PROBLEM SET 4 FOR 18.102, SPRING 2016
DUE FRIDAY 26 FEBRUARY, IN THE USUAL SENSE

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As you are probably realize, there is a test on Tuesday March 1, in class. I think the schedule has been a bit punishing this semester so I have decided to make the test really boring. Namely there will be three questions all drawn from the homework so far, including one from the first five questions below.

Problem 4.1

Combining the original definition with Lebesgue's dominated convergence, show that $f : \mathbb{R} \longrightarrow \mathbb{C}$ is in $\mathcal{L}^1(\mathbb{R})$ if and only if there exists a sequence $u_n \in \mathcal{C}(\mathbb{R})$ and $F \in \mathcal{L}^1(\mathbb{R})$ such that $|u_n(x)| \le F(x)$ a.e. and $u_n(x) \to f(x)$ a.e.

Hint: LDC is your friend to show that if u is continuous and $|u| \le F$ a.e. with $F \in \mathcal{L}^1$ then $u \in \mathcal{L}^1$.

Problem 4.2

Define $\mathcal{L}^\infty(\mathbb{R})$ as the set of functions $g : \mathbb{R} \longrightarrow \mathbb{C}$ such that there exists $C > 0$ and $v_n \in \mathcal{C}(\mathbb{R})$ with $|v_n(x)| \le C$ and $v_n(x) \to g(x)$ a.e. Show that $\mathcal{L}^\infty$ is a linear space, that

$$\|g\|_\infty = \inf\{\sup_{\mathbb{R}\setminus E} |g(x)|; E \text{ has measure zero and } \sup_{\mathbb{R}\setminus E} |g(x)| < \infty\}$$

is a seminorm on $\mathcal{L}^\infty(\mathbb{R})$ and that this makes $L^\infty(\mathbb{R}) = \mathcal{L}^\infty(\mathbb{R})/\mathcal{N}$ into a Banach space, where $\mathcal{N}$ is the space of null functions.

Hint(s): This is not the only way to do it but it is the hint I have given to those who asked.

- It is a good idea, although not essential, to show that for any one element of $\mathcal{L}^\infty$ there is a set of measure zero, E, such that
$$\|g\|_\infty = \sup_{\mathbb{R}\setminus E} |g(x)|.$$
- For completeness first check that it is enough to deal with real-valued Cauchy sequence and then show that such a sequence (or absolutely summable if you prefer) converges a.e. and defines a bounded function u (a.e. as usual) which is the putative limit.
- Now, using e.g. LDC show that for $N \in \mathbb{Z}$, $\chi_{[N-1,N+1]}u \in \mathcal{L}^1(\mathbb{R})$ and hence that there exists a sequence $u_{n.N} \in \mathcal{C}_{\rm c}(\mathbb{R})$ which converges a.e. to $\chi_{[N-,N+1]}u$ and such that $|u_{n,N}(x)| \le R$ where R is fixed (find a sequence which might not be bounded and then cut it off).
- Now, this is the point that hangs most people up, look at the function v_0 defined to be zero on $[-\infty, -3/4]$ linear and positive on $[-3/4, -1/4]$, 1 on $[-1/4, 1/4]$ and even so that it is continuous. Show that $\sum\limits_N v_0(x-N) = 1$ (assuming I got it right but just fix it otherwise) with the sum finite on any

compact set. Now show that $v_n(x) = \sum_N v_0(x-N)u_{n,N}(x)$ is a sequence as needed to show that $u \in \mathcal{L}^\infty(\mathbb{R})$.

- Don't forget to check convergence.

Problem 4.3

Show that if $g \in \mathcal{L}^\infty(\mathbb{R})$ and $f \in \mathcal{L}^1(\mathbb{R})$ then $gf \in \mathcal{L}^1(\mathbb{R})$ and that this defines a map

$$L^\infty \times L^1(\mathbb{R}) \longrightarrow L^1(\mathbb{R})$$

which satisfies $\|gf\|_{L^1} \le \|g\|_{L^\infty}\|f\|_{L^1}$.

Problem 4.4

Define a set $U \subset \mathbb{R}$ to be (Lebesgue) measurable if its characteristic function

$$\chi_U(x) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases}$$

is in $\mathcal{L}^\infty(\mathbb{R})$. Letting $\mathcal{M}$ be the collection of measurable sets, show

(1) $\mathbb{R} \in \mathcal{M}$
(2) $U \in \mathcal{M} \Longrightarrow \mathbb{R} \setminus U \in \mathcal{M}$
(3) $U_j \in \mathcal{M}$ for $j \in \mathbb{N}$ then $\bigcup_{j=1}^\infty U_j \in \mathcal{M}$
(4) If $U \subset \mathbb{R}$ is open then $U \in \mathcal{M}$

Hint: For $U = \bigcup_{j=1}^\infty U_j$. First show that $\chi_{[-N,N]}\chi_U \in \mathcal{L}^1(\mathbb{R})$ for each N and then that it is in $\mathcal{L}^\infty(\mathbb{R})$. Then try the same argument as in the hint for P4.2.

Problem 4.5

If $U \subset \mathbb{R}$ is measurable and $f \in \mathcal{L}^1(\mathbb{R})$ show that

$$\int_U f = \int \chi_U f \in \mathbb{C}$$

is well-defined. Prove that if $f \in \mathcal{L}^1(\mathbb{R})$ then

$$I_f(x) = \begin{cases} \int_{(0,x)} f & x \ge 0 \\ -\int_{(0,-x)} f & x < 0 \end{cases}$$

is a bounded continuous function on $\mathbb{R}$.

Hint: Use the density of $\mathcal{C}_c(\mathbb{R})$ in the last part – given $\epsilon > 0$ choose $u \in \mathcal{C}_c(\mathbb{R})$ such that $\int |f-u| < e/2$. Now use the continuity of the Riemann integral with respect to the ends of the interval.

Problem 4.6 – Extra

Recall (from Rudin's book for instance) that if $F : [a,b] \longrightarrow [A,B]$ is an increasing continuously differentiable map, in the strong sense that $F'(x) > 0$, between finite intervals then for any continuous function $f : [A,B] \longrightarrow \mathbb{C}$, (Rudin shows it for Riemann integrable functions)

$$\int_A^B f(y)dy = \int_a^b f(F(x))F'(x)dx. \tag{1}$$

Prove the correspondng identity for every $f \in \mathcal{L}^1((A,B))$, which in particular requires the right side to make sense.

Problem 4.7 – Extra

Show that if $f \in \mathcal{L}^1(\mathbb{R})$ and I_f in Problem 4.5 vanishes identically then $f \in \mathcal{N}$.

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