.1}$$

Note that the assumption on ϕ means that

$$A\psi(x) = -\int_{x}^{\infty}(\psi(t) - c(\psi)\phi(t))\,dt \tag{4.2}$$

Clearly $A\psi$ is smooth, and in fact it is a Schwartz function since

$$\frac{d}{dx}(A\psi(x)) = \psi(x) - c\phi(x) \in \mathcal{S}(\mathbb{R}) \tag{4.3}$$

so it suffices to show that $x^k A\psi$ is bounded for any k as $|x| \to \pm\infty$. Since $\psi(t) - c\phi(t) \le C_k t^{-k-1}$ in $t \ge 1$ it follows from (4.2) that

$$|x^k A\psi(x)| \le C x^k \int_x^\infty t^{-k-1} dt \le C',\ k > 1,\ \text{in } x > 1.$$

A similar estimate as $x \to -\infty$ follows from (4.1). Now, A is clearly linear, and it follows from the estimates above, including that on the integral, that for any k there exists C and j such that

$$\sup_{\alpha,\beta \le k} |x^\alpha D^\beta A\psi| \le C \sum_{\alpha',\beta' \le j} \sup_{x\in\mathbb{R}} |x^{\alpha'} D^{\beta'} \psi|.$$

Finally then, given $u \in \mathcal{S}'(\mathbb{R})$ define $v(\psi) = -u(A\psi)$. From the continuity of A, $v \in \mathcal{S}(\mathbb{R})$ and from the definition of A, $A(\psi') = \psi$. Thus

$$dv/dx(\psi) = v(-\psi') = u(A\psi') = u(\psi) \Longrightarrow \frac{dv}{dx} = u.$$

Solution 4.6. We have to prove that $\langle\xi\rangle^{m+m'}\widehat{u} \in L_2(\mathbb{R}^n)$, in other words, that

$$\int_{\mathbb{R}^n} \langle\xi\rangle^{2(m+m')} |\widehat{u}|^2 \, d\xi < \infty.$$

But that is true since

$$\int_{\mathbb{R}^n} \langle\xi\rangle^{2(m+m')} |\widehat{u}|^2 \, d\xi = \int_{\mathbb{R}^n} \langle\xi\rangle^{2m'} (1 + \xi_1^2 + \ldots + \xi_n^2)^m |\widehat{u}|^2 \, d\xi =$$

$$= \int_{\mathbb{R}^n} \langle\xi\rangle^{2m'} \left(\sum_{|\alpha|\le m} C_\alpha \xi^{2\alpha} \right) |\widehat{u}|^2 \, d\xi = \sum_{|\alpha|\le m} C_\alpha \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{2m'} \xi^{2\alpha} |\widehat{u}|^2 \, d\xi \right)$$

and since $\langle\xi\rangle^{m'}\xi^\alpha\widehat{u} = \langle\xi\rangle^{m'}\widehat{D^\alpha u}$ is in $L^2(\mathbb{R}^n)$ (note that $u \in H^m(\mathbb{R}^n)$ follows from $D^\alpha u \in H^{m'}(\mathbb{R}^n)$, $|\alpha| \le m$). The converse is also true since C_α in the formula above are strictly positive.

Solution 4.7. Take $v \in L^2(\mathbb{R}^n)$, and define subsets of $\mathbb{R}^n$ by

$$E_0 = \{x \colon |x| \le 1\},$$

$$E_i = \{x \colon |x| \ge 1, |x_i| = \max_j |x_j|\}.$$

Then obviously we have $1 = \sum_{i=0}^n \chi_{E_j}$ a.e., and $v = \sum_{j=0}^n v_j$ for $v_j = \chi_{E_j} v$. Then $\langle x\rangle$ is bounded by $\sqrt{2}$ on E_0, and $\langle x\rangle v_0 \in L^2(\mathbb{R}^n)$; and on E_j, $1 \le j \le n$, we have

$$\frac{\langle x\rangle}{|x_j|} \le \frac{(1 + n|x_j|^2)^{1/2}}{|x_j|} = \left(n + 1/|x_j|^2\right)^{1/2} \le (2n)^{1/2},$$

