CHAPTER 4

Differential equations

The last part of the course includes some applications of Hilbert space and the spectral theorem – the completeness of the Fourier basis, some spectral theory for second-order differential operators on an interval or the circle and enough of a treatment of the eigenfunctions for the harmonic oscillator to show that the Fourier transform is an isomorphism on $L^2(\mathbb{R})$. Once one has all this, one can do a lot more, but there is no time left. Such is life.

1. Fourier series and $L^2(0, 2\pi)$.

Let us now try applying our knowledge of Hilbert space to a concrete Hilbert space such as $L^2(a, b)$ for a finite interval $(a, b) \subset \mathbb{R}$. You showed that this is indeed a Hilbert space. One of the reasons for developing Hilbert space techniques originally was precisely the following result.

THEOREM 4.1. *If $u \in L^2(0, 2\pi)$ then the Fourier series of u,*

$$\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} c_k e^{ikx},\ c_k = \int_{(0,2\pi)} u(x)e^{-ikx}dx \tag{4.1}$$

converges in $L^2(0, 2\pi)$ to u.

Notice that this does not say the series converges pointwise, or pointwise almost everywhere. In fact it is true that the Fourier series of a function in $L^2(0, 2\pi)$ converges almost everywhere to u, but it is hard to prove! In fact it is an important result of L. Carleson. Here we are just claiming that

$$\lim_{n\to\infty}\int |u(x) - \frac{1}{2\pi}\sum_{|k|\le n} c_k e^{ikx}|^2 = 0 \tag{4.2}$$

for any $u \in L^2(0, 2\pi)$.

Our abstract Hilbert space theory has put us quite close to proving this. First observe that if $e'_k(x) = \exp(ikx)$ then these elements of $L^2(0, 2\pi)$ satisfy

$$\int e'_k\overline{e'_j} = \int_0^{2\pi}\exp(i(k-j)x) = \begin{cases} 0 & \text{if } k \ne j \\ 2\pi & \text{if } k = j. \end{cases} \tag{4.3}$$

Thus the functions

$$e_k = \frac{e'_k}{\|e'_k\|} = \frac{1}{\sqrt{2\pi}}e^{ikx} \tag{4.4}$$

form an orthonormal set in $L^2(0, 2\pi)$. It follows that (4.1) is just the Fourier-Bessel series for u with respect to this orthonormal set:-

$$c_k = \sqrt{2\pi}(u, e_k) \Longrightarrow \frac{1}{2\pi}c_k e^{ikx} = (u, e_k)e_k. \tag{4.5}$$

So, we already know that this series converges in $L^2(0, 2\pi)$ thanks to Bessel's inequality. So 'all' we need to show is

PROPOSITION 4.1. *The e_k, $k \in \mathbb{Z}$, form an orthonormal basis of $L^2(0, 2\pi)$, i.e. are complete:*

$$\int ue^{ikx} = 0 \ \forall \ k \Longrightarrow u = 0 \ \textit{in} \ L^2(0, 2\pi). \tag{4.6}$$

This however, is not so trivial to prove. An equivalent statement is that the finite linear span of the e_k is *dense* in $L^2(0, 2\pi)$. I will prove this using Fejér's method. In this approach, we check that any continuous function on $[0, 2\pi]$ satisfying the additional condition that $u(0) = u(2\pi)$ is the uniform limit on $[0, 2\pi]$ of a sequence in the finite span of the e_k. Since uniform convergence of continuous functions certainly implies convergence in $L^2(0, 2\pi)$ and we already know that the continuous functions which vanish near 0 and 2π are dense in $L^2(0, 2\pi)$ this is enough to prove Proposition 4.1. However the proof is a serious piece of analysis, at least it seems so to me! There are other approaches, for instance we could use the Stone-Weierstrass Theorem; rather than do this we will deduce the Stone-Weierstrass Theorem from Proposition 4.1. Another good reason to proceed directly is that Fejér's approach is clever and generalizes in various ways as we will see.

So, the problem is to find the sequence in the span of the e_k which converges to a given continuous function and the trick is to use the Fourier expansion that we want to check! The idea of Cesàro is close to one we have seen before, namely to make this Fourier expansion 'converge faster', or maybe better. For the moment we can work with a general function $u \in L^2(0, 2\pi)$ – or think of it as continuous if you prefer. The truncated Fourier series of u is a finite linear combination of the e_k :

$$U_n(x) = \frac{1}{2\pi} \sum_{|k| \leq n} (\int_{(0,2\pi)} u(t)e^{-ikt}dt)e^{ikx} \tag{4.7}$$

where I have just inserted the definition of the c_k's into the sum. Since this is a finite sum we can treat x as a parameter and use the linearity of the integral to write it as

$$U_n(x) = \int_{(0,2\pi)} D_n(x - t)u(t), \ D_n(s) = \frac{1}{2\pi} \sum_{|k| \leq n} e^{iks}. \tag{4.8}$$

Now this sum can be written as an explicit quotient, since, by telescoping,

$$2\pi D_n(s)(e^{is/2} - e^{-is/2}) = e^{i(n+\frac{1}{2})s} - e^{-i(n+\frac{1}{2})s}. \tag{4.9}$$

So in fact, at least where $s \neq 0$,

$$D_n(s) = \frac{e^{i(n+\frac{1}{2})s} - e^{-i(n+\frac{1}{2})s}}{2\pi(e^{is/2} - e^{-is/2})} \tag{4.10}$$

and the limit as $s \to 0$ exists just fine.

As I said, Cesàro's idea is to speed up the convergence by replacing U_n by its average

$$V_n(x) = \frac{1}{n+1} \sum_{l=0}^{n} U_l. \tag{4.11}$$

Again plugging in the definitions of the U_l's and using the linearity of the integral we see that

$$V_n(x)=\int_{(0,2\pi)} S_n(x-t)u(t),\ S_n(s)=\frac{1}{n+1}\sum_{l=0}^{n} D_l(s). \tag{4.12}$$

So again we want to compute a more useful form for $S_n(s)$ – which is the Fejér kernel. Since the denominators in (4.10) are all the same,

$$2\pi(n+1)(e^{is/2}-e^{-is/2})S_n(s)=\sum_{l=0}^{n} e^{i(l+\frac{1}{2})s}-\sum_{l=0}^{n} e^{-i(l+\frac{1}{2})s}. \tag{4.13}$$

Using the same trick again,

$$(e^{is/2}-e^{-is/2})\sum_{l=0}^{n} e^{i(l+\frac{1}{2})s}=e^{i(n+1)s}-1 \tag{4.14}$$

so

$$\begin{gathered}2\pi(n+1)(e^{is/2}-e^{-is/2})^2S_n(s)=e^{i(n+1)s}+e^{-i(n+1)s}-2\\ \Longrightarrow S_n(s)=\frac{1}{n+1}\frac{\sin^2(\frac{(n+1)}{2}s)}{2\pi\sin^2(\frac{s}{2})}.\end{gathered} \tag{4.15}$$

Now, what can we say about this function? One thing we know immediately is that if we plug $u=1$ into the disucssion above, we get $U_n=1$ for $n\ge 0$ and hence $V_n=1$ as well. Thus in fact

$$\int_{(0,2\pi)} S_n(x-\cdot)=1,\ \forall\ x\in(0,2\pi). \tag{4.16}$$

Looking directly at (4.15) the first thing to notice is that $S_n(s)\ge 0$. Also, we can see that the denominator only vanishes when $s=0$ or $s=2\pi$ in $[0,2\pi]$. Thus if we stay away from there, say $s\in(\delta,2\pi-\delta)$ for some $\delta>0$ then – since $\sin(t)$ is a bounded function

$$|S_n(s)|\le(n+1)^{-1}C_\delta \text{ on } (\delta,2\pi-\delta). \tag{4.17}$$

We are interested in how close $V_n(x)$ is to the given $u(x)$ in supremum norm, where now we will take u to be continuous. Because of (4.16) we can write

$$u(x)=\int_{(0,2\pi)} S_n(x-t)u(x) \tag{4.18}$$

where t denotes the variable of integration (and x is fixed in $[0,2\pi]$). This 'trick' means that the difference is

$$V_n(x)-u(x)=\int_{(0,2\pi)} S_n(x-t)(u(t)-u(x)). \tag{4.19}$$

For each x we split this integral into two parts, the set $\Gamma(x)$ where $x-t\in[0,\delta]$ or $x-t\in[2\pi-\delta,2\pi]$ and the remainder. So

$$|V_n(x)-u(x)|\le\int_{\Gamma(x)} S_n(x-t)|u(t)-u(x)|+\int_{(0,2\pi)\setminus\Gamma(x)} S_n(x-t)|u(t)-u(x)|. \tag{4.20}$$

Now on $\Gamma(x)$ either $|t-x|\le\delta$ – the points are close together – or t is close to 0 and x to 2π so $2\pi-x+t\le\delta$ or conversely, x is close to 0 and t to 2π so $2\pi-t+x\le\delta$.

In any case, by assuming that $u(0) = u(2\pi)$ and using the uniform continuity of a continuous function on $[0, 2\pi]$, given $\epsilon > 0$ we can choose δ so small that

$$|u(x) - u(t)| \le \epsilon/2 \text{ on } \Gamma(x). \tag{4.21}$$

On the complement of $\Gamma(x)$ we have (4.17) and since u is bounded we get the estimate

$$|V_n(x) - u(x)| \le \epsilon/2 \int_{\Gamma(x)} S_n(x-t) + (n+1)^{-1}C'(\delta) \le \epsilon/2 + (n+1)^{-1}C'(\delta). \tag{4.22}$$

Here the fact that S_n is non-negative and has integral one has been used again to estimate the integral of $S_n(x-t)$ over $\Gamma(x)$ by 1. Having chosen δ to make the first term small, we can choose n large to make the second term small and it follows that

$$V_n(x) \to u(x) \text{ uniformly on } [0, 2\pi] \text{ as } n \to \infty \tag{4.23}$$

under the assumption that $u \in \mathcal{C}([0, 2\pi])$ satisfies $u(0) = u(2\pi)$.

So this proves Proposition 4.1 subject to the density in $L^2(0, 2\pi)$ of the continuous functions which vanish near (but not of course in a fixed neighbourhood of) the ends. In fact we know that the L^2 functions which vanish near the ends are dense since we can chop off and use the fact that

$$\lim_{\delta \to 0} \left(\int_{(0,\delta)} |f|^2 + \int_{(2\pi-\delta, 2\pi)} |f|^2 \right) = 0. \tag{4.24}$$

This proves Theorem 4.1.

Notice that from what we have shown it follows that the finite linear combinations of the $\exp(ikx)$ are dense in $\mathcal{C}^0([0, 2\pi])$. Taking a general element $u \in \mathcal{C}^0([0, 2\pi]$ we can choose constants so that

$$v = u - c - dx \in \mathcal{C}^0([0, 2\pi]) \text{ satisfies } v(0) = v(2\pi) = 0. \tag{4.25}$$

Indeed we just need to take $c = u(0)$, $d = u(1) - c$. Then we know that v is the uniform limit of a sequence of finite sums of the $\exp(ikx)$. However, the Taylor series

$$e^{ikx} = \sum_l \frac{(ik)^l}{l!} x^k \tag{4.26}$$

converges uniformly to e^{ikx} in any (complex) disk. So it follows in turn that the polynomials are dense

Theorem 4.2 (Stone-Weierstrass). *The polynomials are dense in* $\mathcal{C}^0([a, b])$ *for any* $a < b$, *in the uniform topology.*

2. Dirichlet problem on an interval

I want to do a couple more 'serious' applications of what we have done so far. There are many to choose from, and I will mention some more, but let me first consider the Diriclet problem on an interval. I will choose the interval $[0, 2\pi]$ because we looked at it before but of course we could work on a general bounded interval instead. So, we are supposed to be trying to solve

$$-\frac{d^2u(x)}{dx^2} + V(x)u(x) = f(x) \text{ on } (0, 2\pi),\ u(0) = u(2\pi) = 0 \tag{4.27}$$

where the last part are the Dirichlet boundary conditions. I will assume that the 'potential'

$$V : [0, 2\pi] \longrightarrow \mathbb{R} \text{ is continuous and real-valued.} \tag{4.28}$$

Now, it certainly makes sense to try to solve the equation (4.27) for say a given $f \in \mathcal{C}^0([0, 2\pi])$, looking for a solution which is twice continuously differentiable on the interval. It may not exist, depending on V but one thing we can shoot for, which has the virtue of being explicit, is the following:

PROPOSITION 4.2. *If $V \geq 0$ as in* (4.28) *then for each $f \in \mathcal{C}^0([0, 2\pi])$ there exists a unique twice continuously differentiable solution, u, to* (4.27).

There are in fact various approaches to this but we want to go through L^2 theory – not surprisingly of course. How to start?

Well, we do know how to solve (4.27) if $V \equiv 0$ since we can use (Riemann) integration. Thus, ignoring the boundary conditions for the moment, we can find a solution to $-d^2v/dx^2 = f$ on the interval by integrating twice:

$$v(x) = -\int_0^x \int_0^y f(t)dtdy \text{ satifies } -d^2v/dx^2 = f \text{ on } (0, 2\pi). \tag{4.29}$$

Moroever v really is twice continuously differentiable if f is continuous. So, what has this got to do with operators? Well, we can change the order of integration in (4.29) to write v as

$$v(x) = -\int_0^x \int_t^x f(t)dydt = \int_0^{2\pi} a(x,t)f(t)dt,\ a(x,t) = (t-x)H(x-t) \tag{4.30}$$

where the Heaviside function $H(y)$ is 1 when $y \geq 0$ and 0 when $y < 0$. Thus $a(x,t)$ is actually continuous on $[0, 2\pi] \times [0, 2\pi]$ since the $t - x$ factor vanishes at the jump in $H(t - x)$. So (4.30) shows that v is given by applying an integral operator, with continuous kernel on the square, to f.

Before thinking more seriously about this, recall that there is also the matter of the boundary conditions. Clearly, $v(0) = 0$ since we integrated from there. On the other hand, there is no particular reason why

$$v(2\pi) = \int_0^{2\pi} (t - 2\pi)f(t)dt \tag{4.31}$$

should vanish. However, we can always add to v any linear function and still satify the differential equation. Since we do not want to spoil the vanishing at $x = 0$ we can only afford to add cx but if we choose the constant c correctly this will work. Namely consider

$$c = \frac{1}{2\pi}\int_0^{2\pi} (2\pi - t)f(t)dt, \text{ then } (v + cx)(2\pi) = 0. \tag{4.32}$$

So, finally the solution we want is

$$w(x) = \int_0^{2\pi} b(x,t)f(t)dt,\ b(x,t) = \min(t,x) - \frac{tx}{2\pi} \in \mathcal{C}([0, 2\pi]^2) \tag{4.33}$$

with the formula for b following by simple manipulation from

$$b(x,t) = a(x,t) + x - \frac{tx}{2\pi} \tag{4.34}$$

Thus there is a unique, twice continuously differentiable, solution of $-d^2w/dx^2 = f$ in $(0, 2\pi)$ which vanishes at both end points and it is given by the *integral operator* (4.33).

LEMMA 4.1. *The integral operator* (4.33) *extends by continuity from* $\mathcal{C}^0([0, 2\pi])$ *to a compact, self-adjoint operator on* $L^2(0, 2\pi)$.

PROOF. Since w is given by an integral operator with a continuous real-valued kernel which is even in the sense that (check it)

$$b(t, x) = b(x, t) \tag{4.35}$$

we might as well give a more general result. □

PROPOSITION 4.3. *If* $b \in \mathcal{C}^0([0, 2\pi]^2)$ *then*

$$Bf(x) = \int_0^{2\pi} b(x, t)f(t)dt \tag{4.36}$$

defines a compact operator on $L^2(0, 2\pi)$ *if in addition* b *satisfies*

$$\overline{b(t, x)} = b(x, t) \tag{4.37}$$

then B *is self-adjoint.*

PROOF. If $f \in L^2((0, 2\pi))$ and $v \in \mathcal{C}([0, 2\pi])$ then the product $vf \in L^2((0, 2\pi))$ and $\|vf\|_{L^2} \le \|v\|_\infty \|f\|_{L^2}$. This can be seen for instance by taking an absolutely summable approximation to f, which gives a sequence of continuous functions converging a.e. to f and bounded by a fixed L^2 function and observing that $vf_n \to vf$ a.e. with bound a constant multiple, $\sup |v|$, of that function. It follows that for $b \in \mathcal{C}([0, 2\pi]^2)$ the product

$$b(x, y)f(y) \in L^2(0, 2\pi) \tag{4.38}$$

for each $x \in [0, 2\pi]$. Thus $Bf(x)$ is well-defined by (4.37) since $L^2((0, 2\pi) \subset L^1((0, 2\pi))$.

