Question 1
Show that if $A \in \mathcal{B}(H)$, where $H$ is a separable Hilbert space, has the property that $u_n \rightharpoonup u$ (converges weakly) implies that $Au_n$ is Cauchy. Show that $A(\{u \in H; \|u\| \leq 1\})$ is compact.

Question 2
If $H$ is an infinite-dimensional separable Hilbert space, define a set of operators $T(H)$ by the property $A \in T(H) \text{ iff } A \in \mathcal{B}(H) \text{ and } |A| \in \mathcal{K}(H), \sum_i \lambda_i(|A|) < \infty$
where the $\lambda_i(|A|)$ are the eigenvalues of $|A|$ repeated with multiplicity. Show that $T(H)$ is a 2-sided $*$-closed ideal.

Question 3
Suppose $A \in \mathcal{B}(H)$ is a bounded operator on a Hilbert space such that $A^*A$ is a compact operator. Show that $A$ is a compact operator.

Question 4
If $H$ is a separable, infinite dimensional, Hilbert space set

$$l^2(H) = \{ u : \mathbb{N} \rightarrow H; \|u\|^2_{l^2(H)} = \sum_i \|u_i\|^2_H < \infty \}.$$ 

Show that $l^2(H)$ has a Hilbert space structure and construct an explicit isometric (norm-preserving) isomorphism (bijection) from $l^2(H)$ to $H$.

Question 5
Starting from the definition of weak convergence of a sequence in a separable Hilbert space, $v_n \rightharpoonup v$, that $\langle v_n, w \rangle \rightarrow \langle v, w \rangle$ in $\mathbb{C}$ for each $w \in H$, show that a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if

$$\|v\|_H = \lim_{n \rightarrow \infty} \|v_n\|_H.$$
Question 6

Let $e_k, k \in \mathbb{N}$, be an orthonormal basis in a separable Hilbert space, $H$. Show that there is a unique bounded linear operator $T : H \to H$ satisfying

$$T e_j = e_{j-1} \quad \forall j \geq 2, \quad T e_1 = 0,$$

and that if $B \in \mathcal{B}(H)$ has $\|B\| < 1$ then $T + B$ has one-dimensional null space.

Question 7

Suppose $H$ is an infinite dimensional separable Hilbert space with an orthonormal basis $\{e_k\}_{k=1}^{\infty}$. Show that a continuous function $K : [0,1] \to H$ has the property that the Fourier-Bessel series of $K(x) \in H$, for $x \in [0,1]$, converges uniformly in the sense that if $K_n(x) = \sum_{k \leq n} \langle K(x), e_k \rangle e_k$ then $K_n : [0,1] \to H$ is also continuous and

$$\sup_{x \in [0,1]} \|K(x) - K_n(x)\|_H \to 0.$$  

Question 8

If $A \in \mathcal{B}(H)$ is a bounded operator on a separable, infinite-dimensional, Hilbert space $H$, explain why $|A| = (A^* A)^{1/2} \in \mathcal{B}(H)$ is well-defined. Suppose that $|A|$ is compact and for a orthonormal basis of $H$ consisting of eigenvectors of $|A|$,

$$\sum_{i} \|A e_i\|^2 < \infty.$$  

Show that

$$\sum_{i} \|A f_i\|^2 < \infty$$

for any orthonormal sequence $f_j$.

Question 9

Show that a separable Hilbert space $H$ with the property that every bounded operator from $H$ to itself is compact is necessarily finite dimensional.

Question 10

Show that if $B$ is a compact operator on a separable Hilbert space $H$ and $A$ is an invertible operator then

$$\{u \in H; Bu = Au\}$$

is finite dimensional.

Question 11
Assuming the Stone-Weierstrass theorem, show that there is a complete orthonormal basis of $L^2([0, 2\pi])$ consisting of polynomials.

Question 12

Let $H$ be a separable Hilbert space and let $C_c(\mathbb{R}; H)$ be the linear space of continuous maps from $\mathbb{R}$ to $H$ which vanish outside some interval $[-R, R]$ depending on the function. Show that

\begin{equation}
\|u\|^2 = \int_{\mathbb{R}} \|u(x)\|^2_H
\end{equation}

defines a norm which comes from a pre-Hilbert structure on $C_c(\mathbb{R}; H)$. Show that if $u_n$ is a Cauchy sequence in this pre-Hilbert space and $h \in H$ then $\langle u_n(x), h \rangle_H$ converges in $L^2(\mathbb{R})$. 

Department of Mathematics, Massachusetts Institute of Technology

Email address: rbm@math.mit.edu