so $\langle x\rangle v_j = x_j w_j$ for $w_j \in L^2(\mathbb{R}^n)$. But that means that $\langle x\rangle v = w_0 + \sum_{j=1}^n x_j w_j$ for $w_j \in L^2(\mathbb{R}^n)$.
If u is in $L^2(\mathbb{R}^n)$ then $\widehat{u} \in L^2(\mathbb{R}^n)$, and so there exist $w_0, \dots, w_n \in L^2(\mathbb{R}^n)$ so that

$$\langle\xi\rangle\widehat{u} = w_0 + \sum_{j=1}^n \xi_j w_j,$$

in other words

$$\widehat{u} = \widehat{u}_0 + \sum_{j=1}^n \xi_j \widehat{u}_j$$

where $\langle\xi\rangle\widehat{u}_j \in L^2(\mathbb{R}^n)$. Hence

$$u = u_0 + \sum_{j=1}^n D_j u_j$$

where $u_j \in H^1(\mathbb{R}^n)$.

Solution 4.8. Since

$$D_x H(\varphi) = H(-D_x\varphi) = i\int_{-\infty}^{\infty} H(x)\varphi'(x)\,dx = i\int_0^{\infty}\varphi'(x)\,dx = i(0-\varphi(0)) = -i\delta(\varphi),$$

we get $D_x H = C\delta$ for $C = -i$.

Solution 4.9. It is equivalent to ask when $\langle\xi\rangle^m\widehat{\delta_0}$ is in $L^2(\mathbb{R}^n)$. Since

$$\widehat{\delta_0}(\psi) = \delta_0(\widehat{\psi}) = \widehat{\psi}(0) = \int_{\mathbb{R}^n}\psi(x)\,dx = 1(\psi),$$

this is equivalent to finding m such that $\langle\xi\rangle^{2m}$ has a finite integral over $\mathbb{R}^n$. One option is to write $\langle\xi\rangle = (1+r^2)^{1/2}$ in spherical coordinates, and to recall that the Jacobian of spherical coordinates in n dimensions has the form $r^{n-1}\Psi(\varphi_1, \dots, \varphi_{n-1})$, and so $\langle\xi\rangle^{2m}$ is integrable if and only if

$$\int_0^\infty \frac{r^{n-1}}{(1+r^2)^m}\,dr$$

converges. It is obvious that this is true if and only if $n-1-2m < -1$, ie. if and only if $m > n/2$.

Solution 4.10 (Solution to Problem31). We know that $\delta \in H^m(\mathbb{R}^n)$ for any $m < -n/1$. Thus is just because $\langle\xi\rangle^p \in L^2(\mathbb{R}^n)$ when $p < -n/2$. Now, divide $\mathbb{R}^n$ into $n+1$ regions, as above, being $A_0 = \{\xi; |\xi| \le 1$ and $A_i = \{\xi; |\xi_i| = \sup_j |\xi_j|,\ |\xi| \ge 1\}$. Let v_0 have Fourier transform χ_{A_0} and for $i = 1, \dots, n$, $v_i \in \mathcal{S}; (\mathbb{R}^n)$ have Fourier transforms $\xi_i^{-n-1}\chi_{A_i}$. Since $|\xi_i| > c\langle\xi\rangle$ on the support of $\widehat{v}_i$ for each $i = 1, \dots, n$, each term is in H^m for any $m < 1 + n/2$ so, by the Sobolev embedding theorem, each $v_i \in \mathcal{C}_0^0(\mathbb{R}^n)$ and

$$1 = \hat{v}_0 \sum_{i=1}^{n} \xi_i^{n+1} \widehat{v_i} \Longrightarrow \delta = v_0 + \sum_i D_i^{n+1} v_i. \tag{4.4}$$

How to see that this cannot be done with n or less derivatives? For the moment I do not have a proof of this, although I believe it is true. Notice that we are actually proving that δ can be written

$$\delta = \sum_{|\alpha| \le n+1} D^\alpha u_\alpha, \; u_\alpha \in H^{n/2}(\mathbb{R}^n). \tag{4.5}$$

This cannot be improved to n from $n+1$ since this would mean that $\delta \in H^{-n/2}(\mathbb{R}^n)$, which it isn't. However, what I am asking is a little more subtle than this.

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