Not only that, but $Bf \in \mathcal{C}([0, 2\pi])$ as can be seen from the Cauchy-Schwarz inequality,

(4.39)

$$|Bf(x') - Bf(x)| = |\int (b(x', y) - b(x, y))f(y)| \le \sup_y |b(x', y - b(x, y)|(2\pi)^{\frac{1}{2}} \|f\|_{L^2}.$$

Essentially the same estimate shows that

$$\sup_x \|Bf(x)\| \le (2\pi)^{\frac{1}{2}} \sup_{(x,y)} |b| \|f\|_{L^2} \tag{4.40}$$

so indeed, $B : L^2(0, 2\pi) \longrightarrow \mathcal{C}([0, 2\pi])$ is a bounded linear operator.

When b satisfies (4.37) and f and g are continuous

$$\int Bf(x)\overline{g(x)} = \int f(x)\overline{Bg(x)} \tag{4.41}$$

and the general case follows by approximation in L^2 by continuous functions.

So, we need to see the compactness. If we fix x then $b(x, y) \in \mathcal{C}([0, 2\pi])$ and then if we let x vary,

$$[0, 2\pi] \ni x \longmapsto b(x, \cdot) \in \mathcal{C}([0, 2\pi]) \tag{4.42}$$

is continuous as a map into this Banach space. Again this is the uniform continuity of a continuous function on a compact set, which shows that

$$\sup_y |b(x',y)-b(x,y)| \to 0 \text{ as } x' \to x. \tag{4.43}$$

Since the inclusion map $\mathcal{C}([0,2\pi]) \longrightarrow L^2((0,2\pi))$ is bounded, i.e continuous, it follows that the map (I have reversed the variables)

$$[0,2\pi] \ni y \longmapsto b(\cdot,y) \in L^2((0,2\pi)) \tag{4.44}$$

is continuous and so has a compact range.

Take the Fourier basis e_k for $[0,2\pi]$ and expand b in the first variable. Given $\epsilon > 0$ the compactness of the image of (4.44) implies that for some N

$$\sum_{|k|>N} |(b(x,y),e_k(x))|^2 < \epsilon \ \forall\ y \in [0,2\pi]. \tag{4.45}$$

The finite part of the Fourier series is continuous as a function of both arguments

$$b_N(x,y) = \sum_{|k|\le N} e_k(x)c_k(y),\ \ c_k(y) = (b(x,y),e_k(x)) \tag{4.46}$$

and so defines another bounded linear operator B_N as before. This operator can be written out as

$$B_N f(x) = \sum_{|k|\le N} e_k(x) \int c_k(y)f(y)dy \tag{4.47}$$

and so is of finite rank – it always takes values in the span of the first $2N+1$ trigonometric functions. On the other hand the remainder is given by a similar operator with corresponding to $q_N = b - b_N$ and this satisfies

$$\sup_y \|q_N(\cdot,y)\|_{L^2((0,2\pi))} \to 0 \text{ as } N \to \infty. \tag{4.48}$$

Thus, q_N has small norm as a bounded operator on $L^2((0,2\pi))$ so B is compact – it is the norm limit of finite rank operators. □

Now, recall from Problem# that $u_k = c\sin(kx/2)$, $k \in \mathbb{N}$, is also an orthonormal basis for $L^2(0,2\pi)$ (it is not the Fourier basis!) Moreover, differentiating we find straight away that

$$-\frac{d^2u_k}{dx^2} = \frac{k^2}{4}u_k. \tag{4.49}$$

Since of course $u_k(0) = 0 = u_k(2\pi)$ as well, from the uniqueness above we conclude that

$$Bu_k = \frac{4}{k^2}u_k \ \forall\ k. \tag{4.50}$$

Thus, in this case we know the orthonormal basis of eigenfunctions for B. They are the u_k, each eigenspace is 1 dimensional and the eigenvalues are $4k^{-2}$. So, this happenstance allows us to decompose B as the square of another operator defined directly on the othornormal basis. Namely

$$Au_k = \frac{2}{k}u_k \Longrightarrow B = A^2. \tag{4.51}$$

Here again it is immediate that A is a compact self-adjoint operator on $L^2(0,2\pi)$ since its eigenvalues tend to 0. In fact we can see quite a lot more than this.

Lemma 4.2. *The operator A maps $L^2(0,2\pi)$ into $\mathcal{C}^0([0,2\pi])$ and $Af(0) = Af(2\pi) = 0$ for all $f \in L^2(0,2\pi)$.*

Proof. If $f \in L^2(0,2\pi)$ we may expand it in Fourier-Bessel series in terms of the u_k and find

$$f = \sum_k c_k u_k,\ \{c_k\} \in l^2. \tag{4.52}$$

Then of course, by definition,

$$Af = \sum_k \frac{2c_k}{k} u_k. \tag{4.53}$$

Here each u_k is a bounded continuous function, with the bound on u_k being independent of k. So in fact (4.53) converges uniformly and absolutely since it is uniformly Cauchy, for any $q > p$,

$$|\sum_{k=p}^{q} \frac{2c_k}{k} u_k| \le 2|c| \sum_{k=p}^{q} |c_k| k^{-1} \le 2|c| \left(\sum_{k=p}^{q} k^{-2} \right)^{\frac{1}{2}} \|f\|_{L^2} \tag{4.54}$$

where Cauchy-Schwarz has been used. This proves that

$$A : L^2(0,2\pi) \longrightarrow \mathcal{C}^0([0,2\pi])$$

is bounded and by the uniform convergence $u_k(0) = u_k(2\pi) = 0$ for all k implies that $Af(0) = Af(2\pi) = 0$. □

So, going back to our original problem we try to solve (4.27) by moving the Vu term to the right side of the equation (don't worry about regularity yet) and hope to use the observation that

$$u = -A^2(Vu) + A^2 f \tag{4.55}$$

should satisfy the equation and boundary conditions. In fact, let's anticipate that $u = Av$, which has to be true if (4.55) holds with $v = -AVu + Af$, and look instead for

$$v = -AVAv + Af \Longrightarrow (\mathrm{Id} + AVA)v = Af. \tag{4.56}$$

So, we know that multiplication by V, which is real and continuous, is a bounded self-adjoint operator on $L^2(0,2\pi)$. Thus AVA is a self-adjoint compact operator so we can apply our spectral theory to it and so examine the invertibility of $\mathrm{Id} + AVA$. Working in terms of a complete orthonormal basis of eigenfunctions e_i of AVA we see that $\mathrm{Id} + AVA$ is invertible if and only if it has trivial null space, i.e. if -1 is *not* an eigenvalue of AVA. Indeed, an element of the null space would have to satisfy $u = -AVAu$. On the other hand we know that AVA is *positive* since

$$(AVAw, w) = (VAv, Av) = \int_{(0,2\pi)} V(x)|Av|^2 \ge 0 \Longrightarrow \int_{(0,2\pi)} |u|^2 = 0, \tag{4.57}$$

using the non-negativity of V. So, there can be no null space – all the eigenvalues of AVA are at least non-negative and the inverse is the bounded operator given by its action on the basis

$$(\mathrm{Id} + AVA)^{-1} e_i = (1 + \tau_i)^{-1},\ AVAe_i = \tau_i e_i. \tag{4.58}$$

Thus $\mathrm{Id}+AVA$ is invertible on $L^2(0,2\pi)$ with inverse of the form $\mathrm{Id}+Q$, Q again compact and self-adjoint since $(1+\tau_i)^1-1\to 0$. Now, to solve (4.56) we just need to take

$$v=(\mathrm{Id}+Q)Af \Longleftrightarrow v+AVAv=Af \text{ in } L^2(0,2\pi). \tag{4.59}$$

Then indeed

$$u=Av \text{ satisfies } u+A^2Vu=A^2f. \tag{4.60}$$

In fact since $v\in L^2(0,2\pi)$ from (4.59) we already know that $u\in\mathcal{C}^0([0,2\pi])$ vanishes at the end points.

Moreover if $f\in\mathcal{C}^0([0,2\pi])$ we know that $Bf=A^2f$ is twice continuously differentiable, since it is given by two integrations – that is where B came from. Now, we know that u in L^2 satisfies $u=-A^2(Vu)+A^2f$. Since $Vu\in L^2((0,2\pi)$ so is $A(Vu)$ and then, as seen above, $A(A(Vu)$ is continuous. So combining this with the result about A^2f we see that u itself is continuous and hence so is Vu. But then, going through the routine again

$$u=-A^2(Vu)+A^2f \tag{4.61}$$

is the sum of two twice continuously differentiable functions. Thus it is so itself. In fact from the properties of $B=A^2$ it satisifes

$$-\frac{d^2u}{dx^2}=-Vu+f \tag{4.62}$$

which is what the result claims. So, we have proved the existence part of Proposition 4.2.

The uniqueness follows pretty much the same way. If there were two twice continuously differentiable solutions then the difference w would satisfy

$$-\frac{d^2w}{dx^2}+Vw=0,\ w(0)=w(2\pi)=0\Longrightarrow w=-Bw=-A^2Vw. \tag{4.63}$$

Thus $w=A\phi$, $\phi=-AVw\in L^2(0,2\pi)$. Thus ϕ in turn satisfies $\phi=AVA\phi$ and hence is a solution of $(\mathrm{Id}+AVA)\phi=0$ which we know has none (assuming $V\ge 0$). Since $\phi=0$, $w=0$.

This completes the proof of Proposition 4.2. To summarize, what we have shown is that $\mathrm{Id}+AVA$ is an invertible bounded operator on $L^2(0,2\pi)$ (if $V\ge 0$) and then the solution to (4.27) is precisely

$$u=A(\mathrm{Id}+AVA)^{-1}Af \tag{4.64}$$

which is twice continuously differentiable and satisfies the Dirichlet conditions for each $f\in\mathcal{C}^0([0,2\pi])$.

Now, even if we do not assume that $V\ge 0$ we pretty much know what is happening.

Proposition 4.4. *For any $V\in\mathcal{C}^0([0,2\pi])$ real-valued, there is an orthonormal basis w_k of $L^2(0,2\pi)$ consisting of twice-continuously differentiable functions on $[0,2\pi]$, vanishing at the end-points and satisfying $-\frac{d^2w_k}{dx^2}+Vw_k=T_kw_k$ where $T_k\to\infty$ as $k\to\infty$. The equation (4.27) has a (twice continuously differentiable) solution for given $f\in\mathcal{C}^0([0,2\pi])$ if and only if*

$$T_k=0\Longrightarrow\int_{(0,2\pi)}fw_k=0, \tag{4.65}$$

i.e. f *is orthogonal to the null space of* $\mathrm{Id} + A^2V$, *which is the image under* A *of the null space of* $\mathrm{Id} + AVA$, *in* $L^2(0, 2\pi)$.

PROOF. Notice the form of the solution in case $V \geq 0$ in (4.64). In general, we can choose a constant c such that $V + c \geq 0$. Then the equation can be rewritten

$$-\frac{d^2w}{dx^2} + Vw = Tw_k \Longleftrightarrow -\frac{d^2w}{dx^2} + (V+c)w = (T+c)w. \tag{4.66}$$

Thus, if w satisfies this eigen-equation then it also satisfies

$$\begin{aligned} w = (T+c)A(\mathrm{Id} + A(V+c)A)^{-1}Aw \Longleftrightarrow \\ Sw = (T+c)^{-1}w,\ S = A(\mathrm{Id} + A(V+c)A)^{-1}A. \end{aligned} \tag{4.67}$$

Now, we have shown that S is a compact self-adjoint operator on $L^2(0, 2\pi)$ so we know that it has a complete set of eigenfunctions, e_k, with eigenvalues $\tau_k \neq 0$. From the discussion above we then know that each e_k is actually continuous – since it is Aw' with $w' \in L^2(0, 2\pi)$ and hence also twice continuously differentiable. So indeed, these e_k satisfy the eigenvalue problem (with Dirichlet boundary conditions) with eigenvalues

$$T_k = \tau_k^{-1} + c \to \infty \text{ as } k \to \infty. \tag{4.68}$$

The solvability part also follows in much the same way. □

3. Friedrichs' extension

Next I will discuss an abstract Hilbert space set-up which covers the treatment of the Dirichlet problem above and several other applications to differential equations and indeed to other problems. I am attributing this method to Friedrichs and he certainly had a hand in it.

Instead of just one Hilbert space we will consider two at the same time. First is a 'background' space, H, a separable infinite-dimensional Hilbert space which you can think of as being something like $L^2(I)$ for some interval I. The inner product on this I will denote $(\cdot, \cdot)_H$ or maybe sometimes leave off the 'H' since this is the basic space. Let me denote a second, separable infinite-dimensional, Hilbert space as D, which maybe stands for 'domain' of some operator. So D comes with its own inner product $(\cdot, \cdot)_D$ where I will try to remember not to leave off the subscript. The relationship between these two Hilbert spaces is given by a linear map

$$i : D \longrightarrow H. \tag{4.69}$$

This is denoted 'i' because it is supposed to be an 'inclusion'. In particular I will always require that

$$i \text{ is injective.} \tag{4.70}$$

Since we will not want to have parts of H which are inaccessible, I will also assume that

$$i \text{ has dense range } i(D) \subset H. \tag{4.71}$$

In fact because of these two conditions it is quite safe to identify D with $i(D)$ and think of each element of D as really being an element of H. The subspace '$i(D) = D$' will not be closed, which is what we are used to thinking about (since it is dense) but rather has its own inner product $(\cdot, \cdot)_D$. Naturally we will also suppose that i is continuous and to avoid too many constants showing up I will suppose that i has norm at most 1 so that

$$\|i(u)\|_H \le \|u\|_D. \tag{4.72}$$

If you are comfortable identifying $i(D)$ with D this just means that the 'D-norm' on D is *bigger* than the H norm restricted to D. A bit later I will assume one more thing about i.

What can we do with this setup? Well, consider an arbitrary element $f \in H$. Then consider the linear map

$$T_f : D \ni u \longrightarrow (i(u), f)_H \in \mathbb{C}. \tag{4.73}$$

where I have put in the identification i but will leave it out from now on, so just write $T_f(u) = (u, f)_H$. This is in fact a continuous linear functional on D since by Cauchy-Schwarz and then (4.72),

$$|T_f(u)| = |(u, f)_H| \le \|u\|_H \|f\|_H \le \|f\|_H \|u\|_D. \tag{4.74}$$

So, by the Riesz' representation – so using the assumed completeness of D (with respect to the D-norm of course) there exists a unique element $v \in D$ such that

$$(u, f)_H = (u, v)_D \ \forall \ u \in D. \tag{4.75}$$

Thus, v only depends on f and always exists, so this defines a map

$$B : H \longrightarrow D, \ Bf = v \text{ iff } (f, u)_H = (v, u)_D \ \forall \ u \in D \tag{4.76}$$

where I have taken complex conjugates of both sides of (4.75).

Lemma 4.3. *The map B is a continuous linear map $H \longrightarrow D$ and restricted to D is self-adjoint:*

$$(Bw, u)_D = (w, Bu)_D \ \forall \ u, w \in D. \tag{4.77}$$

The assumption that $D \subset H$ is dense implies that $B : H \longrightarrow D$ is injective.

Proof. The linearity follows from the uniqueness and the definition. Thus if $f_i \in H$ and $c_i \in \mathbb{C}$ for $i = 1, 2$ then

$$\begin{aligned} (c_1 f_1 + c_2 f_2, u)_H &= c_1(f_1, u)_H + c_2(f_2, u)_H \\ &= c_1(Bf_1, u)_D + c_2(Bf_2, u)_D = (c_1 Bf_1 + c_2 Bf_2, u)_D \ \forall \ u \in D \end{aligned} \tag{4.78}$$

shows that $B(c_1 f_1 + c_2 f_2) = c_1 Bf_1 + c_2 Bf_2$. Moreover from the estimate (4.74),

$$|(Bf, u)_D| \le \|f\|_H \|u\|_D \tag{4.79}$$

and setting $u = Bf$ it follows that $\|Bf\|_D \le \|f\|_H$ which is the desired continuity.

To see the self-adjointness suppose that u, $w \in D$, and hence of course since we are erasing i, u, $w \in H$. Then, from the definitions

$$(Bu, w)_D = (u, w)_H = \overline{(w, u)_H} = \overline{(Bw, u)_D} = (u, Bw)_D \tag{4.80}$$

so B is self-adjoint.

