You may freely use the facts established this week, that the Fourier transform

\[ \hat{f}(\xi) = \int e^{-ix\xi} f(x) dx, \quad f \in L^1(\mathbb{R}) \]

restricts to the Schwartz space as an isomorphism and extends by continuity to an isomorphism of \( L^2(\mathbb{R}) \). Since it is the end of the semester there is only one question – except it has five parts!

Define the first Sobolev space, \( H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \) as consisting of those \( f \in L^2(\mathbb{R}) \) such that \( \xi \hat{f} \in L^2(\mathbb{R}) \).

1. Show that this is a Hilbert space with the norm

\[ \| f \|_{H^1}^2 = \int_{\mathbb{R}} (|\hat{f}(\xi)|^2 + |\xi|^2 |\hat{f}(\xi)|^2) d\xi \]

Remark. There is not very much to show, but you should do it carefully.

2. Show that \( H^1(\mathbb{R}) \) consists of bounded continuous functions:-

3. \( H^1(\mathbb{R}) \subset C_{\infty}(\mathbb{R}), \sup |f| \leq C\|f\|_{H^1} \)

3. Define

\[ D : H^1(\mathbb{R}) \to L^2(\mathbb{R}), \quad \widehat{Df} = \xi \hat{f} \]

and show that it is a bounded linear operator.

4. Suppose \( g \in L^2(\mathbb{R}) \), show that there is a well-defined element \( Qg \in H^1(\mathbb{R}) \) such that

\[ \langle f, g \rangle_{L^2} = \langle f, Qg \rangle_{H^1} \quad \forall f \in H^1(\mathbb{R}) \]

5. Show that as a map \( Q : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) this is a bounded self-adjoint operator.

\[ \text{Department of Mathematics, Massachusetts Institute of Technology} \]
\[ \text{Email address: rwm@math.mit.edu} \]