

18.102/1021 QUESTIONS FOR FINAL EXAM

No notes, books, reference material or communication equipment will be permitted!

Unless otherwise defined, H is an infinite-dimensional separable Hilbert space.

PROBLEM 1

Let A be a compact, self-adjoint operator on a H which is positive in the sense that $\langle Au, u \rangle > 0$ for all $0 \neq u \in H$. Show that the range $R(A)$ is a dense subspace of H and that

$$\|v\|_A = \|u\|_H, \quad v \in R(A), \quad v = Au, \quad u \in H$$

defines a norm on $R(A)$ with respect to which it is a Hilbert space.

PROBLEM 2

Let a be a continuous function on the square $[0, 2\pi]^2$. Show that $[0, 2\pi] \ni x \mapsto a(x, \cdot) \in \mathcal{C}^0([0, 2\pi])$ is a continuous map into the continuous functions with supremum norm. Using this, or otherwise, show that the Fourier coefficients with respect to the second variable

$$(1) \quad c_k(x) = \frac{1}{2\pi} \int_0^{2\pi} a(x, t) e^{ikt} dt$$

are continuous functions on $[0, 2\pi]$.

PROBLEM 3

Let $H_i, i = 1, 2$ be two Hilbert spaces with inner products $(\cdot, \cdot)_i$ and suppose that $I : H_1 \rightarrow H_2$ is a continuous linear map between them.

- (1) Show that there is a continuous linear map $Q : H_2 \rightarrow H_1$ such that $(u, If)_2 = (Qu, f)_1 \quad \forall f \in H_1$.
- (2) Show that as a map from H_1 to itself, $Q \circ I$ is bounded and self-adjoint
- (3) Show that the spectrum of $Q \circ I$ is contained in $[0, \|I\|^2]$.

PROBLEM 4

Let $u_n : [0, 2\pi] \rightarrow \mathbb{C}$ be a sequence of continuously differentiable functions which is uniformly bounded, with bounded derivatives i.e. $\sup_n \sup_{x \in [0, 2\pi]} |u_n(x)| < \infty$ and $\sup_n \sup_{x \in [0, 2\pi]} |u'_n(x)| < \infty$. Show that u_n has a subsequence which converges in $L^2([0, 2\pi])$.

PROBLEM 5

Consider the subspace $H \subset \mathcal{C}[0, 2\pi]$ consisting of those continuous functions on $[0, 2\pi]$ which satisfy

$$(2) \quad u(x) = \int_0^x U, \quad \forall x \in [0, 2\pi]$$

for some $U \in L^2(0, 2\pi)$ (depending on u of course).

- (1) Show that the function U is determined by u (given that it exists).
 (2) Show that

$$(3) \quad \|u\|_H^2 = \int_{(0,2\pi)} |U|^2$$

turns H into a Hilbert space.

- (3) If $\int_0^{2\pi} U = 0$, determine the Fourier series of u in terms of that of U .

PROBLEM 6

Consider the space of those complex-valued functions on $[0, 1]$ for which there is a constant C (depending on the function) such that

$$(4) \quad |u(x) - u(y)| \leq C|x - y|^{\frac{1}{2}} \quad \forall x, y \in [0, 1].$$

Show that this is a Banach space with norm

$$(5) \quad \|u\|_{\frac{1}{2}} = \sup_{[0,1]} |u(x)| + \inf_{(4) \text{ holds}} C.$$

PROBLEM 7

Let $A_j \subset \mathbb{R}$ be a sequence of subsets with the property that the characteristic function, χ_j of A_j , is integrable for each j . Show that the characteristic function of $\mathbb{R} \setminus A$, where $A = \bigcup_j A_j$ is locally integrable.

PROBLEM 8

A bounded operator $A \in \mathcal{B}(H)$ on a separable Hilbert space is a Hilbert-Schmidt operator if for some orthonormal basis $\{e_i\}$

$$(6) \quad \sum_i \|Ae_i\|^2 < \infty.$$

Show that if A and B are Hilbert-Schmidt operators then the sum

$$(7) \quad \text{Tr}(AB) = \sum_i \langle Ae_i, B^*e_i \rangle$$

exists and is independent of the orthonormal basis used to define it and that

$$(8) \quad \text{Tr}(AB) = \text{Tr}(BA).$$

PROBLEM 9

Let B_n be a sequence of bounded linear operators on a Hilbert space H such that for each u and $v \in H$ the sequence $(B_n u, v)$ converges in \mathbb{C} . Show that there is a uniquely defined bounded operator B on H such that

$$(Bu, v) = \lim_{n \rightarrow \infty} (B_n u, v) \quad \forall u, v \in H.$$

PROBLEM 10

Suppose $P \subset H$ is a closed linear subspace of a Hilbert space, with $\pi_P : H \rightarrow P$ the orthogonal projection onto P . If H is separable and A is a compact self-adjoint operator on H , show that there is a complete orthonormal basis of H each element of which satisfies $\pi_P A \pi_P e_i = t_i e_i$ for some $t_i \in \mathbb{R}$.

