PROBLEM SET 2 FOR 18.102, SPRING 2018 DUE FRIDAY 23 FEB.

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Here is what I said before about collaboration and all that. I do not mind who you talk to, what you read or where you find information. However, I expect that you will devise and write out the answers yourself. This means precisely no direct copying, you must first assimilate the material then rewrite it.

You can get full marks by doing any five of these problems, probably the first five are the most straightforward and you cannot get more than 50 marks – only your 'best' five solutions will count.

Problem 2.1

Show that if $K \in \mathcal{C}([0,1]^2)$ is a continuous function of two variables, then the integral operator

$$Au(x) = \int_0^1 K(x,y)u(y)dy \tag{1}$$

(given by a Riemann integral) is a bounded operator, i.e. a continous linear map, from $\mathcal{C}([0,1])$ to itself with respect to the supremum norm.

Problem 2.2

(1) Show that the 'Dirac delta function at $y \in [0,1]$' is well-defined as a continuous linear map

$$\delta_y : \mathcal{C}([0,1]) \ni u \longmapsto u(y) \in \mathbb{C} \tag{2}$$

with respect to the supremum norm on $\mathcal{C}([0,1])$.

(2) Show that δ_y is *not* continuous with respect to the L^1 norm $\int_0^1 |u|$.

Problem 2.3

As we already know, a subset $E \subset \mathbb{R}$ is said to be *of measure zero* if there exists an absolutely summable sequence $f_n \in \mathcal{C}_\mathrm{c}(\mathbb{R})$ (so $\sum\limits_n \int |f_n| < \infty$) such that

$$E \subset \{x \in \mathbb{R}; \sum_n |f_n(x)| = +\infty\}. \tag{3}$$

Show that if E is of measure zero and $\epsilon > 0$ is given then there exists $f_n \in \mathcal{C}_\mathrm{c}(\mathbb{R})$ satisfying (3) and in addition

$$\sum_n \int |f_n| < \epsilon. \tag{4}$$

Problem 2.4

Using the previous problem (or otherwise ...) show that a countable union of sets of measure zero is a set of measure zero.

Problem 2.5

Suppose $E \subset \mathbb{R}$ has the following (well-known) property:-

$$\forall\ \epsilon > 0\ \exists \text{ a countable collection of intervals } (a_i, b_i) \text{ s.t.}$$

$$\sum_i (b_i - a_i) < \epsilon,\ E \subset \bigcup_i (a_i, b_i). \tag{5}$$

Show that E is a set of measure zero in the sense used in lectures and above.

Problem 2.6 – Extra

Let's generalize the theorem about $\mathcal{B}(V, W)$ to bilinear maps.

(1) Check that if U and V are normed spaces then $U \times V$ (the linear space of all pairs (u, v) where $u \in U$ and $v \in V$) is a normed space where addition and scalar multiplication is 'componentwise' and the norm is the sum

$$\|(u, v)\|_{U\times V} = \|u\|_U + \|v\|_V. \tag{6}$$

(2) Show that $U \times V$ is a Banach space if both U and V are Banach spaces.

(3) Consider three normed spaces U, V and W. Let

$$B : U \times V \longrightarrow W \tag{7}$$

be a *bilinear* map. This means that

$$B(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 B(u_1, v) + \lambda_2 B(u_2, v),$$
$$B(u, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 B(u, v_1) + \lambda_2 B(u, v_2)$$

for all u, u_1, $u_2 \in U$, v, v_1, $v_2 \in V$ and λ_1, $\lambda_2 \in \mathbb{C}$. Show that B is continuous if and only if it satisfies

$$\|B(u, v)\|_W \le C\|u\|_U \|v\|_V\ \forall\ u \in U,\ v \in V. \tag{8}$$

(4) Let $\mathcal{M}(U, V; W)$ be the space of all such continuous bilinear maps. Show that this is a linear space and that

$$\|B\| = \sup_{\|u\|=1, \|v\|=1} \|B(u, v)\|_W \tag{9}$$

is a norm.

(5) Show that $\mathcal{M}(U, V; W)$ is a Banach space if W is a Banach space.

Problem 2.7 – Extra

Consider the space $\mathcal{C}_c(\mathbb{R}^n)$ of continuous functions $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ which vanish outside a compact set, i.e. in $|x| > R$ for some R (depending on u). Check (quickly) that this is a linear space.

Show that if $y \in \mathbb{R}^{n-1}$ and $u \in \mathcal{C}_c(\mathbb{R}^n)$ then

$$U_y : \mathbb{R} \ni t \longmapsto u(y, t) \in \mathbb{C} \tag{10}$$

defines an element $U_y \in \mathcal{C}_c(\mathbb{R})$. Fix an overall 'rectangle' $[-R, R]^n$ and only consider functions $\mathcal{C}_{c,R}(\mathbb{R})$ vanishing outside this rectangle. With this restriction on supports show for each R that $\mathbb{R}^{n-1} \ni y \longmapsto U_y$ is a continuous map into $\mathcal{C}_{c,R}(\mathbb{R})$ *with respect to the supremum norm* which vanishes for $|y| > R$, i.e. has compact support. Conclude that 'integration in the last variable' gives a continuous linear map (with respect to supremum norms)

$$\mathcal{C}_{c,R}(\mathbb{R}^n) \ni u \longrightarrow v \in \mathcal{C}_{c,R}(\mathbb{R}^{n-1}),\ v(y) = \int U_y. \tag{11}$$

By iterating this statement show that the iterated Riemann integral is well defined

$$\int : \mathcal{C}_{\mathrm{c},R}(\mathbb{R}^n) \longrightarrow \mathbb{C} \tag{12}$$

and that $\int |u|$ is a norm which is independent of R – so defined on the whole of $\mathcal{C}_{\mathrm{c}}(\mathbb{R}^n)$.