SOLUTIONS TO 18.102/18.1021 FINAL EXAM, SPRING 2016

Problem 1

Consider the subspace $H \subset \mathcal{C}[0, 2\pi]$ consisting of those continuous functions on $[0, 2\pi]$ which satisfy

$$u(x) = \int_0^x U(x), \ \forall \ x \in [0, 2\pi]$$

for some $U \in L^2(0, 2\pi)$ (depending on u of course). Show that the function U is determined by u (given that it exists) and that

$$\|u\|_H^2 = \int_{(0,2\pi)} |U|^2$$

turns H into a Hilbert space.

Solution: Since $L^2(0, 2\pi) \subset L^1(0, 2\pi)$ if $U \in L^2(0, 2\pi)$ then

$$u(x) = \int_0^x U \text{ is well-defined and for } x \geq y$$

$$|u(x) - u(y)| \leq \int_y^x |U| \leq |x - y|^{\frac{1}{2}} \|U\|_{L^2}$$

by Cauchy-Schwartz. Thus u is continuous and this defines a bounded linear operator $T : L^2(0, 2\pi) \longrightarrow \mathcal{C}[0, 2\pi]$ by linearity of the integral. To say that U is determined by u if it exists is to say this operator is injective. If $TU = u = 0$ then from the estimate above

$$\int U\chi_{(x,y)} = 0 \ \forall \ (x, y) \subset (0, 2\pi).$$

Thus U is orthogonal to all characteristic functions of intervals and hence to all step functions. However, we showed that the step functions are dense in $L^2(0, 2\pi)$ (by showing they were dense in continuous functions in the supremum norm) so $U = 0$.

Thus T is injective and so a bijection onto its range which is H. By definition $\|u\|_H = \|U\|_{L^2}$ if $u = TU$ so H is isometrically isomorphic to $L^2(0, 2\pi)$ and hence is a Hilbert space.

Remark: Usual problem is failure to show injectivity of T – differentiation is not an option. Even the surjectivity of T requires a comment.

Problem 2

Suppose $A \in \mathcal{B}(H)$ has closed range. Show that A has a generalized inverse, $B \in \mathcal{B}(H)$ such that

$$AB = \Pi_R, \ BA = \operatorname{Id} - \Pi_N \tag{1}$$

where Π_N and Π_R are the orthonormal projections onto the null space and range of A respectively.

Solution: Since R, the range of A, is closed it is a Hilbert space and $A : N^\perp \longrightarrow R$, where N is the null space of A, is a bounded linear bijection between Hilbert spaces. By the Open Mapping Theorem it has a bounded inverse $E : R \longrightarrow N$. Define

$$Bu = E\Pi_R u \in N^\perp \subset H,\ B : H \longrightarrow H.$$

Then B is bounded as the composite of bounded operators and

$$BAu = E\Pi_R Au = EAu = u,\ \text{if } u \in N^\perp,\ BAu = 0 \text{ if } u \in N,$$
$$ABu = AEu = u \text{ if } u \in R,\ ABu = 0 \text{ if } u \in R^\perp$$

which is just (1).

Remark: Usual problem was failure to show boundedness of the generalized inverse.

Problem 3

Let $A_j \subset \mathbb{R}$ be a sequence of subsets with the property that the characteristic function, χ_j of A_j, is integrable for each j. Show that the characteristic function of $\mathbb{R} \setminus A$, where $A = \bigcup_j A_j$ is locally integrable.

Solution: The χ_j are real-valued integrable functions hence so is $\chi_{[k]} = \max_{j\le k} \chi_j$ which is the characteristic function of $A_{[k]} = \bigcup j \le k A_j$. If $\chi_{[-R,R]}$ then $f_k = \chi_{[-R,R]}\chi_{[k]} \le \chi_{[-R,R]}$ is the characteristic function of $A_{[k]} \cap [-R,R]$ and these form an increasing sequence which converges pointwise to the characteristic function, $\chi_A\chi_{[-R,R]}$ of $A \cap [-R,R]$. By Monotone Convergence it follows that this is integrable, hence so is $(1-\chi_A)\chi_{[-R,R]}$ which is the characteristic function of $(\mathbb{R} \setminus A) \cap [-R,R]$. So the characteristic function of $\mathbb{R} \setminus A$ is locally integrable.