Finally observe that $Bf = 0$ implies that $(Bf, u)_D = 0$ for all $u \in D$ and hence that $(f, u)_H = 0$, but since D is dense, this implies $f = 0$ so B is injective. □

To go a little further we will assume that the inclusion i is *compact.* Explicitly this means

$$u_n \rightharpoonup_D u \Longrightarrow u_n (= i(u_n)) \to_H u \tag{4.81}$$

where the subscript denotes which space the convergence is in. Thus compactness means that a weakly convergent sequence in D is, or is mapped to, a strongly convergent sequence in H.

LEMMA 4.4. *Under the assumptions* (4.69), (4.70), (4.71), (4.72) *and* (4.81) *on the inclusion of one Hilbert space into another, the operator B in* (4.76) *is compact as a self-adjoint operator on D and has only positive eigenvalues.*

PROOF. Suppose $u_n \rightharpoonup u$ is weakly convergent in D. Then, by assumption it is strongly convergent in H. But B is continuous as a map from H to D so $Bu_n \to Bu$ in D and it follows that B is compact as an operator on D.

So, we know that D has an orthonormal basis of eigenvectors of B. None of the eigenvalues λ_j can be zero since B is injective. Moreover, from the definition if $Bu_j = \lambda_j u_j$ then

$$\|u_j\|_H^2 = (u_j, u_j)_H = (Bu_j, u_j)_D = \lambda_j \|u_j\|_D^2 \tag{4.82}$$

showing that $\lambda_j > 0$. □

Now, in view of this we can define another compact operator on D by

$$Au_j = \lambda_j^{\frac{1}{2}} u_j \tag{4.83}$$

taking the positive square-roots. So of course $A^2 = B$. In fact $A : H \longrightarrow D$ is also a bounded operator.

LEMMA 4.5. *If u_j is an orthonormal basis of D of eigenvectors of B then $f_j = \lambda^{-\frac{1}{2}} u_j$ is an orthonormal basis of H and $A : D \longrightarrow D$ extends by continuity to an isometric isomorphism $A : H \longrightarrow D$.*

PROOF. The identity (4.82) extends to pairs of eigenvectors

$$(u_j, u_k)_H = (Bu_j, u_k)_D = \lambda_j \delta_{jk} \tag{4.84}$$

which shows that the f_j form an orthonormal sequence in H. The span is dense in D (in the H norm) and hence is dense in H so this set is complete. Thus A maps an orthonormal basis of H to an orthonormal basis of D, so it is an isometric isomorphism. □

If you think about this a bit you will see that this is an abstract version of the treatment of the 'trivial' Dirichlet problem above, except that I did not describe the Hilbert space D concretely in that case.

There are various ways this can be extended. One thing to note is that the failure of injectivity, i.e. the loss of (4.70) is not so crucial. If i is not injective, then its null space is a closed subspace and we can take its orthocomplement in place of D. The result is the same except that the operator D is only defined on this orthocomplement.

An additional thing to observe is that the completeness of D, although used crucially above in the application of Riesz' Representation theorem, is not really such a big issue either

PROPOSITION 4.5. *Suppose that $\tilde{D}$ is a* pre-Hilbert *space with inner product $(\cdot,\cdot)_D$ and $i : \tilde{A} \longrightarrow H$ is a linear map into a Hilbert space. If this map is injective, has dense range and satisfies* (4.72) *in the sense that*

$$\|i(u)\|_H \le \|u\|_D \ \forall\ u \in \tilde{D} \tag{4.85}$$

then it extends by continuity to a map of the completion, D, of $\tilde{D}$, satisfying (4.70), (4.71) *and* (4.72) *and if bounded sets in $\tilde{D}$ are mapped by i into precompact sets in H then* (4.81) *also holds.*

PROOF. We know that a completion exists, $\tilde{D} \subset D$, with inner product restricting to the given one and every element of D is then the limit of a Cauchy sequence in $\tilde{D}$. So we denote without ambiguity the inner product on D again as $(\cdot,\cdot)_D$. Since i is continuous with respect to the norm on D (and on H of course) it extends by continuity to the closure of $\tilde{D}$, namely D as $i(u) = \lim_n i(u_n)$ if u_n is Cauchy in $\tilde{D}$ and hence converges in D; this uses the completeness of H since $i(u_n)$ is Cauchy in H. The value of $i(u)$ does not depend on the choice of approximating sequence, since if $v_n \to 0$, $i(v_n) \to 0$ by continuity. So, it follows that $i : D \longrightarrow H$ exists, is linear and continuous and its norm is no larger than before so (4.69) holds. □

The map extended map may not be injective, i.e. it might happen that $i(u_n) \to 0$ even though $u_n \to u \neq 0$.

The general discussion of the set up of Lemmas 4.4 and 4.5 can be continued further. Namely, having defined the operators B and A we can define a new positive-definite Hermitian form on H by

$$(u,v)_E = (Au, Av)_H,\ u,\ v \in H \tag{4.86}$$

with the same relationship as between $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_D$. Now, it follows directly that

$$\|u\|_H \le \|u\|_E \tag{4.87}$$

so if we let E be the completion of H with respect to this new norm, then $i : H \longrightarrow E$ is an injection with dense range and A extends to an isometric isomorphism $A : E \longrightarrow H$. Then if u_j is an orthonormal basis of H of eigenfunctions of A with eigenvalues $\tau_j > 0$ it follows that $u_j \in D$ and that the $\tau_j^{-1} u_j$ form an orthonormal basis for D while the $\tau_j u_j$ form an orthonormal basis for E.

LEMMA 4.6. *With E defined as above as the completion of H with respect to the inner product* (4.86)*, B extends by continuity to an isomoetric isomorphism $B : E \longrightarrow D$.*

PROOF. Since $B = A^2$ on H this follows from the properties of the eigenbases above. □

The typical way that Friedrichs' extension arises is that we are actually given an explicit 'operator', a linear map $P : \tilde{D} \longrightarrow H$ such that $(u,v)_D = (u, Pv)_H$ satisfies the conditions of Proposition 4.5. Then P extends by continuity to an isomorphism $P : D \longrightarrow E$ which is precisely the inverse of B as in Lemma 4.6. We shall see examples of this below.

4. Dirichlet problem revisited

So, does the setup of the preceding section work for the Dirichlet problem? We take $H = L^2((0, 2\pi))$. Then, and this really is Friedrichs' extension, we take as a subspace $\tilde{D} \subset H$ the space of functions which are once continuously differentiable and vanish outside a compact subset of $(0, 2\pi)$. This just means that there is some smaller interval, depending on the function, $[\delta, 2\pi - \delta]$, $\delta > 0$, on which we have a continuously differentiable function f with $f(\delta) = f'(\delta) = f(2\pi-\delta) = f'(2\pi-\delta) = 0$ and then we take it to be zero on $(0, \delta)$ and $(2\pi - \delta, 2\pi)$. There are lots of these, let's call the space $\tilde{D}$ as above

$$\begin{aligned} \tilde{D} = \{u \in \mathcal{C}^0[0, 2\pi]; u \text{ continuously differentiable on } [0, 2\pi], \\ u(x) = 0 \text{ in } [0, \delta] \cup [2\pi - \delta, 2\pi] \text{ for some } \delta > 0\}. \end{aligned} \tag{4.88}$$

Then our first claim is that

$$\tilde{D} \text{ is dense in } L^2(0, 2\pi) \tag{4.89}$$

with respect to the norm on L^2 of course.

What inner product should we take on $\tilde{D}$? Well, we can just integrate formally by parts and set

$$(u, v)_D = \frac{1}{2\pi} \int_{[0,2\pi]} \frac{du}{dx} \overline{\frac{dv}{dx}} dx. \tag{4.90}$$

This is a pre-Hilbert inner product. To check all this note first that $(u, u)_D = 0$ implies $du/dx = 0$ by Riemann integration (since $|du/dx|^2$ is continuous) and since $u(x) = 0$ in $x < \delta$ for some $\delta > 0$ it follows that $u = 0$. Thus $(u, v)_D$ makes $\tilde{D}$ into a pre-Hilbert space, since it is a positive definite sesquilinear form. So, what about the completion? Observe that, the elements of $\tilde{D}$ being continuously differentiable, we can always integrate from $x = 0$ and see that

$$u(x) = \int_0^x \frac{du}{dx} dx \tag{4.91}$$

as $u(0) = 0$. Now, to say that $u_n \in \tilde{D}$ is Cauchy is to say that the continuous functions $v_n = du_n/dx$ are Cauchy in $L^2(0, 2\pi)$. Thus, from the completeness of L^2 we know that $v_n \to v \in L^2(0, 2\pi)$. On the other hand (4.91) applies to each u_n so

$$|u_n(x) - u_m(x)| = |\int_0^x (v_n(s) - v_m(s))ds| \le \sqrt{2\pi}\|v_n - v_m\|_{L^2} \tag{4.92}$$

by applying Cauchy-Schwarz. Thus in fact the sequence u_n is uniformly Cauchy in $C([0, 2\pi])$ if u_n is Cauchy in $\tilde{D}$. From the completeness of the Banach space of continuous functions it follows that $u_n \to u$ in $C([0, 2\pi])$ so each element of the completion, $\tilde{D}$, 'defines' (read 'is') a continuous function:

$$u_n \to u \in D \Longrightarrow u \in \mathcal{C}([0, 2\pi]),\ u(0) = u(2\pi) = 0 \tag{4.93}$$

where the Dirichlet condition follows by continuity from (4.92).

Thus we do indeed get an injection

$$D \ni u \longrightarrow u \in L^2(0, 2\pi) \tag{4.94}$$

where the injectivity follows from (4.91) that if $v = \lim du_n/dx$ vanishes in L^2 then $u = 0$.

Now, you can go ahead and check that with these definitions, B and A are the same operators as we constructed in the discussion of the Dirichlet problem.

5. Harmonic oscillator

As a second 'serious' application of our Hilbert space theory I want to discuss the harmonic oscillator, the corresponding Hermite basis for $L^2(\mathbb{R})$. Note that so far we have not found an explicit orthonormal basis on the whole real line, even though we know $L^2(\mathbb{R})$ to be separable, so we certainly know that such a basis exists. How to construct one explicitly and with some handy properties? One way is to simply orthonormalize – using Gram-Schmidt – some countable set with dense span. For instance consider the basic Gaussian function

$$\exp(-\frac{x^2}{2}) \in L^2(\mathbb{R}). \tag{4.95}$$

This is so rapidly decreasing at infinity that the product with any polynomial is also square integrable:

$$x^k \exp(-\frac{x^2}{2}) \in L^2(\mathbb{R}) \ \forall \ k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}. \tag{4.96}$$

Orthonormalizing this sequence gives an orthonormal basis, where completeness can be shown by an appropriate approximation technique but as usual is not so simple. This is in fact the Hermite basis as we will eventually show.

Rather than proceed directly we will work up to this by discussing the eigenfunctions of the harmonic oscillator

$$P = -\frac{d^2}{dx^2} + x^2 \tag{4.97}$$

which we want to think of as an operator – although for the moment I will leave vague the question of what it operates *on*.

As you probably already know, and we will show later, it is straightforward to show that P has a lot of eigenvectors using the 'creation' and 'annihilation' operators. We want to know a bit more than this and in particular I want to apply the abstract discussion above to this case but first let me go through the 'formal' theory. There is nothing wrong here, just that we cannot easily conclude the completeness of the eigenfunctions.

The first thing to observe is that the Gaussian is an eigenfunction of H

$$\begin{aligned} Pe^{-x^2/2} &= -\frac{d}{dx}(-xe^{-x^2/2} + x^2 e^{-x^2/2}) \\ &= -(x^2-1)e^{-x^2/2} + x^2 e^{-x^2/2} = e^{-x^2/2} \end{aligned} \tag{4.98}$$

with eigenvalue 1. It is an eigenfunction but not, for the moment, of a bounded operator on any Hilbert space – in this sense it is only a formal eigenfunction.

In this special case there is an essentially algebraic way to generate a whole sequence of eigenfunctions from the Gaussian. To do this, write

$$\begin{aligned} Pu &= (-\frac{d}{dx} + x)(\frac{d}{dx} + x)u + u = (\operatorname{Cr}\operatorname{An} + 1)u, \\ &\operatorname{Cr} = (-\frac{d}{dx} + x),\ \operatorname{An} = (\frac{d}{dx} + x) \end{aligned} \tag{4.99}$$

again formally as operators. Then note that

$$\operatorname{An} e^{-x^2/2} = 0 \tag{4.100}$$

which again proves (4.98). The two operators in (4.99) are the 'creation' operator and the 'annihilation' operator. They almost commute in the sense that

$$[\operatorname{An}, \operatorname{Cr}]u = (\operatorname{An}\operatorname{Cr} - \operatorname{Cr}\operatorname{An})u = 2u \tag{4.101}$$

for say any twice continuously differentiable function u.

Now, set $u_0 = e^{-x^2/2}$ which is the 'ground state' and consider $u_1 = \operatorname{Cr} u_0$. From (4.101), (4.100) and (4.99),

$$Pu_1 = (\operatorname{Cr}\operatorname{An}\operatorname{Cr} + \operatorname{Cr})u_0 = \operatorname{Cr}^2 \operatorname{An} u_0 + 3\operatorname{Cr} u_0 = 3u_1. \tag{4.102}$$

Thus, u_1 is an eigenfunction with eigenvalue 3.

LEMMA 4.7. *For $j \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ the function $u_j = \operatorname{Cr}^j u_0$ satisfies $Pu_j = (2j+1)u_j$.*

PROOF. This follows by induction on j, where we know the result for $j = 0$ and $j = 1$. Then

$$P \operatorname{Cr} u_j = (\operatorname{Cr}\operatorname{An} + 1)\operatorname{Cr} u_j = \operatorname{Cr}(P-1)u_j + 3\operatorname{Cr} u_j = (2j+3)u_j. \tag{4.103}$$

□

Again by induction we can check that $u_j = (2^j x^j + q_j(x))e^{-x^2/2}$ where q_j is a polynomial of degree at most $j-2$. Indeed this is true for $j = 0$ and $j = 1$ (where $q_0 = q_1 \equiv 0$) and then

$$\operatorname{Cr} u_j = (2^{j+1}x^{j+1} + \operatorname{Cr} q_j)e^{-x^2/2}. \tag{4.104}$$

and $q_{j+1} = \operatorname{Cr} q_j$ is a polynomial of degree at most $j-1$ – one degree higher than q_j.

From this it follows in fact that the finite span of the u_j consists of all the products $p(x)e^{-x^2/2}$ where $p(x)$ is any polynomial.

Now, all these functions are in $L^2(\mathbb{R})$ and we want to compute their norms. First, a standard integral computation[1] shows that

$$\int_{\mathbb{R}} (e^{-x^2/2})^2 = \int_{\mathbb{R}} e^{-x^2} = \sqrt{\pi} \tag{4.105}$$

For $j > 0$, integration by parts (easily justified by taking the integral over $[-R, R]$ and then letting $R \to \infty$) gives

$$\int_{\mathbb{R}} (\operatorname{Cr}^j u_0)^2 = \int_{\mathbb{R}} \operatorname{Cr}^j u_0(x) \operatorname{Cr}^j u_0(x) dx = \int_{\mathbb{R}} u_0 \operatorname{An}^j \operatorname{Cr}^j u_0. \tag{4.106}$$

[1]To compute the Gaussian integral, square it and write as a double integral then introduce polar coordinates

$$(\int_{\mathbb{R}} e^{-x^2} dx)^2 = \int_{\mathbb{R}^2} e^{-x^2-y^2} dxdy = \int_0^\infty \int_0^{2\pi} e^{-r^2} r dr d\theta = \pi\big[-e^{-r^2}\big]_0^\infty = \pi.$$

Now, from (4.101), we can move one factor of An through the j factors of Cr until it emerges and 'kills' u_0

$$\begin{aligned}(4.107)\quad \operatorname{An}\operatorname{Cr}^j u_0 &= 2\operatorname{Cr}^{j-1} u_0 + \operatorname{Cr}\operatorname{An}\operatorname{Cr}^{j-1} u_0 \\ &= 2\operatorname{Cr}^{j-1} u_0 + \operatorname{Cr}^2\operatorname{An}\operatorname{Cr}^{j-2} u_0 = 2j\operatorname{Cr}^{j-1} u_0.\end{aligned}$$

So in fact,

$$\int_{\mathbb{R}} (\operatorname{Cr}^j u_0)^2 = 2j\int_{\mathbb{R}} (\operatorname{Cr}^{j-1} u_0)^2 = 2^j j!\sqrt{\pi}. \tag{4.108}$$

A similar argument shows that

$$\int_{\mathbb{R}} u_k u_j = \int_{\mathbb{R}} u_0 \operatorname{An}^k \operatorname{Cr}^j u_0 = 0 \text{ if } k \neq j. \tag{4.109}$$

Thus the functions

$$e_j = 2^{-j/2}(j!)^{-\frac{1}{2}}\pi^{-\frac{1}{4}}C^j e^{-x^2/2} \tag{4.110}$$

form an orthonormal sequence in $L^2(\mathbb{R})$.

