For problem 7.5 below you can use the fact, which I hope will be proved in time, namely that the trigonometric functions

\[ e^{ikx} \sqrt{2\pi}, \quad k \in \mathbb{Z} \]

form an orthonormal basis of \( L^2(0, 2\pi) \) – it is the completeness which is not obvious. The Fourier coefficients of a function \( a \in L^2(0, 2\pi) \) are normalized below to be

\[ a_j = \int_{(0, 2\pi)} a(x)e^{-ijx} \]

so there are some factors of \( \sqrt{2\pi} \) to take care of.

**Problem 7.1**

Suppose that \( E \in \mathcal{B}(H) \) is a compact self-adjoint operator on a separable Hilbert space and that \( E \) is non-negative in the sense that \( (Eu, u) \geq 0 \forall u \in H \).

Show that \( E \) has no negative eigenvalues and that the positive eigenvalues can be arranged in a (weakly) decreasing sequence

\[ s_1 \geq s_2 \geq \cdots \to 0 \]

either finite, or decreasing to zero, such that if \( F \subset H \) has dimension \( N \) then

\[ \min_{u \in F, \|u\|=1} (Eu, u) \leq s_N, \forall N. \]

NB. The \( s_j \) have to be repeated corresponding to the dimension of the associated eigenspace.

**Problem 7.2**

Extend this further to show that under the same conditions on \( E \) the eigenvalues are give by the minimax formula:-

\[ s_j(E) = \max_{F \subset H, \dim F=j} \left( \min_{u \in F, \|u\|=1} (Eu, u) \right). \]

**Problem 7.3**

With \( E \) as above, suppose that \( D \in \mathcal{B}(H) \) is a bounded self-adjoint operator.

Show that

\[ s_j(DED) \leq \|D\|^2 s_j(E) \forall j. \]

NB. Be a bit careful about the minimax argument.
Problem 7.4
Let $A$ be a self-adjoint Hilbert–Schmidt operator (see an earlier problem set). Explain why the eigenspaces for non-zero eigenvalues, $\lambda_j$, of $A$ are finite dimensional and show that
\[ \sum_j \lambda_j^2 < \infty. \]

Problem 7.5
Suppose $a \in C^0([0, 2\pi]^2)$ is a continuous function of two variables. Show that the Fourier coefficients of $a$ in the second variable are continuous functions of the first variable and hence that the double Fourier coefficients
\[ a_{jk} = \int_0^{2\pi} \int_0^{2\pi} a(x, y)e^{-ijx-iky} dy dx \]
are well-defined. If $A$ is the integral operator ‘with kernel $a$’, so
\[ (Af)(x) = \int_0^{2\pi} a(x, y)f(y)dy, \quad f \in L^2(0, 2\pi) \]
show that
\[ \sum_{k \in \mathbb{Z}} \| Ae^{iky} \|^2_{L^2(0, 2\pi)} < \infty \]
and so conclude that $A$ is a Hilbert–Schmidt operator. What does this imply about the coefficients $a_{jk}$?

Hint: Think about
\[ \sum_{k=1}^{N} |c_k(x)|^2 \]
where $x$ is fixed and $c_k(x) = A(e^{ik\cdot})$. From the definition of $A$ you can think of this as an inner product and so it can be bounded by an integral using Bessel’s inequality. Integrate both sides in $x$ and deduce that the integrated sum has a bound independent of $N$.

Problem 7.6 – extra
Consider the notion of an unbounded self-adjoint operator (since so far an operator is bounded, you should think of this as unbounded-self-adjoint-operator, a new notion which does include bounded self-adjoint operators). Namely, if $H$ is a separable Hilbert space and $D \subset H$ is a dense linear subspace then a linear map $A : D \rightarrow H$ is an unbounded self-adjoint operator if

1. For all $v, w \in D$, $\langle Av, w \rangle_H = \langle v, Aw \rangle_H$.
2. $\{ u \in H; D \ni v \mapsto \langle Av, u \rangle \in \mathbb{C} \}$ extends to a continuous map on $H$.

Show that
\
\[ \text{Gr}(A) = \{(u, Au) \in H \times H; u \in D\} \]
is a closed subspace of $H \times H$ and that $A + i\text{Id} : D \rightarrow H$ is surjective with a bounded two-sided inverse $B : H \rightarrow H$ (with range $D$ of course).
Problem 7.7 – extra

Suppose $A$ is a compact self-adjoint operator on a separable Hilbert space and that $\text{Nul}(A) = \{0\}$. Define a dense subspace $D \subset H$ in such a way that $A^{-1} : D \to H$ is an unbounded self-adjoint operator which is a two-sided inverse of $A$.

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