Version 2. Problem 3.2 replaced.

Problem 3.1
Suppose that \( f: \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function with Riemann integral satisfying
\[
\sup_R \int_{-R}^R |f(x)| \, dx < \infty.
\]
Show that \( f \in L^1(\mathbb{R}) \).

Problem 3.2
Show that the function \( \frac{\sin x}{1+|x|} \) is not an element of \( L^1(\mathbb{R}) \).

Problem 3.3
Recall that a function \( f: \mathbb{R} \rightarrow \mathbb{C} \) is in \( L^2(\mathbb{R}) \) if there exists a sequence \( f_n \in C(\mathbb{R}) \) such that \( f_n(x) \rightarrow f(x) \) a.e. and there exists \( F \in L^1(\mathbb{R}) \) such that \( |f_n|^2 \leq F(x) \) a.e. Show that \( \chi_{[-R,R]}f \in L^1(\mathbb{R}) \) and that
\[
(\int \chi_{[-R,R]}|f|)^2 \leq (2R) \int |f|^2.
\]

Problem 3.4
Show that the function with \( F(0) = 0 \) and
\[
F(x) = \begin{cases} 
0 & x > 1 \\
\exp(i/x) & 0 < |x| \leq 1 \\
0 & x < -1,
\end{cases}
\]
is an element of \( L^1(\mathbb{R}) \).
Problem 3.5
Suppose \( f \in L^1(\mathbb{R}) \) is real-valued. Show that there is a sequence \( f_n \in C_c(\mathbb{R}) \) and another element \( F \in L^1(\mathbb{R}) \) such that
\[
f_n(x) \to f(x) \ a.e. \ on \ \mathbb{R}, \ |f_n(x)| \leq F(x) \ a.e.
\]

Problem 3.6 – extra

1. Suppose that \( O \subset \mathbb{R} \) is a bounded open subset, so \( O \subset (-R, R) \) for some \( R \). Show that the characteristic function of \( O \)
\[
\chi_O(x) = \begin{cases} 1 & x \in O \\ 0 & x \notin O \end{cases}
\]

is an element of \( L^1(\mathbb{R}) \).

2. If \( O \) is bounded and open define the length (or Lebesgue measure) of \( O \) to be \( l(O) = \int \chi_O \). Show that if \( U = \bigcup_j O_j \) is a (n at most) countable union of bounded open sets such that \( \sum_j l(O_j) < \infty \) then \( \chi_U \in L^1(\mathbb{R}) \); again we set \( l(U) = \int \chi_U \).

3. Conversely show that if \( U \) is open and \( \chi_U \in L^1(\mathbb{R}) \) then \( U = \bigcup_j O_j \) is the union of a countable collection of bounded open sets with \( \sum_j l(O_j) < \infty \).

4. Show that if \( K \subset \mathbb{R} \) is compact then its characteristic function is an element of \( L^1(\mathbb{R}) \).

Problem 3.7 – extra
Prove the converse of Problem 3.2, that for any \( \epsilon > 0 \) any set of measure zero is covered by a countable collection of open intervals the sum of whose lengths is less than \( \epsilon \).

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