Remark: A gift.

Problem 4

Let $A \in \mathcal{B}(H)$ be such that

$$\sup \sum_i |\langle Ae_i, f_i\rangle| < \infty$$

where the supremum is over orthonormal bases e_i and f_j. Use the polar decomposition to show that $A = B_1B_2$ where the $B_k \in \mathcal{B}(H)$ are Hilbert-Schmidt operators, i.e. $\sum_i \|B_k e_i\|^2 < \infty$, $k = 1, 2$.

Solution: Let C be this supremum. Since all terms are positive it follows that

$$\sum_i |\langle Ae_i, f_i\rangle| \le C$$

for all finite, and hence for all, orthonormal *sequences* $\{e_i\}$ and $\{f_j\}$.

Consider the polar decomposition of $A = UE$, $E = (A^*A)^{\frac{1}{2}}$ where U is a partial isometry from $\operatorname{Nul}(A)^\perp = \operatorname{Nul}(E)^\perp$ to R, the closure of the range of A. Thus if e_i is an orthonormal basis for $\operatorname{Nul}(E)^\perp$ then $e_i = U^* f_i$ where f_i is an orthonormal sequence (in fact an orthonormal basis of R). It follows that

$$\sum_i |\langle UEe_i, f_i\rangle| = \sum_i \langle Ee_i, e_i\rangle \le C$$

since E is non-negative. Taking $B_2 = E^{\frac{1}{2}}$ using the Functional Calculus,

$$\sum_i \|B_2 e_i\|^2 \le C$$

for an orthonormal basis of $\mathrm{Nul}(B_2)^\perp$ and hence for some orthonormal basis of H. Thus B_2 is Hilbert-Schmidt and since the Hilbert-Schmidt operators form an ideal, so is $B_1 = UB_2$. Thus $A = B_1 B_2$ is the product of two Hilbert-Schmidt operators.

Remark: There is a problem just taking e_i to include an onb of $\mathrm{Nul}(A)^\perp$ such that f_i is an onb which includes the $U^* e_i$ – mainly numbering since for instance $\mathrm{Nul}(A)$ might be $\{0\}$ but R could have infinite codimension. The initial discussion above is to avoid this small issue – I did not take off marks for this.

Problem 5

Let $a \in \mathcal{C}([0, 1])$ be a real-valued continouous function. Show that multiplication by a defines a self-adjoint operator on $L^2(0, 1)$ which has spectrum exactly the range $a([0, 1]) \subset \mathbb{R}$.

Solution: Multiplication by a continuous function is a bounded operator on $L^2(0, 1)$ since if $u \in L^2(0, 1)$ and $a \in \mathcal{C}([0, 1])$ then $au \in L^2(0, 1)$ (almost from the original definition), depends linearly on u and

$$\int \|au\|^2 \le \sup |a|^2 \int |u|^2.$$

If $\lambda \notin a([0, 1])$ then $\inf_x |a(x) - \lambda| > 0$ so $(a - \lambda)^{-1} \in \mathcal{C}([0, 1])$ and B_λ, defined as multiplication by $(a - \lambda)^{-1}$ is a 2-sided inverse of $A - \lambda$. Thus the spectrum of A is contained in $a([0, 1])$.

So consider $\lambda \in a([0, 1])$, thus by compactness of $[0, 1]$ there exists $x_0 \in [0, 1]$ with $a(x_0) = \lambda$. If $A - \lambda$ was invertible there would be a bounded operator B such that $B(A - \lambda)u = u$ for all $u \in L^2(0, 1)$ and hence a constant C such that

$$\|u\| \le C\|(A - \lambda)u\|.$$

By continuity of a there is some $\delta > 0$ such that on one of the intervals $I = [x_o, x_0 + \delta]$ or $I = [x_0 - \delta, x_0]$ $|a(x) - \lambda| \le 1/2C$. If u has support in U and norm 1 then $\|(A - \lambda)u\| \le \frac{1}{2}C\|u\|$ which contradicts the estimate above. Thus $\mathrm{Spec}(A) = a([0, 1])$.

Remark: Self-adjointness (and indeed reality of a) is not used here. Main problem was failing to properly show that every λ in the range of a is in the spectrum. Note that even then $a - \lambda$ is usually injective (unless a takes the value λ on a set of positive measure) but it is not invertible.