PROBLEM SET 5 FOR 18.102, SPRING 2018
DUE FRIDAY 30 MARCH IN THE USUAL SENSE.

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Problem 5.1

Recall the space $h^{2,1}$ (discussed in the preceding problem set) consisting of the complex valued sequences c_i such that

$$\|c\|^2 = \sum_i (1+|i|^2)|c_i|^2 < \infty.$$

Show that the unit ball in this space, considered as a subset of l^2, has compact closure.

Problem 5.2

Define the space $\mathcal{L}^2(0,1)$ as consisting of those elements of $\mathcal{L}^2(\mathbb{R})$ which vanish outside $(0,1)$ and show that the quotient $L^2(0,1) = \mathcal{L}^2(0,1)/\mathcal{N}(0,1)$ by the null functions in $\mathcal{L}^2(0,1)$ is a Hilbert space.

Remark: This is indeed easy, but make sure you do it properly (for instance identify $L^2(0,1)$ with a closed subspace of $L^2(\mathbb{R})$ by treating the null functions properly). You can use the fact that $L^2(\mathbb{R})$ is a Hilbert space.

Problem 5.3

Identify $\mathcal{C}[0,1]$, the space of continuous functions on the closed interval, as a subspace of $L^2(0,1)$. For each $n \in \mathbb{N}$ let $F_n \subset L^2(0,1)$ be the subspace of functions which are constant on each interval $((m-1)/n, m/n]$ for $m = 1, \dots, n$ Show that if $f \in \mathcal{C}[0,1]$ there exists a sequence $g_n \in F_n$ such that

$$\delta_n = \sup_{|t-s|\le 1/n} |f(t)-f(s)| \Longrightarrow \|f-g_n\|_{L^2} \le \delta_n.$$

Problem 5.4

Show that a bounded and equicontinuous subset of $\mathcal{C}[0,1]$ has compact closure in $L^2(0,1)$. Note that equicontinuity means 'uniform equicontinuity' so for each $\epsilon > 0$ there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \epsilon$ for all elements f of the set.

Hint: Show that A defines a bounded linear map from $L^2(0,1)$ to $\mathcal{C}[0,1]$ and that the image of the unit ball is *equicontinuous* using the uniform continuity of K.

Problem 5.5

Suppose that H_1 and H_2 are two different Hilbert spaces and $A : H_1 \longrightarrow H_2$ is a bounded linear operator. Show that there is a unique bounded linear operator (the adjoint) $A^* : H_2 \longrightarrow H_1$ with the property

$$\langle Au_1, u_2 \rangle_{H_2} = \langle u_1, A^* u_2 \rangle_{H_1} \ \forall \ u_1 \in H_1, \ u_2 \in H_2. \tag{1}$$

Problem 5.6 – extra

Show that a closed and bounded subset of $L^2(\mathbb{R})$ is compact if and only if it is 'uniformly equicontinuous in the mean' and 'uniformly small at infinity' so that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathbb{R}\setminus[-1/\delta,1/\delta]} |f|^2 < \epsilon^2 \text{ and } |t| < \delta \Longrightarrow \int |f(x) - f(x-t)|^2 < \epsilon^2$$

for all elements of the set.

Problem 5.7 – Extra

Consider the space of continuous functions on $\mathbb{R}$ vanishing outside $(0,1)$ which are of the form

$$u(x) = \int_0^x v, \ v \in L^2(0,1).$$

Show that these form a Hilbert space and that the unit ball of this space has compact closure in $L^2(0,1)$.

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