

**PROBLEM SET 6 FOR 18.102**  
**DUE FRIDAY MARCH 25, 2015**  
**OF COURSE YOU CAN SUBMIT IT EARLIER!.**

Problem 6.1

Recall the space  $h^{2,1}$  (discussed in the preceding problem set) consisting of the complex valued sequences  $c_i$  such that

$$\|c\|^2 = \sum_i (1 + |i|^2) |c_i|^2 < \infty.$$

Show that the unit ball in this space, considered as a subset of  $l^2$ , has compact closure.

Problem 6.2

Define the space  $\mathcal{L}^2(0, 1)$  as consisting of those elements of  $L^2(\mathbb{R})$  which vanish outside  $(0, 1)$  and show that the quotient  $L^2(0, 1) = \mathcal{L}^2(0, 1)/\mathcal{N}(0, 1)$  by the null functions in  $\mathcal{L}^2(0, 1)$  is a Hilbert space.

Remark: This is indeed easy, but make sure you do it properly (for instance identify  $L^2(0, 1)$  with a closed subspace of  $L^2(\mathbb{R})$  by treating the null functions properly).

Problem 6.3

Identify  $\mathcal{C}[0, 1]$ , the space of continuous functions on the closed interval, as a subspace of  $L^2(0, 1)$ . For each  $n \in \mathbb{N}$  let  $F_n \subset L^2(0, 1)$  be the subspace of functions which are constant on each interval  $((m-1)/n, m/n]$  for  $m = 1, \dots, n$ . Show that if  $f \in \mathcal{C}[0, 1]$  there exists  $g_n \in F_n$  such that

$$\delta_n = \sup_{|t-s| \leq 1/n} |f(t) - f(s)| \implies \|f - g_n\|_{L^2} \leq \delta_n.$$

Problem 6.4

Show that a bounded and equicontinuous subset of  $\mathcal{C}[0, 1]$  has compact closure in  $L^2(0, 1)$ . Note that equicontinuity means ‘uniform equicontinuity’ so for each  $\epsilon > 0$  there exists  $\delta > 0$  such that  $|x - y| < \delta$  implies  $|f(x) - f(y)| < \epsilon$  for all elements  $f$  of the set.

Problem 6.5

If  $K \in \mathcal{C}([0, 1] \times [0, 1])$  is a continuous function of two variables, show that

$$(6.1) \quad Af(x) = \int K(x, y)f(y)$$

defines a compact linear operator on  $L^2(0, 1)$ .

Hint: Show that  $A$  defines a bounded linear map from  $L^2(0, 1)$  to  $\mathcal{C}[0, 1]$  and that the image of the unit ball is *equicontinuous* using the uniform continuity of  $K$ .

## Problem 6.6 – extra

Show that a closed and bounded subset of  $L^2(\mathbb{R})$  is compact if and only if it is ‘uniformly equicontinuous in the mean’ and ‘uniformly small at infinity’ so that for each  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$\int_{-1/\delta}^{1/\delta} |f|^2 < \epsilon^2 \text{ and } |t| < \delta \implies \int |f(x) - f(x-t)|^2 < \epsilon^2$$

for all elements of the set.

## Problem 6.7 – extra

Consider the space of continuous functions on  $\mathbb{R}$  vanishing outside  $(0, 1)$  which are of the form

$$u(x) = \int_0^x v, \quad v \in L^2(0, 1).$$

Show that these form a Hilbert space and that the unity ball of this space has compact closure in  $L^2(0, 1)$ .