PROBLEM 11

Let $e_j = c_j C^j e^{-x^2/2}$, $c_j > 0$, where $j = 1, 2, \dots$, and $C = -\frac{d}{dx} + x$ is the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. You may assume completeness in $L^2(\mathbb{R})$ and use the facts established in class that $-\frac{d^2 e_j}{dx^2} + x^2 e_j = (2j + 1)e_j$, that $c_j = 2^{-j/2} (j!)^{-\frac{1}{2}} \pi^{-\frac{1}{4}}$ and that $e_j = p_j(x) e_0$ for a polynomial of degree j . Compute $C e_j$ and $A e_j$ in terms of the basis and hence arrive at formulae for $d e_j / dx$ and $x e_j$. Show that

$$(9) \quad H_{\text{iso}}^1 = \left\{ u \in L^2(\mathbb{R}); \sum_{j \geq 1} j | (u, e_j) |^2 < \infty \right\}$$

is a Hilbert space and that there is a bounded operator

$$C : H_{\text{iso}}^1 \longrightarrow L^2(\mathbb{R})$$

reducing to $-\frac{d}{dx} + x$ on the basis.

PROBLEM 12

Suppose that $f \in \mathcal{L}^2(0, 2\pi)$ is such that there exists a function $v \in \mathcal{L}^2(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} f \phi' = \int v \hat{\phi} \quad \forall \phi \in \mathcal{S}(\mathbb{R})$$

where $\hat{\phi}$ is the Fourier transform of ϕ . Using if desired the density of $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$ show that there is a bounded continuous function g such that $f = g$ a.e.

PROBLEM 13

Suppose $A \in \mathcal{B}(H)$ has closed range. Show that A has a generalized inverse, $B \in \mathcal{B}(H)$ such that

$$AB = \Pi_R, \quad BA = \text{Id} - \Pi_N$$

where Π_N and Π_R are the orthonormal projections onto the null space and range of A respectively.

PROBLEM 14

For $u \in L^2(0, 1)$ show that

$$Iu(x) = \int_0^x u(t) dt, \quad x \in (0, 1)$$

is a bounded linear operator on $L^2(0, 1)$. If $V \in \mathcal{C}([0, 1])$, is real-valued and $V \geq 0$, show that there is a bounded linear operator B on $L^2(0, 1)$ such that

$$(10) \quad B^2 u = u + I^* M_V I u \quad \forall u \in L^2(0, 1)$$

where M_V denotes multiplication by V .

PROBLEM 15

Let $A \in \mathcal{B}(H)$ be such that

$$(11) \quad \sup \sum_i |\langle Ae_i, f_i \rangle| < \infty$$

where the supremum is over orthonormal sequence e_i and f_j . Use the polar decomposition to show that $A = B_1 B_2$ where the $B_k \in \mathcal{B}(H)$ are Hilbert-Schmidt operators, i.e. $\sum_i \|B_k e_i\|^2 < \infty$, $k = 1, 2$.

PROBLEM 16

Suppose $f \in \mathcal{L}^1(\mathbb{R})$ and the Fourier transform

$$\hat{f}(\xi) = \int e^{-ix\xi} f(x) \in \mathcal{L}^2(\mathbb{R}),$$

show that $f \in \mathcal{L}^2(\mathbb{R})$.

PROBLEM 17

Show that the functions $c_k \cos(kx)$, $k = 0, 1, \dots$, for appropriate constants c_k , form an orthonormal basis of $L^2(0, \pi)$. Using these, or otherwise, show that for each function $f \in \mathcal{C}([0, \pi])$ the Neumann problem

$$(12) \quad -\frac{d^2 u}{dx^2} + u = f \text{ on } (0, \pi), \quad \frac{du}{dx}(0) = \frac{du}{dx}(\pi) = 0$$

has a unique twice continuously differentiable solution and the resulting map $f \mapsto u$ extends by continuity to a compact self-adjoint operator on $L^2(0, \pi)$.

PROBLEM 18

Let $a \in \mathcal{C}([0, 1])$ be a real-valued continuous function. Show that multiplication by a defines a self-adjoint operator A_a on $L^2(0, 1)$ which has spectrum exactly the range $a([0, 1]) \subset \mathbb{R}$. If $h \in \mathcal{C}(\mathbb{R})$ is real-valued, describe the operator $h(A_a)$ given by the functional calculus.

PROBLEM 19

Explain carefully (but briefly ..) why the Schwartz space $\mathcal{S}(\mathbb{R})$ gives a well-defined subspace of $L^2(\mathbb{R})$. Show that if $u \in L^2(\mathbb{R})$ has a weak derivative in $\mathcal{S}(\mathbb{R})$ in the sense that there exists $v \in \mathcal{S}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} u \frac{d\phi}{dx} = - \int_{\mathbb{R}} v \phi \quad \forall \phi \in \mathcal{S}(\mathbb{R})$$

then $u \in \mathcal{S}(\mathbb{R})$.

PROBLEM 20

Suppose that $B : H \rightarrow H$ is a linear map such that $u_n \rightarrow u$ in H and $Bu_n \rightarrow v$ in H implies that $v = Bu$. Show that B is bounded.

PROBLEM 21

Show that the Schwartz space $\mathcal{S}(\mathbb{R})$ is dense in $L^1(\mathbb{R})$.