We would like to show this orthonormal sequence is complete. Rather than argue through approximation, we can guess that in some sense the operator

$$\operatorname{An}\operatorname{Cr} = (\frac{d}{dx} + x)(-\frac{d}{dx} + x) = -\frac{d^2}{dx^2} + x^2 + 1 \tag{4.111}$$

should be invertible, so one approach is to use the ideas above of Friedrichs' extension to construct its 'inverse' and show this really exists as a compact, self-adjoint operator on $L^2(\mathbb{R})$ and that its only eigenfunctions are the e_i in (4.110). Another, more indirect approach is described below.

6. Isotropic space

There are some functions which should be in the domain of P, namely the twice continuously differentiable functions on $\mathbb{R}$ with compact support, those which vanish outside a finite interval. Recall that there are actually a lot of these, they are dense in $L^2(\mathbb{R})$. Following what we did above for the Dirichlet problem set

$$\begin{aligned}(4.112)\quad \tilde{D} = \{u : \mathbb{R} \longmapsto \mathbb{C}; \exists\ R \text{ s.t. } u = 0 \text{ in } |x| > R, \\ u \text{ is twice continuously differentiable on } \mathbb{R}\}.\end{aligned}$$

Now for such functions integration by parts on a large enough interval (depending on the functions) produces no boundary terms so

$$(Pu, v)_{L^2} = \int_{\mathbb{R}} (Pu)\overline{v} = \int_{\mathbb{R}} \left(\frac{du}{dx}\overline{\frac{dv}{dx}} + x^2 u\overline{v}\right) = (u, v)_{\text{iso}} \tag{4.113}$$

is a positive definite hermitian form on $\tilde{D}$. Indeed the vanishing of $\|u\|_S$ implies that $\|xu\|_{L^2} = 0$ and so $u = 0$ since $u \in \tilde{D}$ is continuous. The suffix 'iso' here stands for 'isotropic' and refers to the fact that xu and du/dx are essentially on the same footing here. Thus

$$(u, v)_{\text{iso}} = (\frac{du}{dx}, \frac{dv}{dx})_{L^2} + (xu, xv)_{L^2}. \tag{4.114}$$

This may become a bit clearer later when we get to the Fourier transform.

Definition 4.1. Let $H^1_{\text{iso}}(\mathbb{R})$ be the completion of $\tilde{D}$ in (4.112) with respect to the inner product $(\cdot, \cdot)_{\text{iso}}$.

Proposition 4.6. *The inclusion map $i : \tilde{D} \longrightarrow L^2(\mathbb{R})$ extends by continuity to $i : H^1_{\text{iso}} \longrightarrow L^2(\mathbb{R})$ which satisfies (4.69), (4.70), (4.71), (4.72) and (4.81) with $D = H^1_{\text{iso}}$ and $H = L^2(\mathbb{R})$ and the derivative and multiplication maps define an injection*

$$H^1_{\text{iso}} \longrightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R}). \tag{4.115}$$

Proof. Let us start with the last part, (4.115). The map here is supposed to be the continuous extension of the map

$$\tilde{D} \ni u \longmapsto (\frac{du}{dx}, xu) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \tag{4.116}$$

where du/dx and xu are both compactly supported continuous functions in this case. By definition of the inner product $(\cdot,\cdot)_{\text{iso}}$ the norm is precisely

$$\|u\|^2_{\text{iso}} = \|\frac{du}{dx}\|^2_{L^2} + \|xu\|^2_{L^2} \tag{4.117}$$

so if u_n is Cauchy in $\tilde{D}$ with respect to $\|\cdot\|_{\text{iso}}$ then the sequences du_n/dx and xu_n are Cauchy in $L^2(\mathbb{R})$. By the completeness of L^2 they converge defining an element in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ as in (4.115). Moreover the elements so defined only depend on the element of the completion that the Cauchy sequence defines. The resulting map (4.115) is clearly continuous.

Now, we need to show that the inclusion i extends to H^1_{iso} from $\tilde{D}$. This follows from another integration identity. Namely, for $u \in \tilde{D}$ the Fundamental theorem of calculus applied to

$$\frac{d}{dx}(ux\overline{u}) = |u|^2 + \frac{du}{dx}x\overline{u} + ux\overline{\frac{du}{dx}}$$

gives

$$\|u\|^2_{L^2} \le \int_{\mathbb{R}} |\frac{du}{dx}x\overline{u}| + \int |ux\overline{\frac{du}{dx}}| \le \|u\|^2_{\text{iso}}. \tag{4.118}$$

Thus the inequality (4.72) holds for $u \in \tilde{D}$.

It follows that the inclusion map $i : \tilde{D} \longrightarrow L^2(\mathbb{R})$ extends by continuity to H^1_{iso} since if $u_n \in \tilde{D}$ is Cauchy with respect in H^1_{iso} it is Cauchy in $L^2(\mathbb{R})$. It remains to check that i is injective and compact, since the range is already dense on $\tilde{D}$.

If $u \in H^1_{\text{iso}}$ then to say $i(u) = 0$ (in $L^2(\mathbb{R})$) is to say that for any $u_n \to u$ in H^1_{iso}, with $u_n \in \tilde{D}$, $u_n \to 0$ in $L^2(\mathbb{R})$ and we need to show that this means $u_n \to 0$ in H^1_{iso} to conclude that $u = 0$. To do so we use the map (4.115). If $u_n\tilde{D}$ converges in H^1_{iso} then it follows that the sequence $(\frac{du}{dx}, xu)$ converges in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. If v is a continuous function of compact support then $(xu_n, v)_{L^2} = (u_n, xv) \to (u, xv)_{L^2}$, for if $u = 0$ it follows that $xu_n \to 0$ as well. Similarly, using integration by parts the limit U of $\frac{du_n}{dx}$ in $L^2(\mathbb{R})$ satisfies

$$(U, v)_{L^2} = \lim_n \int \frac{du_n}{dx}\overline{v} = -\lim_n \int u_n \overline{\frac{dv}{dx}} = -(u, \frac{dv}{dx})_{L^2} = 0 \tag{4.119}$$

if $u = 0$. It therefore follows that $U = 0$ so in fact $u_n \to 0$ in H^1_{iso} and the injectivity of i follows. □

We can see a little more about the metric on H^1_{iso}.

Lemma 4.8. *Elements of H^1_{iso} are continuous functions and convergence with respect to $\|\cdot\|_{\text{iso}}$ implies uniform convergence on bounded intervals.*

PROOF. For elements of the dense subspace $\tilde{D}$, (twice) continuously differentiable and vanishing outside a bounded interval the Fundamental Theorem of Calculus shows that

$$\begin{gathered} u(x)=e^{x^2/2}\int_{-\infty}^{x}(\frac{d}{dt}(e^{-t^2/2}u)=e^{x^2/2}\int_{-\infty}^{x}(e^{-t^2/2}(-tu+\frac{du}{dt}))\Longrightarrow \\ |u(x)|\le e^{x^2/2}(\int_{-\infty}^{x}e^{-t^2})^{\frac12}\|u\|_{\mathrm{iso}} \end{gathered} \tag{4.120}$$

where the estimate comes from the Cauchy-Schwarz applied to the integral. It follows that if $u_n\to u$ with respect to the isotropic norm then the sequence converges uniformly on bounded intervals with

$$\sup_{[-R,R]}|u(x)|\le C(R)\|u\|_{\mathrm{iso}}. \tag{4.121}$$

□

Now, to proceed further we either need to apply some 'regularity theory' or do a computation. I choose to do the latter here, although the former method (outlined below) is much more general. The idea is to show that

LEMMA 4.9. *The linear map* $(P+1):\mathcal{C}^2_c(\mathbb{R})\longrightarrow\mathcal{C}_c(\mathbb{R})$ *is injective with range dense in* $L^2(\mathbb{R})$ *and if* $f\in L^2(\mathbb{R})\cap\mathcal{C}(\mathbb{R})$ *there is a sequence* $u_n\in\mathcal{C}^2_c(\mathbb{R})$ *such that* $u_n\to u$ *in* H^1_{iso}, $u_n\to u$ *locally uniformly with its first two derivatives and* $(P+1)u_n\to f$ *in* $L^2(\mathbb{R})$ *and locally uniformly.*

PROOF. Why $P+1$ and not P? The result is actually true for P but not so easy to show directly. The advantage of $P+1$ is that it factorizes

$$(P+1)=\operatorname{An}\operatorname{Cr}\ \text{ on } \mathcal{C}^2_{\mathrm{c}}(\mathbb{R}).$$

so we proceed to solve the equation $(P+1)u=f$ in two steps.

First, if $f\in\mathrm{c}(\mathbb{R})$ then using the natural integrating factor

$$v(x)=e^{x^2/2}\int_{-\infty}^{x}e^{t^2/2}f(t)dt+ae^{-x^2/2}\ \text{satisfies}\ \operatorname{An}v=f. \tag{4.122}$$

The integral here is not in general finite if f does not vanish in $x<-R$, which by assumption it does. Note that $\operatorname{An}e^{-x^2/2}=0$. This solution is of the form

$$v\in\mathcal{C}^1(\mathbb{R}),\ v(x)=a_\pm e^{-x^2/2}\ \text{in}\ \pm x>R \tag{4.123}$$

where R depends on f and the constants can be different.

In the second step we need to solve away such terms – in general one cannot. However, we can always choose a in (4.122) so that

$$\int_{\mathbb{R}}e^{-x^2/2}v(x)=0. \tag{4.124}$$

Now consider

$$u(x)=e^{x^2/2}\int_{-\infty}^{x}e^{-t^2/2}v(t)dt. \tag{4.125}$$

Here the integral does make sense because of the decay in v from (4.123) and $u\in\mathcal{C}^2(\mathbb{R})$. We need to understand how it behaves as $x\to\pm\infty$. From the second part of (4.123),

$$u(x) = a_- \operatorname{erf}_-(x),\ x < -R,\ \operatorname{erf}_-(x) = \int_{(-\infty,x]} e^{x^2/2-t^2} \tag{4.126}$$

is an incomplete error function. It's derivative is e^{-x^2} but it actually satisfies

$$|x \operatorname{erf}_-(x)| \le Ce^{x^2},\ x < -R. \tag{4.127}$$

In any case it is easy to get an estimate such as Ce^{-bx^2} as $x \to -\infty$ for any $0 < b < 1$ by Cauchy-Schwarz.

As $x \to \infty$ we would generally expect the solution to be rapidly increasing, but precisely because of (4.124). Indeed the vanishing of this integral means we can rewrite (4.125) as an integral from $+\infty$:

$$u(x) = -e^{x^2/2} \int_{[x,\infty)} e^{-t^2/2} v(t) dt \tag{4.128}$$

and then the same estimates analysis yields

$$u(x) = -a_+ \operatorname{erf}_+(x),\ x < -R,\ \operatorname{erf}_+(x) = \int_{[x,\infty)} e^{x^2/2-t^2} \tag{4.129}$$

So for any $f \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ we have found a solution of $(P+1)u = f$ with u satisfying the rapid decay conditions (4.126) and (4.129). These are such that if $\chi \in \mathcal{C}^2_{\mathrm{c}}(\mathbb{R})$ has $\chi(t) = 1$ in $|t| < 1$ then the sequence

$$u_n = \chi(\frac{x}{n})u(x) \to u,\ u_n' \to u',\ u_n'' \to u'' \tag{4.130}$$

in all cases with convergence in $L^2(\mathbb{R})$ and uniformly and even such that $x^2 u_n \to xu$ uniformly and in $L^2(\mathbb{R})$.

This yields the first part of the Lemma, since if $f \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ and u is the solution just constructed to $(P+1)u = f$ then $(P+1)u_n \to f$ in L^2. So the closure $L^2(\mathbb{R})$ in range of $(P+1)$ on $\mathcal{C}^2_{\mathrm{c}}(\mathbb{R})$ includes $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$ so is certainly dense in $L^2(\mathbb{R})$.

The second part also follows from this construction. If $f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ then the sequence

$$f_n = \chi(\frac{x}{n})f(x) \in \mathcal{C}_{\mathrm{c}}(\mathbb{R}) \tag{4.131}$$

converges to f both in $L^2(\mathbb{R})$ and locally uniformly. Consider the solution, u_n to $(P+1)u_n = f_n$ constructed above. We want to show that $u_n \to u$ in L^2 and locally uniformly with its first two derivatives. The decay in u_n is enough to allow integration by parts to see that

$$\int_{\mathbb{R}} (P+1)u_n \overline{u_n} = \|u_n\|^2_{\mathrm{iso}} + \|u\|^2_{L^2} = |(f_n, u_n)| \le \|f_n\|_{l^2} \|u_n\|_{L^2}. \tag{4.132}$$

This shows that the sequence is bounded in H^1_{iso} and applying the same estimate to $u_n - u_m$ that it is Cauchy and hence convergent in H^1_{iso}. This implies $u_n \to u$ in H^1_{iso} and so both in $L^2(\mathbb{R})$ and locally uniformly. The differential equation can be written

$$(u_n)'' = x^2 u_n - u_n - f_n \tag{4.133}$$

where the right side converges locally uniformly. It follows from a standard result on uniform convergence of sequences of derivatives that in fact the uniform limit u

is twice continuously differentiable and that $(u_n)'' \to u''$ locally uniformly. So in fact $(P+1)u = f$ and the last part of the Lemma is also proved. $\square$

7. Fourier transform

The Fourier transform for functions on $\mathbb{R}$ is in a certain sense the limit of the definition of the coefficients of the Fourier series on an expanding interval, although that is not generally a good way to approach it. We know that if $u \in L^1(\mathbb{R})$ and $v \in \mathcal{C}_\infty(\mathbb{R})$ is a bounded continuous function then $vu \in L^1(\mathbb{R})$ – this follows from our original definition by approximation. So if $u \in L^1(\mathbb{R})$ the integral

$$\hat{u}(\xi) = \int e^{-ix\xi} u(x)dx,\ \xi \in \mathbb{R} \tag{4.134}$$

exists for each $\xi \in \mathbb{R}$ as a Legesgue integral. Note that there are many different normalizations of the Fourier transform in use. This is the standard 'analysts' normalization.

Proposition 4.7. *The Fourier tranform,* (4.134)*, defines a bounded linear map*

$$\mathcal{F} : L^1(\mathbb{R}) \ni u \longmapsto \hat{u} \in \mathcal{C}_0(\mathbb{R}) \tag{4.135}$$

into the closed subspace $\mathcal{C}_0(\mathbb{R}) \subset \mathcal{C}_\infty(\mathbb{R})$ of continuous functions which vanish at infinity (with respect to the supremum norm).

Proof. We know that the integral exists for each ξ and from the basic properties of the Lebesgue integal

$$|\hat{u}(\xi)| \le \|u\|_{L^1},\ \text{since } |e^{-ix\xi}u(x)| = |u(x)|. \tag{4.136}$$

To investigate its properties we restrict to $u \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$, a compactly-supported continuous function. Then the integral becomes a Riemann integral and the integrand is a continuous function of both variables. It follows that the result is uniformly continuous:-

$$|\hat{u}(\xi) - \hat{u}(\xi')| \le \int_{|x|\le R} |e^{-ix\xi} - e^{-ix\xi'}||u(x)|dx \le C(u) \sup_{|x|\le R} |e^{-ix\xi} - e^{-ix\xi'}| \tag{4.137}$$

with the right side small by the uniform continuity of continuous functions on compact sets. From (4.136), if $u_n \to u$ in $L^1(\mathbb{R})$ with $u_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ it follows that $\hat{u}_n \to \hat{u}$ uniformly on $\mathbb{R}$. Thus the Fourier transform is uniformly continuous on $\mathbb{R}$ for any $u \in L^1(\mathbb{R})$ (you can also see this from the continuity-in-the-mean of L^1 functions).

