PROBLEM SET 9 FOR 18.102, SPRING 2018
DUE FRIDAY MAY 10.

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You may freely use the facts established this week, that the Fourier transform

$$\hat{f}(\xi) = \int e^{-ix\xi} f(x)dx, \; f \in L^1(\mathbb{R}) \tag{1}$$

restricts to the Schwartz space as an isomorphism and extends by continuity to an isomorphism of $L^2(\mathbb{R})$. Since it is the end of the semester there is only one question – except it has five parts!

Define the first Sobolev space, $H^1(\mathbb{R}) \subset L^2(\mathbb{R})$ as consisting of those $f \in L^2(\mathbb{R})$ such that $\xi \hat{f} \in L^2(\mathbb{R})$.

(1) Show that this is a Hilbert space with the norm

$$\|f\|^2_{H^1} = \int_{\mathbb{R}} (|\hat{f}(\xi)|^2 + |\xi|^2 |\hat{f}(\xi)|^2) d\xi \tag{2}$$

Remark. There is not very much to show, but you should do it carefully.

(2) Show that $H^1(\mathbb{R})$ consists of bounded continuous functions:-

$$H^1(\mathbb{R}) \subset \mathcal{C}_\infty(\mathbb{R}), \; \sup |f| \le C\|f\|_{H^1} \tag{3}$$

(3) Define

$$D : H^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \; \widehat{Df} = \xi \hat{f} \tag{4}$$

and show that it is a bounded linear operator.

(4) Suppose $g \in L^2(\mathbb{R})$, show that there is a well-defined element $Qg \in H^1(\mathbb{R})$ such that

$$\langle f, g \rangle_{L^2} = \langle f, Qg \rangle_{H^1} \; \forall \; f \in H^1(\mathbb{R}). \tag{5}$$

(5) Show that as a map $Q : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ this is a bounded self-adjoint operator.

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