PROBLEM SET 3 FOR 18.102, SPRING 2018
DUE MARCH 2 (IN THE USUAL SENSE)

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Version 2. Problem 3.2 replaced.

Problem 3.1

Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function with Riemann integral satisfying

$$\sup_R \int_{-R}^{R} |f(x)| dx < \infty. \tag{1}$$

Show that $f \in \mathcal{L}^1(\mathbb{R})$.

Problem 3.2

Show that the function $\frac{\sin x}{(1+|x|)}$ is not an element of $L^1(\mathbb{R})$.

Problem 3.3

Recall that a function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is in $\mathcal{L}^2(\mathbb{R})$ if there exists a sequence $f_n \in \mathcal{C}(\mathbb{R})$ such that $f_n(x) \longrightarrow f(x)$ a.e. and there exists $F \in \mathcal{L}^1(\mathbb{R})$ such that $|f_n|^2 \le F(x)$ a.e. Show that $\chi_{[-R,R]} f \in \mathcal{L}^1(\mathbb{R})$ and that

$$(\int \chi_{[-R,R]} |f|)^2 \le (2R) \int |f|^2.$$

Problem 3.4

Show that the function with $F(0) = 0$ and

$$F(x) = \begin{cases} 0 & x > 1 \\ \exp(i/x) & 0 < |x| \le 1 \\ 0 & x < -1, \end{cases}$$

is an element of $\mathcal{L}^1(\mathbb{R})$.

Problem 3.5

Suppose $f \in \mathcal{L}^1(\mathbb{R})$ is real-valued. Show that there is a sequence $f_n \in \mathcal{C}_c(\mathbb{R})$ and another element $F \in \mathcal{L}^1(\mathbb{R})$ such that

$$f_n(x) \to f(x) \text{ a.e. on } \mathbb{R},\ |f_n(x)| \le F(x) \text{ a.e.}$$

Problem 3.6 – extra

(1) Suppose that $O \subset \mathbb{R}$ is a *bounded* open subset, so $O \subset (-R, R)$ for some R. Show that the characteristic function of O

$$\chi_O(x) = \begin{cases} 1 & x \in O \\ 0 & x \notin O \end{cases} \tag{2}$$

is an element of $\mathcal{L}^1(\mathbb{R})$.

(2) If O is bounded and open define the length (or Lebesgue measure) of O to be $l(O) = \int \chi_O$. Show that if $U = \bigcup_j O_j$ is a (n at most) countable union of bounded open sets such that $\sum\limits_j l(O_j) < \infty$ then $\chi_U \in \mathcal{L}^1(\mathbb{R})$; again we set $l(U) = \int \chi_U$.

(3) Conversely show that if U is open and $\chi_U \in \mathcal{L}^1(\mathbb{R})$ then $U = \bigcup_j O_j$ *is* the union of a countable collection of bounded open sets with $\sum\limits_j l(O_j) < \infty$.

(4) Show that if $K \subset \mathbb{R}$ is compact then its characteristic function is an element of $\mathcal{L}^1(\mathbb{R})$.

Problem 3.7 – extra

Prove the converse of Problem 3.2, that for any $\epsilon > 0$ any set of measure zero is covered by a countable collection of open intervals the sum of whose lengths is less than ϵ.

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