Now, we know that even the compactly-supported once continuously differentiable functions, forming $\mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ are dense in $L^1(\mathbb{R})$ so we can also consider (4.134) where $u \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$. Then the integration by parts as follows is justified

$$\xi\hat{u}(\xi) = i\int (\frac{de^{-ix\xi}}{dx})u(x)dx = -i\int e^{-ix\xi}\frac{du(x)}{dx}dx. \tag{4.138}$$

Since $du/dx \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ (by assumption) the estimate (4.136) now gives

$$\sup_{\xi\in\mathbb{R}} |\xi\hat{u}(\xi)| \le \sup_{x\in\mathbb{R}} |\frac{du}{dx}|. \tag{4.139}$$

This certainly implies the weaker statement that

$$\lim_{|\xi|\to\infty} |\hat{u}(\xi)| = 0 \tag{4.140}$$

which is 'vanishing at infinity'. Now we again use the density, this time of $\mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$, in $L^1(\mathbb{R})$ and the uniform estimate (4.136), plus the fact that if a sequence of continuous functions on $\mathbb{R}$ converges uniformly on $\mathbb{R}$ and each element vanishes at infinity then the limit vanishes at infinity to complete the proof of the Proposition. □

8. Fourier inversion

We could use the completeness of the orthonormal sequence of eigenfunctions for the harmonic oscillator discussed above to show that the Fourier tranform extends by continuity from $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$ to define an isomorphism

$$\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \tag{4.141}$$

with inverse given by the corresponding continuous extension of

$$\mathcal{G}v(x) = (2\pi)^{-1} \int e^{ix\xi} v(\xi). \tag{4.142}$$

Instead, we will give a direct proof of the Fourier inversion formula, via Schwartz space and an elegant argument due to Hörmander. Then we will use this to prove the completeness of the eigenfunctions we have found.

We have shown above that the Fourier transform is defined as an integral if $u \in L^1(\mathbb{R})$. Suppose that in addition we know that $xu \in L^1(\mathbb{R})$. We can summarize the combined information as

$$(1 + |x|)u \in L^1(\mathbb{R}). \tag{4.143}$$

Lemma 4.10. *If u satisfies* (4.143) *then $\hat{u}$ is continuously differentiable and $d\hat{u}/d\xi = \mathcal{F}(-ixu)$ is bounded.*

Proof. Consider the difference quotient for the Fourier transform:

$$\frac{\hat{u}(\xi + s) - \hat{u}(\xi)}{s} = \int D(x, s) e^{-ix\xi} u(x),\ D(x, s) = \frac{e^{-ixs} - 1}{s}. \tag{4.144}$$

We can use the standard proof of Taylor's formula to write the difference quotient inside the integral as

$$D(x, s) = -ix \int_0^1 e^{-itxs} dt \Longrightarrow |D(x, s)| \le |x|. \tag{4.145}$$

It follows that as $s \to 0$ (along a sequence if you prefer) $D(x, s)e^{-ix\xi} f(x)$ is bounded by the $L^1(\mathbb{R})$ function $|x||u(x)|$ and converges pointwise to $-ie^{-ix\xi}xu(x)$. Dominated convergence therefore shows that the integral converges showing that the derivative exists and that

$$\frac{d\hat{u}(\xi)}{d\xi} = \mathcal{F}(-ixu). \tag{4.146}$$

From the earlier results it follows that the derivative is continuous and bounded, proving the lemma. □

Now, we can iterate this result and so conclude:

$$\begin{gathered} (1 + |x|)^k u \in L^1(\mathbb{R})\ \forall\ k \Longrightarrow \\ \hat{u} \text{ is infinitely differentiable with bounded derivatives and} \\ \frac{d^k \hat{u}}{d\xi^k} = \mathcal{F}((-ix)^k u). \end{gathered} \tag{4.147}$$

This result shows that from 'decay' of u we deduce smoothness of $\hat{u}$. We can go the other way too. One way to ensure the assumption in (4.147) is to make the stronger assumption that

$$x^k u \text{ is bounded and continuous } \forall\ k. \tag{4.148}$$

Indeed, Dominated Convergence shows that if u is continuous and satisfies the bound

$$|u(x)| \le (1+|x|)^{-r},\ r>1$$

then $u \in L^1(\mathbb{R})$. So the integrability of $x^j u$ follows from the bounds in (4.148) for $k \le j+2$. This is throwing away information but simplifies things below.

In the opposite direction, suppose that u is continuously differentiable and satisfies the estimates (4.148) and

$$|\frac{u(x)}{dx}| \le (1+|x|)^{-r},\ r>1.$$

Then consider

$$\xi\hat{u} = i\int \frac{de^{-ix\xi}}{dx}u(x) = \lim_{R\to\infty} i\int_{-R}^{R} \frac{de^{-ix\xi}}{dx}u(x). \tag{4.149}$$

We may integrate by parts in this integral to get

$$\xi\hat{u} = \lim_{R\to\infty}\left(i\big[e^{-ix\xi}u(x)\big]_{-R}^{R} - i\int_{-R}^{R} e^{-ix\xi}\frac{du}{dx}\right). \tag{4.150}$$

The decay of u shows that the first term vanishes in the limit so

$$\xi\hat{u} = \mathcal{F}(-i\frac{du}{dx}). \tag{4.151}$$

Iterating this in turn we see that if u has continuous derivatives of all orders and for all j

$$|\frac{d^j u}{dx^j}| \le C_j(1+|x|)^{-r},\ r>1 \text{ then the } \xi^j\hat{u} = \mathcal{F}((-i)^j\frac{d^j u}{dx^j}) \tag{4.152}$$

are all bounded.

Laurent Schwartz defined a space which handily encapsulates these results.

Definition 4.2. Schwartz space, $\mathcal{S}(\mathbb{R})$, consists of all the infinitely differentiable functions $u : \mathbb{R} \longrightarrow \mathbb{C}$ such that

$$\|u\|_{j,k} = \sup |x^j \frac{d^k u}{dx^k}| < \infty\ \forall\ j,\ k \ge 0. \tag{4.153}$$

This is clearly a linear space. In fact it is a complete metric space in a natural way. All the $\|\cdot\|_{j,k}$ in (4.153) are norms on $\mathcal{S}(\mathbb{R})$, but none of them is stronger than the others. So there is no natural norm on $\mathcal{S}(\mathbb{R})$ with respect to which it is complete. In the problems below you can find some discussion of the fact that

$$d(u,v) = \sum_{j,k\ge 0} 2^{-j-k}\frac{\|u-v\|_{j,k}}{1+\|u-v\|_{j,k}} \tag{4.154}$$

is a complete metric. We will not use this here but it is the right way to understand what is going on.

Notice that there is some prejudice on the order of multiplication by x and differentiation in (4.153). This is only apparent, since these estimates (taken together) are equivalent to

$$\sup|\frac{d^k(x^j u)}{dx^k}| < \infty \ \forall\ j,\ k \geq 0. \tag{4.155}$$

To see the equivalence we can use induction over N where the inductive statement is the equivalence of (4.153) and (4.155) for $j + k \leq N$. Certainly this is true for $N = 0$ and to carry out the inductive step just differentiate out the product to see that

$$\frac{d^k(x^j u)}{dx^k} = x^j \frac{d^k u}{dx^k} + \sum_{l+m<k+j} c_{l,m,k,j} x^m \frac{d^l u}{dx^l}$$

where one can be much more precise about the extra terms, but the important thing is that they all are lower order (in fact both degrees go down). If you want to be careful, you can of course prove this identity by induction too! The equivalence of (4.153) and (4.155) for $N + 1$ now follows from that for N.

THEOREM 4.3. *The Fourier transform restricts to a bijection on $\mathcal{S}(\mathbb{R})$ with inverse*

$$\mathcal{G}(v)(x) = \frac{1}{2\pi} \int e^{ix\xi} v(\xi). \tag{4.156}$$

PROOF. The proof (due to Hörmander as I said above) will take a little while because we need to do some computation, but I hope you will see that it is quite clear and elementary.

First we need to check that $\mathcal{F} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$, but this is what I just did the preparation for. Namely the estimates (4.153) imply that (4.152) applies to all the $\frac{d^k(x^j u)}{dx^k}$ and so

$$\xi^k \frac{d^j \hat{u}}{d\xi^j} \text{ is continuous and bounded } \forall\ k,\ j \Longrightarrow \hat{u} \in \mathcal{S}(\mathbb{R}). \tag{4.157}$$

This indeed is why Schwartz introduced this space.

So, what we want to show is that with $\mathcal{G}$ defined by (4.156), $u = \mathcal{G}(\hat{u})$ for all $u \in \mathcal{S}(\mathbb{R})$. Notice that there is only a sign change and a constant factor to get from $\mathcal{F}$ to $\mathcal{G}$ so certainly $\mathcal{G} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$. We start off with what looks like a small part of this. Namely we want to show that

$$I(\hat{u}) = \int \hat{u} = 2\pi u(0). \tag{4.158}$$

Here, $I : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ is just integration, so it is certainly well-defined. To prove (4.158) we need to use a version of Taylor's formula and then do a little computation.

LEMMA 4.11. *If $u \in \mathcal{S}(\mathbb{R})$ then*

$$u(x) = u(0) \exp(-\frac{x^2}{2}) + xv(x),\ v \in \mathcal{S}(\mathbb{R}). \tag{4.159}$$

PROOF. Here I will leave it to you (look in the problems) to show that the Gaussian

$$\exp(-\frac{x^2}{2}) \in \mathcal{S}(\mathbb{R}). \tag{4.160}$$

Observe then that the difference

$$w(x) = u(x) - u(0)\exp(-\frac{x^2}{2}) \in \mathcal{S}(\mathbb{R}) \text{ and } w(0) = 0.$$

This is clearly a necessary condition to see that $w = xv$ with $v \in \mathcal{S}(\mathbb{R})$ and we can then see from the Fundamental Theorem of Calculus that

$$w(x) = \int_0^x w'(y)dy = x\int_0^1 w'(tx)dt \Longrightarrow v(x) = \int_0^1 w'(tx) = \frac{w(x)}{x}. \tag{4.161}$$

From the first formula for v it follows that it is infinitely differentiable and from the second formula the derivatives decay rapidly since each derivative can be written in the form of a finite sum of terms $p(x)\frac{d^l w}{dx^l}/x^N$ where the ps are polynomials. The rapid decay of the derivatives of w therefore implies the rapid decay of the derivatives of v. So indeed we have proved Lemma 4.11. □

Let me set $\gamma(x) = \exp(-\frac{x^2}{2})$ to simplify the notation. Taking the Fourier transform of each of the terms in (4.159) gives

$$\hat{u} = u(0)\hat{\gamma} + i\frac{d\hat{v}}{d\xi}. \tag{4.162}$$

Since $\hat{v} \in \mathcal{S}(\mathbb{R})$,

$$\int \frac{d\hat{v}}{d\xi} = \lim_{R\to\infty}\int_{-R}^{R} \frac{d\hat{v}}{d\xi} = \lim_{R\to\infty}\big[\hat{v}(\xi)\big]_{-R}^{R} = 0. \tag{4.163}$$

So now we see that

$$\int \hat{u} = cu(0),\ c = \int \hat{\gamma}$$

being a constant that we still need to work out!

Lemma 4.12. *For the Gaussian,* $\gamma(x) = \exp(-\frac{x^2}{2})$,

$$\hat{\gamma}(\xi) = \sqrt{2\pi}\gamma(\xi). \tag{4.164}$$

Proof. Certainly, $\hat{\gamma} \in \mathcal{S}(\mathbb{R})$ and from the identities for derivatives above

$$\frac{d\hat{\gamma}}{d\xi} = -i\mathcal{F}(x\gamma),\ \xi\hat{\gamma} = \mathcal{F}(-i\frac{d\gamma}{dx}). \tag{4.165}$$

Thus, $\hat{\gamma}$ satisfies the same differential equation as γ :

$$\frac{d\hat{\gamma}}{d\xi} + \xi\hat{\gamma} = -i\mathcal{F}(\frac{d\gamma}{dx} + x\gamma) = 0.$$

This equation we can solve and so we conclude that $\hat{\gamma} = c'\gamma$ where c' is also a constant that we need to compute. To do this observe that

$$c' = \hat{\gamma}(0) = \int \gamma = \sqrt{2\pi} \tag{4.166}$$

which gives (4.164). The computation of the integral in (4.166) is a standard clever argument which you probably know. Namely take the square and work in polar coordinates in two variables:

$$\begin{aligned}(\int \gamma)^2 &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy \\ &= \int_0^{2\pi}\int_0^\infty e^{-r^2/2}rdrd\theta = 2\pi\big[-e^{-r^2/2}\big]_0^\infty = 2\pi.\end{aligned} \tag{4.167}$$

□

So, finally we need to get from (4.158) to the inversion formula. Changing variable in the Fourier transform we can see that for any $y \in \mathbb{R}$, setting $u_y(x) = u(x+y)$, which is in $\mathcal{S}(\mathbb{R})$ if $u \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{F}(u_y) = \int e^{-ix\xi} u_y(x) dx = \int e^{-i(s-y)\xi} u(s) ds = e^{iy\xi}\hat{u}. \tag{4.168}$$

Now, plugging u_y into (4.158) we see that

$$\int \hat{u}_y(0) = 2\pi u_y(0) = 2\pi u(y) = \int e^{iy\xi}\hat{u}(\xi)d\xi \Longrightarrow u(y) = \mathcal{G}u, \tag{4.169}$$

the Fourier inversion formula. So we have proved the Theorem. □

9. Convolution

There is a discussion of convolution later in the notes, I have inserted a new (but not very different) treatment here to cover the density of $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$ needed in the next section.

Consider two continuous functions of compact support $u,\ v \in \mathcal{C}_c(\mathbb{R})$. Their convolution is

$$u * v(x) = \int u(x-y)v(y)dy = \int u(y)v(x-y)dy. \tag{4.170}$$

The first integral is the definition, clearly it is a well-defined Riemann integral since the integrand is continuous as a function of y and vanishes whenever $v(y)$ vanishes – so has compact support. In fact if both u and v vanish outside $[-R, R]$ then $u * v = 0$ outside $[-2R, 2R]$.

From standard properties of the Riemann integral (or Dominated convergence if you prefer!) it follows easily that $u*v$ is continuous. What we need to understand is what happens if (at least) one of u or v is smoother. In fact we will want to take a very smooth function, so I pause here to point out

Lemma 4.13. *There exists a ('bump') function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ which is infinitely differentiable, i.e. has continuous derivatives of all orders, vanishes outside $[-1, 1]$, is strictly positive on $(-1, 1)$ and has integral 1.*

Proof. We start with an explicit function,

$$\phi(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0. \end{cases} \tag{4.171}$$

The exponential function grows faster than any polynomial at $+\infty$, since

$$\exp(x) > \frac{x^k}{k!} \text{ in } x > 0\ \forall\ k. \tag{4.172}$$

This can be seen directly from the Taylor series which converges on the whole line (indeed in the whole complex plane)

$$\exp(x) = \sum_{k \ge 0} \frac{x^k}{k!}.$$

From (4.172) we deduce that

$$\lim_{x\downarrow 0}\frac{e^{-1/x}}{x^k}=\lim_{R\to\infty}\frac{R^k}{e^R}=0\ \forall\ k \tag{4.173}$$

where we substitute $R = 1/x$ and use the properties of exp. In particular ϕ in (4.171) is continuous across the origin, and so everywhere. We can compute the derivatives in $x > 0$ and these are of the form

$$\frac{d^l\phi}{dx^l}=\frac{p_l(x)}{x^{2l}}e^{-1/x},\ x>0,\ p_l \text{ a polynomial.} \tag{4.174}$$

As usual, do this by induction since it is true for $l = 0$ and differetiating the formula for a given l one finds

$$\frac{d^{l+1}\phi}{dx^{l+1}}=\left(\frac{p_l(x)}{x^{2l+2}}-2l\frac{p_l(x)}{x^{2l+1}}+\frac{p_l'(x)}{x^{2l}}\right)e^{-1/x} \tag{4.175}$$

where the coefficient function is of the desired form p_{l+1}/x^{2l+2}.

