PROBLEM SET 6 FOR 18.102
DUE FRIDAY APRIL 6, 2018

Problem 6.1

Give an example of a closed subset in an infinite-dimensional Hilbert space which is not weakly closed.

Problem 6.2

Let $A \in \mathcal{B}(H)$, H a separable Hilbert space, be such that for some orthonormal basis $\{e_i\}$

$$\sum_i \|Ae_i\|_H^2 < \infty. \tag{6.1}$$

Show that the same inequality is true for any other orthonormal basis and that A^* satisfies the same inequality.

Hint: For another basis, expand each norm $\|Ae_i\|^2$ using Bessel's identity and then use the adjoint identity and undo the double sum the opposite way.

Problem 6.3

The elements of $A \in \mathcal{B}(H)$ as in Problem 7.2 are called 'Hilbert-Schmidt operators'. Show that these form a 2-sided $*$-closed ideal $\mathrm{HS}(H)$, inside the compact operators and that

$$\langle A, B\rangle = \sum_i \langle Ae_i, Be_i\rangle, \tag{6.2}$$

for any choice of orthonormal basis, makes this into a Hilbert space.

Problem 6.4

Consider the 'shift' operators $S : l^2 \longrightarrow l^2$ and $T : l^2 \longrightarrow l^2$ defined by

$$S(\{a_k\}_{k=1}^\infty) = \{b_k\}_{k=1}^\infty,\ b_k = a_{k+1}, k \geq 1,$$
$$T(\{a_k\}_{k=1}^\infty) = \{c_k\}_{k=1}^\infty,\ c_1 = 0,\ c_k = a_{k-1},\ k \geq 2.$$

Compute the norms of these operators and show that $TS = \mathrm{Id} - \Pi_1$ and $ST = \mathrm{Id}$ where $\Pi_1(\{a_k\}_{k=1}^\infty) = \{d_k\}$, $d_1 = a_1$, $d_k = 0$, $k \geq 2$.

Problem 6.5

If $K \in \mathcal{C}([0,1] \times [0,1])$ is a continuous function of two variables, show that

$$Af(x) = \int K(x,y)f(y) \tag{6.3}$$

defines a compact linear operator on $L^2(0,1)$.

Hint: Show that A defines a bounded linear map from $L^2(0,1)$ to $\mathcal{C}[0,1]$ and that the image of the unit ball is *equicontinuous* using the uniform continuity of K; see the preceding Problem set.

Problem 6.6-extra

With the operators as defined above, show that for any $B \in \mathcal{B}(l^2)$ with $\|B\| < 1$, $S+B$ is not invertible (and so conclude that the invertible operators are not dense in $\mathcal{H}(l^2)$).

Hint: Show that $(S+B)T = \mathrm{Id} + BT$ is invertible and so if $S+B$ was invertible, with inverse G then so is $G(S+B)T = T$, as the product of invertibles.

Problem 6.7-extra

Suppose $K : L^2(0,1) \longrightarrow L^2(0,1)$ is an integral operator (as considered above) with a continuous kernel $k \in \mathcal{C}([0,1]^2)$,

$$Kf(x) = \int_{(0,1)} K(x,y)f(y)dy. \tag{6.4}$$

Show that K is Hilbert-Schmidt.