PROBLEM SET 7 FOR 18.102
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For problem 7.5 below you can use the fact, which I hope will be proved in time, namely that the trigonometric functions

$$\frac{e^{ikx}}{\sqrt{2\pi}},\ k \in \mathbb{Z} \tag{7.1}$$

form an orthonormal basis of $L^2(0,2\pi)$ – it is the completeness which is not obvious. The Fourier coefficients of a function $a \in L^2(0,2\pi)$ are normalized below to be

$$a_j = \int_{(0,2\pi)} a(x)e^{-ijx} \tag{7.2}$$

so there are some factors of $\sqrt{2\pi}$ to take care of.

Problem 7.1

Suppose that $E \in \mathcal{B}(H)$ is a compact self-adjoint operator on a separable Hilbert space and that E is non-negative in the sense that

$$(Eu,u) \geq 0\ \forall\ u \in H.$$

Show that E has no negative eigenvalues and that the positive eigenvalues can be arranged in a (weakly) decreasing sequence

$$s_1 \geq s_2 \geq \cdots \to 0$$

either finite, or decreasing to zero, such that if $F \subset H$ has dimension N then

$$\min_{u\in F, \|u\|=1}(Eu,u) \leq s_N,\ \forall\ N.$$

NB. The s_j have to be repeated corresponding to the dimension of the associated eigenspace.

Problem 7.2

Extend this further to show that under the same conditions on E the eigenvalues are give by the minimax formula:-

$$s_j(E) = \max_{F\subset H; \dim F = j}\left(\min_{u\in F; \|u\|=1}(Eu,u)\right).$$

Problem 7.3

With E as above, suppose that $D \in \mathcal{B}(H)$ is a bounded self-adjoint operator. Show that

$$s_j(DED) \leq \|D\|^2 s_j(E)\ \forall\ j.$$

NB. Be a bit careful about the minimax argument.

Problem 7.4

Let A be a self-adjoint Hilbert-Schmidt operator (see an earlier problem set). Explain why the eigenspaces for non-zero eigenvalues, λ_j, of A are finite dimensional and show that

$$\sum_j \lambda_j^2 < \infty.$$

Problem 7.5

Suppose $a \in \mathcal{C}^0([0, 2\pi]^2)$ is a continuous function of two variables. Show that the Fourier coefficients of a in the second variable are continuous functions of the first variable and hence that the double Fourier coefficients

$$a_{jk} = \int_0^{2\pi} \int_0^{2\pi} a(x, y) e^{-ijx - iky} dy dx$$

are well-defined. If A is the integral operator 'with kernel a', so

$$(Af)(x) = \int_0^{2\pi} a(x, y) f(y) dy, \ f \in L^2(0, 2\pi)$$

show that

$$\sum_{k \in \mathbb{Z}} \|Ae^{iky}\|^2_{L^2(0,2\pi)} < \infty$$

and so conclude that A is a Hilbert-Schmidt operator. What does this imply about the coefficients a_{jk}?

Hint: Think about

$$\sum_{k=1}^{N} |c_k(x)|^2$$

where x is fixed and $c_k(x) = A(e^{ik\cdot})$. From the definition of A you can think of this as an inner product and so it can be bounded by an integral using Bessel's inequality. Integrate both sides in x and deduce that the integrated sum has a bound independent of N.

Problem 7.6 – extra

Consider the notion of an *unbounded* self-adjoint operator (since so far an operator is bounded, you should think of this as unbounded-self-adjoint-operator, a new notion which does include bounded self-adjoint operators). Namely, if H is a separable Hilbert space and $D \subset H$ is a *dense* linear subspace then a linear map $A : D \longrightarrow H$ is an unbounded self-adjoint operator if

(1) For all $v, w \in D$, $\langle Av, w \rangle_H = \langle v, Aw \rangle_H$.
(2) $\{u \in H; D \ni v \longmapsto \langle Av, u \rangle \in \mathbb{C}$ extends to a continuous map on $H\} = D$.

Show that

$$\mathrm{Gr}(A) = \{(u, Au) \in H \times H; u \in D\} \tag{7.3}$$

is a closed subspace of $H \times H$ and that $A + i\,\mathrm{Id} : D \longrightarrow H$ is surjective with a bounded two-sided inverse $B : H \longrightarrow H$ (with range D of course).

Problem 7.7 – extra

Suppose A is a compact self-adjoint operator on a separable Hilbert space and that $\mathrm{Nul}(A) = \{0\}$. Define a dense subspace $D \subset H$ in such a way that $A^{-1} : D \longrightarrow H$ is an unbounded self-adjoint operator which is a two-sided inverse of A.

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