Once we know (4.174) then we see from (4.173) that all these functions are continuous down to 0 where they vanish. From this it follows that ϕ in (4.171) is infinitely differentiable. For ϕ itself we can use the Fundamental Theorem of Calculus to write

$$\phi(x)=\int_\epsilon^x U(t)dt+\phi(\epsilon),\ x>\epsilon>0. \tag{4.176}$$

Here U is the derivative in $x > 0$. Taking the limit as $\epsilon \downarrow 0$ both sides converge, and then we see that

$$\phi(x)=\int_0^x U(t)dt.$$

From this it follows that ϕ is continuously differentiable across 0 and it derivative is U, the continuous extension of the derivative from $x > 0$. The same argument applies to successive derivatives, so indeed ϕ is infinitely differentiable.

From ϕ we can construct a function closer to the desired bump function. Namely

$$\Phi(x)=\phi(x+1)\phi(1-x).$$

The first factor vanishes when $x \le -1$ and is otherwise positive while the second vanishes when $x \ge 1$ but is otherwise positive, so the product is infinitely differentiable on $\mathbb{R}$ and positive on $(-1, 1)$ but otherwise 0. Then we can normalize the integral to 1 by taking

$$\psi(x)=\Phi(x)/\int\Phi. \tag{4.177}$$

□

In particular from Lemma 4.13 we conclude that the space $\mathcal{C}^\infty_c(\mathbb{R})$, of infinitely differentiable functions of compact support, is not empty. Going back to convolution in (4.170) suppose now that v is smooth. Then

$$u\in\mathcal{C}_{\mathrm{c}}(\mathbb{R}),\ v\in\mathcal{C}^\infty_c(\mathbb{R})\Longrightarrow u*v\in\mathcal{C}^\infty_c(\mathbb{R}). \tag{4.178}$$

As usual this follows from properties of the Riemann integral or by looking directly at the difference quotient

$$\frac{u*v(x+t)-u*v(x)}{t}=\int u(y)\frac{v(x+t-y)-v(x-y)}{t}dt.$$

As $t \to 0$, the difference quotient for v converges uniformly (in y) to the derivative and hence the integral converges and the derivative of the convolution exists,

$$\frac{d}{dx} u * v(x) = u * (\frac{dv}{dx}). \tag{4.179}$$

This result allows immediate iteration, showing that the convolution is smooth and we know that it has compact support

Proposition 4.8. *For any $u \in \mathcal{C}_c(\mathbb{R})$ there exists $u_n \to u$ uniformly on $\mathbb{R}$ where $u_n \in \mathcal{C}_c^\infty(\mathbb{R})$ with supports in a fixed compact set.*

Proof. For each $\epsilon > 0$ consider the rescaled bump function

$$\psi_\epsilon = \epsilon^{-1}\psi(\frac{x}{\epsilon}) \in \mathcal{C}_c^\infty(\mathbb{R}). \tag{4.180}$$

In fact, ψ_ϵ vanishes outside the interval (ϵ, ϵ), is positive within this interval and has integral 1 – which is what the factor of ϵ^{-1} does. Now set

$$u_\epsilon = u * \psi_\epsilon \in \mathcal{C}_c^\infty(\mathbb{R}),\ \epsilon > 0, \tag{4.181}$$

from what we have just seen. From the supports of these functions, u_ϵ vanishes outside $[-R-\epsilon, R+\epsilon]$ if u vanishes outside $[-R, R]$. So only the convergence remains. To get this we use the fact that the integral of ψ_ϵ is equal to 1 to write

$$u_\epsilon(x) - u(x) = \int (u(x-y)\psi_\epsilon(y) - u(x)\psi_\epsilon(y))dy. \tag{4.182}$$

Estimating the integral using the positivity of the bump function

$$|u_\epsilon(x) - u(x)| = \int_{-\epsilon}^{\epsilon} |u(x-y) - u(x)|\psi_\epsilon(y)dy. \tag{4.183}$$

By the uniformity of a continuous function on a compact set, given $\delta > 0$ there exists $\epsilon > 0$ such that

$$\sup_{[-\epsilon,\epsilon]} |u(x-y) - y(x)| < \delta \ \forall \ x \in \mathbb{R}.$$

So the uniform convergence follows:-

$$\sup |u_\epsilon(x) - u(x)| \le \delta \int \phi_\epsilon = \delta \tag{4.184}$$

Pass to a sequence $\epsilon_n \to 0$ if you wish, □

Corollary 4.1. *The spaces $\mathcal{C}_c^\infty(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ are dense in $L^2(\mathbb{R})$.*

Uniform convegence of continuous functions with support in a fixed subset is stronger than L^2 convergence, so the result follows from the Proposition above for $\mathcal{C}_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$.

10. Plancherel and Parseval

But which is which?

We proceed to show that $\mathcal{F}$ and $\mathcal{G}$, defined in (4.134) and (4.142), both extend to isomorphisms of $L^2(\mathbb{R})$ which are inverses of each other. The main step is to show that

$$\int u(x)\hat{v}(x)dx = \int \hat{u}(\xi)v(\xi)d\xi,\ u,\ v \in \mathcal{S}(\mathbb{R}). \tag{4.185}$$

Since the integrals are rapidly convergent at infinity we may substitute the definite of the Fourier transform into (4.185), write the result out as a double integral and change the order of integration

$$\begin{aligned}(4.186)\quad \int u(x)\hat{v}(x)dx = \int u(x) \int e^{-ix\xi}v(\xi)d\xi dx \\ = \int v(\xi)\int e^{-ix\xi}u(x)dxd\xi = \int \hat{u}(\xi)v(\xi)d\xi.\end{aligned}$$

Now, if $w \in \mathcal{S}(\mathbb{R})$ we may replace $v(\xi)$ by $\overline{\hat{w}}(\xi)$, since it is another element of $\mathcal{S}(\mathbb{R})$. By the Fourier Inversion formula,

$$(4.187)\quad w(x) = (2\pi)^{-1}\int e^{-ix\xi}\hat{w}(\xi) \Longrightarrow \overline{w(x)} = (2\pi)^{-1}\int e^{ix\xi}\overline{\hat{w}(\xi)} = (2\pi)^{-1}\hat{v}.$$

Substituting these into (4.185) gives Parseval's formula

$$(4.188)\quad \int u\overline{w} = \frac{1}{2\pi}\int \hat{u}\overline{\hat{w}},\ u,\ w \in \mathcal{S}(\mathbb{R}).$$

Proposition 4.9. *The Fourier transform $\mathcal{F}$ extends from $\mathcal{S}(\mathbb{R})$ to an isomorphism on $L^2(\mathbb{R})$ with $\frac{1}{\sqrt{2\pi}}\mathcal{F}$ an isometric isomorphism with adjoint, and inverse, $\sqrt{2\pi}\mathcal{G}$.*

Proof. Setting $u = w$ in (4.188) shows that

$$(4.189)\quad \|\mathcal{F}(u)\|_{L^2} = \sqrt{2\pi}\|u\|_{L^2}$$

for all $u \in \mathcal{S}(\mathbb{R})$. The density of $\mathcal{S}(\mathbb{R})$, established above, then implies that $\mathcal{F}$ extends by continuity to the whole of $L^2(\mathbb{R})$ as indicated. □

This isomorphism of $L^2(\mathbb{R})$ has many implications. For instance, we would like to define the Sobolev space $H^1(\mathbb{R})$ by the conditions that $u \in L^2(\mathbb{R})$ and $\frac{du}{dx} \in L^2(\mathbb{R})$ but to do this we would need to make sense of the derivative. However, we can 'guess' that if it exists, the Fourier transform of du/dx should be $i\xi\hat{u}(\xi)$. For a function in L^2, such as $\hat{u}$ given that $u \in L^2$, we *do* know what it means to require $\xi\hat{u}(\xi) \in L^2(\mathbb{R})$. We can then define the Sobolev spaces of any positive, even non-integral, order by

$$(4.190)\quad H^r(\mathbb{R}) = \{u \in L^2(\mathbb{R}); |\xi|^r\hat{u} \in L^2(\mathbb{R})\}.$$

Of course it would take us some time to investigate the properties of these spaces!

11. Completeness of the Hermite functions

In 2015 I gave a different proof of the completeness of the eigenfunctions of the harmonic operator, reducing it to the spectral theorem, discussed in Section 5 above.

The starting point is to find a (generalized) inverse to the creation operator. Namely $e^{-x^2/2}$ is an integrating factor for it, so acting on once differentiable functions

$$(4.191)\quad \mathrm{Cr}\, u = -\frac{du}{dx} + xu = e^{x^2/2}\frac{d}{dx}(e^{-x^2/2}u).$$

For a function, say $f \in \mathcal{C}_c(\mathbb{R})$, we therefore get a solution by integration

$$(4.192)\quad u(x) = -e^{x^2/2}\int_{-\infty}^{x} e^{-t^2/2}f(t)dt.$$

This function vanishes for $x << 0$ but as $x \to +\infty$, after passing the top of the support of f,

$$u(x) = ce^{x^2/2},\ c = -\int_{\mathbb{R}} e^{-t^2/2} f(t)dt. \tag{4.193}$$

So, to have Sf decay in both directions we need to assume that this integral vanishes.

Proposition 4.10. *The creation operator gives a bijection*

$$\mathrm{Cr} : \mathcal{S}(\mathbb{R}) \longrightarrow \{f \in \mathcal{S}(\mathbb{R}); \int e^{t^2/2} f(t)dt = 0\} \tag{4.194}$$

with two-sided inverse in this sense

$$u = Sf,\ Sf(x) = -e^{x^2/2} \int_{-\infty}^{x} e^{-t^2/2} (\Pi_0 f)(t)dt, \tag{4.195}$$
$$\Pi_0 f = f - (\frac{1}{\sqrt{\pi}} \int e^{-t^2/2} f(t)dt) e^{-x^2/2}.$$

Note that Π_0 is the orthogonal projection *off* the ground state of the harmonic oscillator and gives a map from $\mathcal{S}(\mathbb{R})$ to the right side of (4.194).

Proof. For any $f \in \mathcal{S}(\mathbb{R})$ consider the behaviour of u given by (4.192) as $x \to -\infty$. [This is what I messed up in lecture.] What we wish to show is that

$$|x^k u(x)| \text{ is bounded as } x \to -\infty \tag{4.196}$$

for all k. Now, it is not possible to find an explicit primitive for $e^{t^2/2}$ but we can make do with the identity

$$\frac{d}{dt} \frac{e^{-t^2/2}}{t} = -e^{-t^2/2} - \frac{e^{-t^2/2}}{t^2}. \tag{4.197}$$

Inserting this into the integral defining u and integrating by parts we find

$$u(x) = -f(x)/x - e^{x^2/2} \int_{-\infty}^{x} e^{-t^2/2} (\frac{f'(t)}{t} - \frac{f(t)}{t^2})dt. \tag{4.198}$$

The first term here obviously satisfies the estimate (4.196) and we can substitute in the integral and repeat the procedure. Proceeding inductively we find after N steps

$$u(x) = \sum_{1 \le k \le 2N+1} \frac{h_j}{x^j} + e^{x^2/2} \int_{-\infty}^{x} e^{-t^2/2} \left(\sum_{N \le j \le 2N} \frac{g_{j,n}(t)}{t^j} \right) dt,\ h_j,\ g_{j,N} \in \mathcal{S}(\mathbb{R}). \tag{4.199}$$

The first terms certainly satisfy (4.196) for any k and the integral is bounded by $C|x|^{-N} e^{-x^2/2}$ so indeed (4.196) holds for all k.

For $g \in \mathcal{S}(\mathbb{R})$ such that $\int e^{-t^2/2} g(t) = 0$ we can replace (4.192) by

$$u(x) = -e^{x^2/2} \int_{x}^{\infty} e^{-t^2/2} f(t)dt \tag{4.200}$$

to which the same argument applies as $x \to +\infty$. The effect of Π_0 is to ensure this, so

$$\sup(1 + |x|)^k |Sf| < \infty\ \forall\ k. \tag{4.201}$$

By construction, $\frac{d}{dx}Sf = xSf(x) - \Pi_0 f$ so this also shows rapid decrease of the first derivative. In fact we may differentiate this equation N times and deduce, inductively, that *all* derivatives of Sf are rapidly decaying at infinity.

So, we see that S defined by (4.195) maps $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ with null space containing the span of $e^{-x^2/2}$. Since it solves the differetial equation we find

$$\operatorname{Cr} S = \operatorname{Id} - \Pi_0,\ S \operatorname{Cr} = \operatorname{Id} \text{ on } \mathcal{S}(\mathbb{R}). \tag{4.202}$$

Indeed, the first identity is what we have just shown and this shows that Cr in (4.194) is surjective. We already know it is injective since $\operatorname{Cr} f\|_{L^2} \ge \|f\|_{L^2}$ for $f \in \mathcal{S}(\mathbb{R})$. So S Cr in (4.194) is a bijection and S is the bijection inverting it, so the second identity in (4.202) follows. □

Notice that we can deduce from (4.202) that S extends by continuity to a bounded operator

$$S : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}). \tag{4.203}$$

Namely, it is zero on the span of $e^{-x^2/2}$ and

$$\|\frac{dSf}{dx}\|^2_{L^2} + \|xSf\|^2_{L^2} + \|Sf\|^2_{L^2} = \|\Pi_0 f\|^2_{L^2} \le \|f\|^2_{L^2}. \tag{4.204}$$

This actually shows that the bounded extension of S is compact.

THEOREM 4.4. *The composite SS^* is a compact injective self-adjoint operator on $L^2(\mathbb{R})$ with eigenvalues $(2j+2)^{-1}$ for $f \ge 0$ and associated one-dimesnional eigenspaces $E_j \subset \mathcal{S}(\mathbb{R})$ spanned by $\operatorname{Cr}^j e^{-x^2/2}$; in particular the Hermite functions form an orthonormal basis of $L^2(\mathbb{R})$.*

PROOF. We know that S has dense range (since this is already true when it acts on $\mathcal{S}(\mathbb{R})$) so S^* is injective and has range dense in the orthocomplement of $e^{-x^2/2}$. From this it follows that SS^* is injective. Compactness follows from the discussion of the isotropic space above, showing the compactness of S. By the spectral theorem SS^* has an orthonormal basis of eigenfunctions in $L^2(\mathbb{R})$, say v_j, with eigenvalues $s_j > 0$ which we may assume to be decreasing to 0.

□

12. Mehler's formula and completeness

Starting from the ground state for the harmonic oscillator

$$P = -\frac{d^2}{dx^2} + x^2,\ Hu_0 = u_0,\ u_0 = e^{-x^2/2} \tag{4.205}$$

and using the creation and annihilation operators

$$\operatorname{An} = \frac{d}{dx} + x,\ \operatorname{Cr} = -\frac{d}{dx} + x,\ \operatorname{An}\operatorname{Cr} - \operatorname{Cr}\operatorname{An} = 2\operatorname{Id},\ H = \operatorname{Cr}\operatorname{An} + \operatorname{Id} \tag{4.206}$$

we have constructed the higher eigenfunctions:

$$u_j = \operatorname{Cr}^j u_0 = p_j(x)u_0(c),\ p(x) = 2^j x^j + \text{l.o.ts},\ Hu_j = (2j+1)u_j \tag{4.207}$$

and shown that these are orthogonal, $u_j \perp u_k$, $j \neq k$, and so when normalized give an orthonormal system in $L^2(\mathbb{R})$:

$$e_j = \frac{u_j}{2^{j/2}(j!)^{\frac12}\pi^{\frac14}}. \tag{4.208}$$

Now, what we want to see, is that these e_j form an orthonormal basis of $L^2(\mathbb{R})$, meaning they are complete as an orthonormal sequence. There are various proofs of this, but the only 'simple' ones I know involve the Fourier inversion formula and I want to use the completeness to *prove* the Fourier inversion formula, so that will not do. Instead I want to use a version of Mehler's formula.

