**PROBLEM SET 5 FOR 18.102, SPRING 2018**
**DUE FRIDAY 30 MARCH IN THE USUAL SENSE.**

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Problem 5.1
Recall the space $h^{2,1}$ (discussed in the preceding problem set) consisting of the complex valued sequences $c_i$ such that

$$||c||^2 = \sum_i (1 + |i|^2) |c_i|^2 < \infty.$$ 

Show that the unit ball in this space, considered as a subset of $l^2$, has compact closure.

Problem 5.2
Define the space $L^2(0,1)$ as consisting of those elements of $L^2(\mathbb{R})$ which vanish outside $(0,1)$ and show that the quotient $L^2(0,1) = L^2(0,1)/\mathcal{N}(0,1)$ by the null functions in $L^2(0,1)$ is a Hilbert space.

Remark: This is indeed easy, but make sure you do it properly (for instance identify $L^2(0,1)$ with a closed subspace of $L^2(\mathbb{R})$ by treating the null functions properly). You can use the fact that $L^2(\mathbb{R})$ is a Hilbert space.

Problem 5.3
Identify $C[0,1]$, the space of continuous functions on the closed interval, as a subspace of $L^2(0,1)$. For each $n \in \mathbb{N}$ let $F_n \subset L^2(0,1)$ be the subspace of functions which are constant on each interval $((m-1)/n, m/n]$ for $m = 1, \ldots, n$. Show that if $f \in C[0,1]$ there exists a sequence $g_n \in F_n$ such that

$$\delta_n = \sup_{|t-s| \leq 1/n} |f(t) - f(s)| \implies \|f - g_n\|_{L^2} \leq \delta_n.$$ 

Problem 5.4
Show that a bounded and equicontinuous subset of $C[0,1]$ has compact closure in $L^2(0,1)$. Note that equicontinuity means ‘uniform equicontinuity’ so for each $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all elements $f$ of the set.

Hint: Show that $A$ defines a bounded linear map from $L^2(0,1)$ to $C[0,1]$ and that the image of the unit ball is equicontinuous using the uniform continuity of $K$.

Problem 5.5
Suppose that $H_1$ and $H_2$ are two different Hilbert spaces and $A : H_1 \longrightarrow H_2$ is a bounded linear operator. Show that there is a unique bounded linear operator...
(the adjoint) \( A^*: H_2 \rightarrow H_1 \) with the property

\[
(1) \quad \langle Au_1, u_2 \rangle_{H_2} = \langle u_1, A^* u_2 \rangle_{H_1}, \quad \forall \ u_1 \in H_1, \ u_2 \in H_2.
\]

Problem 5.6 – extra

Show that a closed and bounded subset of \( L^2(\mathbb{R}) \) is compact if and only if it is ‘uniformly equicontinuous in the mean’ and ‘uniformly small at infinity’ so that for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\int_{\mathbb{R}\setminus[-1/\delta,1/\delta]} |f|^2 < \epsilon^2 \text{ and } |t| < \delta \implies \int |f(x) - f(x-t)|^2 < \epsilon^2
\]

for all elements of the set.

Problem 5.7 – Extra

Consider the space of continuous functions on \( \mathbb{R} \) vanishing outside \((0, 1)\) which are of the form

\[
u(x) = \int_0^x v, \ v \in L^2(0,1).
\]

Show that these form a Hilbert space and that the unit ball of this space has compact closure in \( L^2(0,1) \).

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