PROBLEM SET 3 FOR 18.102, SPRING 2016
DUE ELECTRONICALLY BY FRIDAY 19 FEBRUARY

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Recall that we have defined a set $E \subset \mathbb{R}$ to be 'of measure zero' if there exists a sequence $f_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ with $\sum\limits_n \int |f_n| < \infty$ such that

$$E \subset \{x \in \mathbb{R}; \sum_n |f_n(x)| = +\infty\}. \tag{1}$$

By Thursday 18 February I expect to show in class that if $f_n \in \mathcal{L}^1(\mathbb{R})$ is a sequence with $\sum\limits_n \int |f_n| < \infty$ then $\sum\limits_n f_n(x)$ converges a.e. and the limit, defined in any way off the set of convergence, is an element of $\mathcal{L}^1(\mathbb{R})$. You may, and probably should, use this below.

Problem 3.1

Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is such that there is a sequence $v_n \in \mathcal{L}^1(\mathbb{R})$ with real values, such that $v_n(x)$ is increasing for each x, $\int v_n$ is bounded and

$$\lim_n v_n(x) = f(x) \tag{2}$$

whenever the limit exists. Show that $f \in \mathcal{L}^1(\mathbb{R})$. Hint: Write this sequence as the sequence of partial sums of another sequence which is absolutely summable.

Problem 3.2

(1) Suppose that $O \subset \mathbb{R}$ is a *bounded* open subset, so $O \subset (-R, R)$ for some R. Show that the characteristic function of O

$$\chi_O(x) = \begin{cases} 1 & x \in O \\ 0 & x \notin O \end{cases} \tag{3}$$

is an element of $\mathcal{L}^1(\mathbb{R})$. Hint: Recall the structure of an open set in $\mathbb{R}$ and use the previous problem.

(2) If O is bounded and open define the length (or Lebesgue measure) of O to be $l(O) = \int \chi_O$. Show that if $U = \bigcup_j O_j$ is a (n at most) countable union of bounded open sets such that $\sum\limits_j l(O_j) < \infty$ then $\chi_U \in \mathcal{L}^1(\mathbb{R})$; again we set $l(U) = \int \chi_U$.

(3) Conversely show that if U is open and $\chi_U \in \mathcal{L}^1(\mathbb{R})$ then $U = \bigcup_j O_j$ *is* the union of a countable collection of bounded open sets with $\sum\limits_j l(O_j) < \infty$.

(4) Show that if $K \subset \mathbb{R}$ is compact then its characteristic function is an element of $\mathcal{L}^1(\mathbb{R})$.

Problem 3.3

Suppose $F \subset \mathbb{R}$ has the following (well-known) property:-

$$\forall\ \epsilon > 0\ \exists \text{ a countable collection of open sets } O_i \text{ s.t.}$$

$$\sum_i l(O_i) < \epsilon,\ F \subset \bigcup_i O_i. \tag{4}$$

Show that F is a set of measure zero in the sense above (the same sense as in lectures).

Problem 3.4

Suppose $f_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ is an absolutely summable sequence; set

$$E(f) = \{x \in \mathbb{R}; \sum_n |f_n(x)| = \infty\}. \tag{5}$$

(1) Show that if $a > 0$ then the set

$$\{x \in \mathbb{R}; \sum_n |f_n(x)| \le a\} \tag{6}$$

is closed.

(2) Deduce that if $\epsilon > 0$ is given then there is an open set $O_\epsilon \supset E$ with $\sum_n |f_n(x)| > 1/\epsilon$ for each $x \in O_\epsilon$.

(3) Deduce that the characteristic function of O_ϵ is in $\mathcal{L}^1(\mathbb{R})$ and that $l(O_\epsilon) \le \epsilon C$, $C = \sum_n \int |f_n(x)|$.

(4) Conclude that $E(f)$ satisfies the condition (4).

Problem 3.5

Show that the function with $F(0) = 0$ and

$$F(x) = \begin{cases} 0 & x > 1 \\ \exp(i/x) & 0 < |x| < 1 \\ 0 & x < -1, \end{cases}$$

is an element of $\mathcal{L}^1(\mathbb{R})$.

Problem 3.6 – Extra

(1) Recall the definition of a Riemann integrable function $g : [a, b] \longrightarrow \mathbb{R}$ – that it is bounded and there exist a sequence of successively finer partitions for which the upper Riemann sum approaches the lower Riemann sum. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the extension of this function to be zero outside the interval.

(2) Translate this condition into a statement about two sequences of piecewise-constant functions (with respect to the partition), u_n, l_n with $l_n(x) \le f(x) \le u_n(x)$ and conclude that $\int(u_n - l_n) \to 0$.

(3) Deduce that $f \in \mathcal{L}^1(\mathbb{R})$ and the Lebesgue integral of f on $\mathbb{R}$ is equal to the Riemann integral of g on $[a, b]$.

(4) Show that there is a Lebesgue integrable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which vanishes outside $[a, b]$ but that no function equal to it a.e. can be Riemann integrable.

Problem 3.7 – Extra

Prove that the Cantor set has measure zero.

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