PROBLEM SET 4 FOR 18.102
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Problem 4.1

Define a set $U \subset \mathbb{R}$ to be Lebesgue measureable if the characteristic function of $U \cap [-R, R]$

$$\chi_{U\cap[-R,R]}(x) = \begin{cases} 1 & x \in U \cap [-R, R] \\ 0 & x \notin U \cap [-R, R] \end{cases}$$

is in $\mathcal{L}^1(\mathbb{R})$ for every $R > 0$. Letting $\mathcal{M}$ be the collection of measureable sets, show

(1) $\mathbb{R} \in \mathcal{M}$
(2) $U \in \mathcal{M} \Longrightarrow \mathbb{R} \setminus U \in \mathcal{M}$
(3) $U_j \in \mathcal{M}$ for $j \in \mathbb{N}$ then $\bigcup_{j=1}^{\infty} U_j \in \mathcal{M}$
(4) If $U \subset \mathbb{R}$ is open then $U \in \mathcal{M}$

Problem 4.2

Let H be a normed space (over $\mathbb{C}$) in which the norm satisfies the parallelogram law:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)\ \forall\ u, v \in H. \tag{1}$$

Show that

$$\langle u, v\rangle = \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2\right) \tag{2}$$

is a positive-definite Hermitian form which induces the given norm.

Hint: Linearity is a pain. Try to get something, say for a mid-point, first.

Problem 4.3

Let H be a finite dimensional (pre)Hilbert space. So, by definition H has a basis $\{v_i\}_{i=1}^n$, meaning that any element of H can be written

$$v = \sum_i c_i v_i \tag{3}$$

and there is no dependence relation between the v_i's – the presentation of $v = 0$ in the form (3) is unique. Show that H has an orthonormal basis, $\{e_i\}_{i=1}^n$ satisfying $(e_i, e_j) = \delta_{ij}$ ($= 1$ if $i = j$ and 0 otherwise). Check that for the orthonormal basis the coefficients in (3) are given by the inner products $c_i = \langle v, e_i\rangle$ and that the map

$$T : H \ni v \longmapsto (\langle v, e_1\rangle, \langle v, e_2\rangle, \dots, \langle v, e_n\rangle) \in \mathbb{C}^n \tag{4}$$

is a linear isomorphism with the properties

$$\langle u, v\rangle = \sum_i (Tu)_i \overline{(Tv)_i},\ \|u\|_H = \|Tu\|_{\mathbb{C}^n}\ \forall\ u, v \in H. \tag{5}$$

Why is a finite dimensional pre-Hilbert space a Hilbert space?

Problem 4.4

Let e_i, $i \in \mathbb{N}$, be an orthonormal sequence in a separable Hilbert space H. Suppose that for each element u in a dense subset $D \subset H$

$$\sum_i |\langle u, e_i \rangle|^2 = \|u\|^2. \tag{6}$$

Conclude that e_i is an orthonormal basis, i.e. is maximal.

Problem 4.5

Consider the sequence space

$$h^{2,1} = \left\{ c : \mathbb{N} \ni j \longmapsto c_j \in \mathbb{C}; \sum_j (1 + j^2)|c_j|^2 < \infty \right\}. \tag{7}$$

(1) Show that

$$h^{2,1} \times h^{2,1} \ni (c, d) \longmapsto \langle c, d \rangle_{2,1} = \sum_j (1 + j^2) c_j \overline{d_j} \tag{8}$$

is an Hermitian inner form which turns $h^{2,1}$ into a Hilbert space.

(2) Denoting the norm on this space by $\| \cdot \|_{2,1}$ and the norm on l^2 by $\| \cdot \|_2$, show that

$$h^{2,1} \subset l^2, \ \|c\|_2 \leq \|c\|_{2,1} \ \forall \ c \in h^{2,1}. \tag{9}$$

Problem 4.6 – Extra

Show that the complex finite linear combinations of the functions $\sin ax$, $a \in \mathbb{R}$ form a pre-Hilbert space with respect to the norm given by

$$\|f\|^2 = \lim_{R \to \infty} \frac{1}{R} \int_{[-R,R]} |f|^2. \tag{10}$$

Show that the completion is a non-separable Hilbert space. Can you give a concrete description of the elements of the completion?

Problem 4.7 – Extra

Consider the subspace of $\mathcal{L}^2(\mathbb{R})$ which consists of continuous functions u with the additional property that there exists $v \in \mathcal{L}^2(\mathbb{R})$ such that

$$u(x) = \begin{cases} u(0) + \int_{(0,x)} v(t) & \text{if } x > 0 \\ u(0) - \int_{(x,0)} v(t) & \text{if } x < 0. \end{cases} \tag{11}$$

Show that for a given u if there are two such functions v then they differ by a null function. Prove that the set of pairs $(u, [v])$ where $[v] \in L^2(\mathbb{R})$ is a Hilbert space with respect to the inner product

$$\langle (u_1, [v_1]), (u_2, [v_2]) \rangle = \int u_1 \overline{u_2} + \int v_1 \overline{v_2}. \tag{12}$$

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