To show the completeness of the e_j's it is enough to find a compact self-adjoint operator with these as eigenfunctions and no null space. It is the last part which is tricky. The first part is easy. Remembering that all the e_j are real, we can find an operator with the e_j;s as eigenfunctions with corresponding eigenvalues $\lambda_j > 0$ (say) by just defining

$$Au(x) = \sum_{j=0}^{\infty} \lambda_j (u, e_j) e_j(x) = \sum_{j=0}^{\infty} \lambda_j e_j(x) \int e_j(y) u(y). \tag{4.209}$$

For this to be a compact operator we need $\lambda_j \to 0$ as $j \to \infty$, although for boundedness we just need the λ_j to be bounded. So, the problem with this is to show that A has no null space – which of course is just the completeness of the e'_j since (assuming all the λ_j are positive)

$$Au = 0 \Longleftrightarrow u \perp e_j \ \forall \ j. \tag{4.210}$$

Nevertheless, this is essentially what we will do. The idea is to write A as an *integral operator* and then work with that. I will take the $\lambda_j = w^j$ where $w \in (0, 1)$. The point is that we can find an explicit formula for

$$A_w(x, y) = \sum_{j=0}^{\infty} w^j e_j(x) e_j(y) = A(w, x, y). \tag{4.211}$$

To find $A(w, x, y)$ we will need to compute the Fourier transforms of the e_j. Recall that

$$\begin{gathered} \mathcal{F} : L^1(\mathbb{R}) \longrightarrow \mathcal{C}^0_\infty(\mathbb{R}),\ \mathcal{F}(u) = \hat{u}, \\ \hat{u}(\xi) = \int e^{-ix\xi} u(x),\ \sup |\hat{u}| \le \|u\|_{L^1}. \end{gathered} \tag{4.212}$$

Lemma 4.14. *The Fourier transform of u_0 is*

$$(\mathcal{F}u_0)(\xi) = \sqrt{2\pi} u_0(\xi). \tag{4.213}$$

Proof. Since u_0 is both continuous and Lebesgue integrable, the Fourier transform is the limit of a Riemann integral

$$\hat{u}_0(\xi) = \lim_{R\to\infty} \int_{-R}^{R} e^{i\xi x} u_0(x). \tag{4.214}$$

Now, for the Riemann integral we can differentiate under the integral sign with respect to the parameter ξ – since the integrand is continuously differentiable – and see that

$$\begin{aligned}
\frac{d}{d\xi}\hat{u}_0(\xi) &= \lim_{R\to\infty}\int_{-R}^{R} ixe^{i\xi x}u_0(x) \\
&= \lim_{R\to\infty} i\int_{-R}^{R} e^{i\xi x}(-\frac{d}{dx}u_0(x) \\
&= \lim_{R\to\infty} -i\int_{-R}^{R}\frac{d}{dx}\left(e^{i\xi x}u_0(x)\right) - \xi\lim_{R\to\infty}\int_{-R}^{R} e^{i\xi x}u_0(x) \\
&= -\xi\hat{u}_0(\xi).
\end{aligned} \tag{4.215}$$

Here I have used the fact that $\mathrm{An}\, u_0 = 0$ and the fact that the boundary terms in the integration by parts tend to zero rapidly with R. So this means that $\hat{u}_0$ is annihilated by An :

$$(\frac{d}{d\xi} + \xi)\hat{u}_0(\xi) = 0. \tag{4.216}$$

Thus, it follows that $\hat{u}_0(\xi) = c\exp(-\xi^2/2)$ since these are the only functions in annihilated by An. The constant is easy to compute, since

$$\hat{u}_0(0) = \int e^{-x^2/2}dx = \sqrt{2\pi} \tag{4.217}$$

proving (4.213). □

We can use this formula, of if you prefer the argument to prove it, to show that

$$v = e^{-x^2/4} \Longrightarrow \hat{v} = \sqrt{\pi}e^{-\xi^2}. \tag{4.218}$$

Changing the names of the variables this just says

$$e^{-x^2} = \frac{1}{2\sqrt{\pi}}\int_{\mathbb{R}} e^{ixs-s^2/4}ds. \tag{4.219}$$

The definition of the u_j's can be rewritten

$$u_j(x) = (-\frac{d}{dx} + x)^j e^{-x^2/2} = e^{x^2/2}(-\frac{d}{dx})^j e^{-x^2} \tag{4.220}$$

as is easy to see inductively – the point being that $e^{x^2/2}$ is an integrating factor for the creation operator. Plugging this into (4.219) and carrying out the derivatives – which is legitimate since the integral is so strongly convergent – gives

$$u_j(x) = \frac{e^{x^2/2}}{2\sqrt{\pi}}\int_{\mathbb{R}}(-is)^j e^{ixs-s^2/4}ds. \tag{4.221}$$

Now we can use this formula twice on the sum on the left in (4.211) and insert the normalizations in (4.208) to find that

$$\sum_{j=0}^{\infty} w^j e_j(x)e_j(y) = \sum_{j=0}^{\infty}\frac{e^{x^2/2+y^2/2}}{4\pi^{3/2}}\int_{\mathbb{R}^2}\frac{(-1)^j w^j s^j t^j}{2^j j!}e^{isx+ity-s^2/4-t^2/4}dsdt. \tag{4.222}$$

The crucial thing here is that we can sum the series to get an exponential, this allows us to finally conclude:

Lemma 4.15. *The identity* (4.211) *holds with*

$$A(w,x,y) = \frac{1}{\sqrt{\pi}\sqrt{1-w^2}}\exp\left(-\frac{1-w}{4(1+w)}(x+y)^2 - \frac{1+w}{4(1-w)}(x-y)^2\right) \tag{4.223}$$

PROOF. Summing the series in (4.222) we find that

$$A(w,x,y) = \frac{e^{x^2/2+y^2/2}}{4\pi^{3/2}} \int_{\mathbb{R}^2} \exp(-\frac{1}{2}wst + isx + ity - \frac{1}{4}s^2 - \frac{1}{4}t^2) dsdt. \tag{4.224}$$

Now, we can use the same formula as before for the Fourier transform of u_0 to evaluate these integrals explicitly. One way to do this is to make a change of variables by setting

$$\begin{gathered} s = (S+T)/\sqrt{2},\ t = (S-T)/\sqrt{2} \Longrightarrow dsdt = dSdT, \\ -\frac{1}{2}wst + isx + ity - \frac{1}{4}s^2 - \frac{1}{4}t^2 = iS\frac{x+y}{\sqrt{2}} - \frac{1}{4}(1+w)S^2 + iT\frac{x-y}{\sqrt{2}} - \frac{1}{4}(1-w)T^2. \end{gathered} \tag{4.225}$$

Note that the integrals in (4.224) are 'improper' (but rapidly convergent) Riemann integrals, so there is no problem with the change of variable formula. The formula for the Fourier transform of $\exp(-x^2)$ can be used to conclude that

$$\begin{aligned} \int_{\mathbb{R}} \exp(iS\frac{x+y}{\sqrt{2}} - \frac{1}{4}(1+w)S^2) dS &= \frac{2\sqrt{\pi}}{\sqrt{(1+w)}} \exp(-\frac{(x+y)^2}{2(1+w)}) \\ \int_{\mathbb{R}} \exp(iT\frac{x-y}{\sqrt{2}} - \frac{1}{4}(1-w)T^2) dT &= \frac{2\sqrt{\pi}}{\sqrt{(1-w)}} \exp(-\frac{(x-y)^2}{2(1-w)}). \end{aligned} \tag{4.226}$$

Inserting these formulæ back into (4.224) gives

$$A(w,x,y) = \frac{1}{\sqrt{\pi}\sqrt{1-w^2}} \exp\left(-\frac{(x+y)^2}{2(1+w)} - \frac{(x-y)^2}{2(1-w)} + \frac{x^2}{2} + \frac{y^2}{2}\right) \tag{4.227}$$

which after a little adjustment gives (4.223). □

Now, this explicit representation of A_w as an integral operator allows us to show

PROPOSITION 4.11. *For all real-valued $f \in L^2(\mathbb{R})$,*

$$\sum_{j=1}^{\infty} |(u, e_j)|^2 = \|f\|^2_{L^2}. \tag{4.228}$$

PROOF. By definition of A_w

$$\sum_{j=1}^{\infty} |(u, e_j)|^2 = \lim_{w \uparrow 1} (f, A_w f) \tag{4.229}$$

so (4.228) reduces to

$$\lim_{w \uparrow 1} (f, A_w f) = \|f\|^2_{L^2}. \tag{4.230}$$

To prove (4.230) we will make our work on the integral operators rather simpler by assuming first that $f \in \mathcal{C}^0(\mathbb{R})$ is continuous and vanishes outside some bounded interval, $f(x) = 0$ in $|x| > R$. Then we can write out the L^2 inner product as a double integral, which is a genuine (iterated) Riemann integral:

$$(f, A_w f) = \int\int A(w,x,y) f(x) f(y) dydx. \tag{4.231}$$

Here I have used the fact that f and A are real-valued.

Look at the formula for A in (4.223). The first thing to notice is the factor $(1-w^2)^{-\frac{1}{2}}$ which blows up as $w \to 1$. On the other hand, the argument of the exponential has two terms, the first tends to 0 as $w \to 1$ and the becomes very large and negative, at least when $x - y \neq 0$. Given the signs, we see that

$$\text{if } \epsilon > 0,\ X = \{(x,y); |x| \le R, |y| \le R, |x-y| \ge \epsilon\} \text{ then } \sup_X |A(w,x,y)| \to 0 \text{ as } w \to 1. \tag{4.232}$$

So, the part of the integral in (4.231) over $|x - y| \ge \epsilon$ tends to zero as $w \to 1$.

So, look at the other part, where $|x - y| \le \epsilon$. By the (uniform) continuity of f, given $\delta > 0$ there exits $\epsilon > 0$ such that

$$|x - y| \le \epsilon \Longrightarrow |f(x) - f(y)| \le \delta. \tag{4.233}$$

Now we can divide (4.231) up into three pieces:-

$$\begin{aligned}(f, A_w f) = &\int_{S\cap\{|x-y|\ge\epsilon\}} A(w,x,y)f(x)f(y)dydx \\ &+ \int_{S\cap\{|x-y|\le\epsilon\}} A(w,x,y)(f(x)-f(y))f(y)dydx \\ &+ \int_{S\cap\{|x-y|\le\epsilon\}} A(w,x,y)f(y)^2dydx\end{aligned} \tag{4.234}$$

where $S = [-R, R]^2$.

Look now at the third integral in (4.234) since it is the important one. We can change variable of integration from x to $t = \sqrt{\frac{1+w}{1-w}}(x-y)$. Since $|x - y| \le \epsilon$, the new t variable runs over $|t| \le \epsilon\sqrt{\frac{1+w}{1-w}}$ and then the integral becomes

$$\int_{S\cap\{|t|\le\epsilon\sqrt{\frac{1+w}{1-w}}\}} A(w, y + t\sqrt{\frac{1-w}{1+w}}, y)f(y)^2dydt, \text{ where}$$

$$\begin{aligned}A(w, y+t\sqrt{\frac{1-w}{1+w}}, y) \\ = \frac{1}{\sqrt{\pi}(1+w)} \exp\left(-\frac{1-w}{4(1+w)}(2y + t\sqrt{1-w})^2\right)\exp\left(-\frac{t^2}{4}\right).\end{aligned} \tag{4.235}$$

Here y is bounded; the first exponential factor tends to 1 and the t domain extends to $(-\infty, \infty)$ as $w \to 1$, so it follows that for any $\epsilon > 0$ the third term in (4.234) tends to

$$\|f\|^2_{L^2} \text{ as } w \to 1 \text{ since } \int e^{-t^2/4} = 2\sqrt{\pi}. \tag{4.236}$$

Noting that $A \ge 0$ the same argument shows that the second term is bounded by a constant multiple of δ. Now, we have already shown that the first term in (4.234) tends to zero as $\epsilon \to 0$, so this proves (4.230) – given some $\gamma > 0$ first choose $\epsilon > 0$ so small that the first two terms are each less than $\frac{1}{2}\gamma$ and then let $w \uparrow 0$ to see that the lim sup and lim inf as $w \uparrow 0$ must lie in the range $[\|f\|^2 - \gamma, \|f\|^2 + \gamma]$. Since this is true for all $\gamma > 0$ the limit exists and (4.228) follows under the assumption that f is continuous and vanishes outside some interval $[-R, R]$.

This actually suffices to prove the completeness of the Hermite basis. In any case, the general case follows by continuity since such continuous functions vanishing outside compact sets are dense in $L^2(\mathbb{R})$ and both sides of (4.228) are continuous in $f \in L^2(\mathbb{R})$. $\square$

Now, (4.230) certainly implies that the e_j form an orthonormal basis, which is what we wanted to show – but hard work! It is done here in part to remind you of how we did the Fourier series computation of the same sort and to suggest that you might like to compare the two arguments.

13. Weak and strong derivatives

In approaching the issue of the completeness of the eigenbasis for harmonic oscillator more directly, rather than by the kernel method discussed above, we run into the issue of weak and strong solutions of differential equations. Suppose that $u \in L^2(\mathbb{R})$, what does it *mean* to say that $\frac{du}{dx} \in L^2(\mathbb{R})$. For instance, we will want to understand what the 'possible solutions of'

$$\operatorname{An} u = f,\ u,\ f \in L^2(\mathbb{R}),\ \operatorname{An} = \frac{d}{dx} + x \tag{4.237}$$

are. Of course, if we assume that u is continuously differentiable then we know what this means, but we need to consider the possibilities of giving a meaning to (4.237) under more general conditions – without assuming too much regularity on u (or any at all).

Notice that there is a difference between the two terms in $\operatorname{An} u = \frac{du}{dx} + xu$. If $u \in L^2(\mathbb{R})$ we can assign a meaning to the second term, xu, since we know that $xu \in L^2_{\text{loc}}(\mathbb{R})$. This is not a normed space, but it is a perfectly good vector space, in which $L^2(\mathbb{R})$ 'sits' – if you want to be pedantic it naturally injects into it. The point however, is that we *do know* what the statement $xu \in^2 (\mathbb{R})$ means, given that $u \in L^2(\mathbb{R})$, it means that there exists $v \in L^2(\mathbb{R})$ so that $xu = v$ in $L^2_{\text{loc}}(\mathbb{R})$ (or $L^2_{\text{loc}}(\mathbb{R})$). The derivative can actually be handled in a similar fashion using the Fourier transform but I will not do that here.

Rather, consider the following three 'L^2-based notions' of derivative.

Definition 4.3. (1) We say that $u \in L^2(\mathbb{R})$ has a *Sobolev derivative* if there exists a sequence $\phi_n \in \mathcal{C}^1_{\text{c}}(\mathbb{R})$ such that $\phi_n \to u$ in $L^2(\mathbb{R})$ and $\phi'_n \to v$ in $L^2(\mathbb{R})$, $\phi'_n = \frac{d\phi_n}{dx}$ in the usual sense of course.

(2) We say that $u \in L^2(\mathbb{R})$ has a *strong derivative* (in the L^2 sense) if the limit

$$\lim_{0 \neq s \to 0} \frac{u(x+s) - u(x)}{s} = \tilde{v} \text{ exists in } L^2(\mathbb{R}). \tag{4.238}$$

(3) Thirdly, we say that $u \in L^2(\mathbb{R})$ has a *weak derivative* in L^2 if there exists $w \in L^2(\mathbb{R})$ such that

$$(u, -\frac{df}{dx})_{L^2} = (w, f)_{L^2}\ \forall\ f \in \mathcal{C}^1_{\text{c}}(\mathbb{R}). \tag{4.239}$$

In all cases, we will see that it is justified to write $v = \tilde{v} = w = \frac{du}{dx}$ because these defintions turn out to be equivalent. Of course *if* $u \in \mathcal{C}^1_{\text{c}}(\mathbb{R})$ then u is differentiable in each sense and the derivative is always du/dx – note that the integration by parts used to prove (4.239) is justified in that case. In fact we are most interested in the first and third of these definitions, the first two are both called 'strong derivatives.'

It is easy to see that the existence of a Sobolev derivative implies that this is also a weak derivative. Indeed, since ϕ_n, the approximating sequence whose existence is the definition of the Soboleve derivative, is in $\mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ so the integration by parts implicit in (4.239) is valid and so for all $f \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$,

$$(\phi_n, -\frac{df}{dx})_{L^2} = (\phi_n', f)_{L^2}. \tag{4.240}$$

Since $\phi_n \to u$ in L^2 and $\phi_n' \to v$ in L^2 both sides of (4.240) converge to give the identity (4.239).

Before proceeding to the rest of the equivalence of these definitions we need to do some preparation. First let us investigate a little the consequence of the existence of a Sobolev derivative.

Lemma 4.16. *If $u \in L^2(\mathbb{R})$ has a Sobolev derivative then $u \in \mathcal{C}(\mathbb{R})$ and there exists an unquely defined element $w \in L^2(\mathbb{R})$ such that*

$$u(x) - u(y) = \int_y^x w(s)ds \ \forall \ y \geq x \in \mathbb{R}. \tag{4.241}$$

Proof. Suppose u has a Sobolev derivative, determined by some approximating sequence ϕ_n. Consider a general element $\psi \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$. Then $\tilde{\phi}_n = \psi\phi_n$ is a sequence in $\mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ and $\tilde{\phi}_n \to \psi u$ in L^2. Moreover, by the product rule for standard derivatives

$$\frac{d}{dx}\tilde{\phi}_n = \psi'\phi_n + \psi\phi_n' \to \psi' u + \psi w \text{ in } L^2(\mathbb{R}). \tag{4.242}$$

Thus in fact ψu also has a Sobolev derivative, namely $\phi' u + \psi w$ if w is the Sobolev derivative for u given by ϕ_n – which is to say that the product rule for derivatives holds under these conditions.

Now, the formula (4.241) comes from the Fundamental Theorem of Calculus which in this case really does apply to $\tilde{\phi}_n$ and shows that

$$\psi(x)\phi_n(x) - \psi(y)\phi_n(y) = \int_y^x (\frac{d\tilde{\phi}_n}{ds}(s))ds. \tag{4.243}$$

For any given $x = \bar{x}$ we can choose ψ so that $\psi(\bar{x}) = 1$ and then we can take y below the lower limit of the support of ψ so $\psi(y) = 0$. It follows that for this choice of ψ,

$$\phi_n(\bar{x}) = \int_y^{\bar{x}} (\psi'\phi_n(s) + \psi\phi_n'(s))ds. \tag{4.244}$$

Now, we can pass to the limit as $n \to \infty$ and the left side converges for each fixed $\bar{x}$ (with ψ fixed) since the integrand converges in L^2 and hence in L^1 on this compact interval. This actually shows that the limit $\phi_n(\bar{x})$ must exist for each fixed $\bar{x}$. In fact we can always choose ψ to be constant near a particular point and apply this argument to see that

$$\phi_n(x) \to u(x) \text{ locally uniformly on } \mathbb{R}. \tag{4.245}$$

That is, the limit exists locally uniformly, hence represents a continuous function but that continuous function must be equal to the original u almost everywhere (since $\psi\phi_n \to \psi u$ in L^2).

Thus in fact we conclude that '$u \in \mathcal{C}(\mathbb{R})$' (which really means that u *has a representative* which is continuous). Not only that but we get (4.241) from passing to the limit on both sides of

$$u(x) - u(y) = \lim_{n\to\infty} (\phi_n(x) - \phi_n(y)) = \lim_{n\to\infty} \int_y^s (\phi'(s))ds = \int_y^s w(s)ds. \tag{4.246}$$

□

One immediate consequence of this is

(4.247) The Sobolev derivative is unique if it exists.

Indeed, if w_1 and w_2 are both Sobolev derivatives then (4.241) holds for both of them, which means that $w_2 - w_1$ has vanishing integral on any finite interval and we know that this implies that $w_2 = w_1$ a.e.

So at least for Sobolev derivatives we are now justified in writing

$$w = \frac{du}{dx} \tag{4.248}$$

since w is unique and behaves like a derivative in the integral sense that (4.241) holds.

Lemma 4.17. *If u has a Sobolev derivative then u has a stong derivative and if u has a strong derivative then this is also a weak derivative.*

Proof. If u has a Sobolev derivative then (3.17) holds. We can use this to write the difference quotient as

$$\frac{u(x+s) - u(x)}{s} - w(x) = \frac{1}{s}\int_0^s (w(x+t) - w(x))dt \tag{4.249}$$

since the integral in the second term can be carried out. Using this formula twice the square of the L^2 norm, which is finite, is

$$\begin{aligned} &\|\frac{u(x+s) - u(x)}{s} - w(x)\|^2_{L^2} \\ &\qquad = \frac{1}{s^2}\int\int_0^s\int_0^s (w(x+t) - w(x)\overline{(w(x+t') - w(x))}dtdt'dx. \end{aligned} \tag{4.250}$$

There is a small issue of manupulating the integrals, but we can always 'back off a little' and replace u by the approximating sequence ϕ_n and then everything is fine – and we only have to check what happens at the end. Now, we can apply the Cauchy-Schwarz inequality as a triple integral. The two factors turn out to be the same so we find that

$$\|\frac{u(x+s) - u(x)}{s} - w(x)\|^2_{L^2} \le \frac{1}{s^2}\int\int_0^s\int_0^s |w(x+t) - w(x)|^2 dxdtdt'. \tag{4.251}$$

Now, something we checked long ago was that L^2 functions are 'continuous in the mean' in the sense that

$$\lim_{0\neq t\to 0}\int |w(x+t) - w(x)|^2 dx = 0. \tag{4.252}$$

Applying this to (4.251) and then estimating the t and t' integrals shows that

$$\frac{u(x+s) - u(x)}{s} - w(x) \to 0 \text{ in } L^2(\mathbb{R}) \text{ as } s \to 0. \tag{4.253}$$

By definition this means that u has w as a strong derivative. I leave it up to you to make sure that the manipulation of integrals is okay.

So, now suppose that u has a strong derivative, $\tilde{v}$. Obsever that if $f \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ then the limit defining the derivative

$$\lim_{0 \neq s \to 0} \frac{f(x+s) - f(x)}{s} = f'(x) \tag{4.254}$$

is *uniform.* In fact this follows by writing down the Fundamental Theorem of Calculus, as in (4.241), again and using the properties of Riemann integrals. Now, consider

$$\begin{aligned}(u(x), \frac{f(x+s) - f(x)}{s})_{L^2} &= \frac{1}{s}\int u(x)\overline{f(x+s)}dx - \frac{1}{s}\int u(x)\overline{f(x)}dx \\ &= (\frac{u(x-s) - u(x)}{s}, f(x))_{L^2}\end{aligned} \tag{4.255}$$

where we just need to change the variable of integration in the first integral from x to $x+s$. However, letting $s \to 0$ the left side converges because of the uniform convergence of the difference quotient and the right side converges because of the assumed strong differentiability and as a result (noting that the parameter on the right is really $-s$)

$$(u, \frac{df}{dx})_{L^2} = -(w, f)_{L^2} \ \forall \ f \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R}) \tag{4.256}$$

which is weak differentiability with derivative $\tilde{v}$. □

So, at this point we know that Sobolev differentiabilty implies strong differentiability and either of the stong ones implies the weak. So it remains only to show that weak differentiability implies Sobolev differentiability and we can forget about the difference!

Before doing that, note again that a weak derivative, if it exists, is unique – since the difference of two would have to pair to zero in L^2 with all of $\mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ which is dense. Similarly, if u has a weak derivative then so does ψu for any $\psi \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ since we can just move ψ around in the integrals and see that

$$\begin{aligned}(\psi u, -\frac{df}{dx}) &= (u, -\overline{\psi}\frac{df}{dx}) \\ &= (u, -\frac{d\overline{\psi} f}{dx}) + (u, \overline{\psi'} f) \\ &= (w, \overline{\psi} f + (\psi' u, f) = (\psi w + \psi' u, f)\end{aligned} \tag{4.257}$$

which also proves that the product formula holds for weak derivatives.

So, let us consider $u \in L^2_{\mathrm{c}}(\mathbb{R})$ which does have a weak derivative. To show that it has a Sobolev derivative we need to construct a sequence ϕ_n. We will do this by convolution.

Lemma 4.18. *If $\mu \in \mathcal{C}_c(\mathbb{R})$ then for any $u \in L^2_c(\mathbb{R})$,*

$$\mu * u(x) = \int \mu(x-s)u(s)ds \in \mathcal{C}_c(\mathbb{R}) \tag{4.258}$$

and if $\mu \in \mathcal{C}^1_c(\mathbb{R})$ then

$$\mu * u(x) \in \mathcal{C}^1_c(\mathbb{R}), \ \frac{d\mu * u}{dx} = \mu' * u(x). \tag{4.259}$$

It folows that if μ has more continuous derivatives, then so does $\mu * u$.

Proof. Since u has compact support and is in L^2 it in L^1 so the integral in (4.258) exists for each $x \in \mathbb{R}$ and also vanishes if $|x|$ is large enough, since the integrand vanishes when the supports become separate – for some R, $\mu(x-s)$ is supported in $|s-x| \le R$ and $u(s)$ in $|s| < R$ which are disjoint for $|x| > 2R$. It is also clear that $\mu * u$ is continuous using the estimate (from uniform continuity of μ)

$$|\mu * u(x') - \mu * u(x)| \le \sup |\mu(x-s) - \mu(x'-s)| \|u\|_{L^1}. \tag{4.260}$$

Similarly the difference quotient can be written

$$\frac{\mu * u(x') - \mu * u(x)}{t} = \int \frac{\mu(x'-s) - \mu(x-s)}{s} u(s) ds \tag{4.261}$$

and the uniform convergence of the difference quotient shows that

$$\frac{d\mu * u}{dx} = \mu' * u. \tag{4.262}$$

□

One of the key properties of thes convolution integrals is that we can examine what happens when we 'concentrate' μ. Replace the one μ by the family

$$\mu_\epsilon(x) = \epsilon^{-1}\mu(\frac{x}{\epsilon}),\ \epsilon > 0. \tag{4.263}$$

The singular factor here is introduced so that $\int \mu_\epsilon$ is independent of $\epsilon > 0$,

$$\int \mu_\epsilon = \int \mu\ \forall\ \epsilon > 0. \tag{4.264}$$

Note that since μ has compact support, the support of μ_ϵ is concentrated in $|x| \le \epsilon R$ for some fixed R.

Lemma 4.19. *If $u \in L^2_c(\mathbb{R})$ and $0 \le \mu \in \mathcal{C}^1_c(\mathbb{R})$ then*

$$\lim_{0 \ne \epsilon \to 0} \mu_\epsilon * u = (\int \mu) u \text{ in } L^2(\mathbb{R}). \tag{4.265}$$

In fact there is no need to assume that u has compact support for this to work.

Proof. First we can change the variable of integration in the definition of the convolution and write it intead as

$$\mu * u(x) = \int \mu(s) u(x-s) ds. \tag{4.266}$$

Now, the rest is similar to one of the arguments above. First write out the difference we want to examine as

$$\mu_\epsilon * u(x) - (\int \mu)(x) = \int_{|s| \le \epsilon R} \mu_\epsilon(s)(u(x-s) - u(x)) ds. \tag{4.267}$$

Write out the square of the absolute value using the formula twice and we find that

$$\begin{aligned} &\int |\mu_\epsilon * u(x) - (\int \mu)(x)|^2 dx \\ &= \int \int_{|s| \le \epsilon R} \int_{|t| \le \epsilon R} \mu_\epsilon(s)\mu_\epsilon(t)(u(x-s) - u(x))\overline{(u(x-s) - u(x))} ds dt dx \end{aligned} \tag{4.268}$$

Now we can write the integrand as the product of two similar factors, one being

$$\mu_\epsilon(s)^{\frac{1}{2}} \mu_\epsilon(t)^{\frac{1}{2}} (u(x-s) - u(x)) \tag{4.269}$$

using the non-negativity of μ. Applying the Cauchy-Schwarz inequality to this we get two factors, which are again the same after relabelling variables, so

$$\int |\mu_\epsilon * u(x) - (\int \mu)(x)|^2 dx \le \int \int_{|s|\le \epsilon R} \int_{|t|\le \epsilon R} \mu_\epsilon(s)\mu_\epsilon(t)|u(x-s)-u(x)|^2. \tag{4.270}$$

The integral in x can be carried out first, then using continuity-in-the mean bounded by $J(s) \to 0$ as $\epsilon \to 0$ since $|s| < \epsilon R$. This leaves

$$\begin{aligned} &\int |\mu_\epsilon * u(x) - (\int \mu)u(x)|^2 dx \\ &\qquad \le \sup_{|s|\le\epsilon R} J(s) \int_{|s|\le \epsilon R} \int_{|t|\le \epsilon R} \mu_\epsilon(s)\mu_\epsilon(t) = (\int \psi)^2 Y \sup_{|s|\le \epsilon R} \to 0. \end{aligned} \tag{4.271}$$

□

After all this preliminary work we are in a position to to prove the remaining part of 'weak=strong'.

Lemma 4.20. *If $u \in L^2(\mathbb{R})$ has w as a weak L^2-derivative then w is also the Sobolev derivative of u.*

Proof. Let's assume first that u has compact support, so we can use the discussion above. Then set $\phi_n = \mu_{1/n} * u$ where $\mu \in \mathcal{C}^1_c(\mathbb{R})$ is chosen to be non-negative and have integral $\int \mu = 0$; μ_ϵ is defined in (4.263). Now from Lemma 4.19 it follows that $\phi_n \to u$ in $L^2(\mathbb{R})$. Also, from Lemma 4.18, $\phi_n \in \mathcal{C}^1_c(\mathbb{R})$ has derivative given by (4.259). This formula can be written as a pairing in L^2 :

$$(\mu_{1/n})' * u(x) = (u(s), -\frac{d\mu_{1/n}(x-s)}{ds})^2_L = (w(s), \frac{d\mu_{1/n}(x-s)}{ds})_{L^2} \tag{4.272}$$

using the definition of the weak derivative of u. It therefore follows from Lemma 4.19 applied again that

$$\phi_n' = \mu_{/m1/n} * w \to w \text{ in } L^2(\mathbb{R}). \tag{4.273}$$

Thus indeed, ϕ_n is an approximating sequence showing that w is the Sobolev derivative of u.

In the general case that $u \in L^2(\mathbb{R})$ has a weak derivative but is not necessarily compactly supported, consider a function $\gamma \in \mathcal{C}^1_c(\mathbb{R})$ with $\gamma(0) = 1$ and consider the sequence $v_m = \gamma(x)u(x)$ in $L^2(\mathbb{R})$ each element of which has compact support. Moreover, $\gamma(x/m) \to 1$ for each x so by Lebesgue dominated convergence, $v_m \to u$ in $L^2(\mathbb{R})$ as $m \to \infty$. As shown above, v_m has as weak derivative

$$\frac{d\gamma(x/m)}{dx}u + \gamma(x/m)w = \frac{1}{m}\gamma'(x/m)u + \gamma(x/m)w \to w$$

as $m \to \infty$ by the same argument applied to the second term and the fact that the first converges to 0 in $L^2(\mathbb{R})$. Now, use the approximating sequence $\mu_{1/n} * v_m$ discussed converges to v_m with its derivative converging to the weak derivative of v_m. Taking $n = N(m)$ sufficiently large for each m ensures that $\phi_m = \mu_{1/N(m)} * v_m$ converges to u and its sequence of derivatives converges to w in L^2. Thus the weak derivative is again a Sobolev derivative. □

Finally then we see that the three definitions are equivalent and we will freely denote the Sobolev/strong/weak derivative as du/dx or u'.

14. Fourier transform and L^2

Recall that one reason for proving the completeness of the Hermite basis was to apply it to prove some of the important facts about the Fourier transform, which we already know is a linear operator

$$L^1(\mathbb{R}) \longrightarrow \mathcal{C}^0_\infty(\mathbb{R}),\ \hat{u}(\xi) = \int e^{ix\xi} u(x) dx. \tag{4.274}$$

Namely we have already shown the effect of the Fourier transform on the 'ground state':

$$\mathcal{F}(u_0)(\xi) = \sqrt{2\pi} e_0(\xi). \tag{4.275}$$

By a similar argument we can check that

$$\mathcal{F}(u_j)(\xi) = \sqrt{2\pi} i^j u_j(\xi)\ \forall\ j \in \mathbb{N}. \tag{4.276}$$

As usual we can proceed by induction using the fact that $u_j = \operatorname{Cr} u_{j-1}$. The integrals involved here are very rapidly convergent at infinity, so there is no problem with the integration by parts in

(4.277)

$$\begin{aligned}\mathcal{F}(\frac{d}{dx}u_{j-1}) &= \lim_{T\to\infty} \int_{-T}^{T} e^{-ix\xi} \frac{du_{j-1}}{dx} dx \\ &= \lim_{T\to\infty} \left(\int_{-T}^{T} (i\xi) e^{-ix\xi} u_{j-1} dx + \left[e^{-ix\xi} u_{j-1}(x) \right]_{-T}^{T} \right) = (i\xi)\mathcal{F}(u_{j-1}), \\ \mathcal{F}(xu_{j-1}) &= i \int \frac{de^{-ix\xi}}{d\xi} u_{j-1} dx = i\frac{d}{d\xi}\mathcal{F}(u_{j-1}).\end{aligned}$$

Taken together these identities imply the validity of the inductive step:

$$\mathcal{F}(u_j) = \mathcal{F}((-\frac{d}{dx} + x)u_{j-1}) = (i(-\frac{d}{d\xi} + \xi)\mathcal{F}(u_{j-1}) = i \operatorname{Cr}(\sqrt{2\pi} i^{j-1} u_{j-1}) \tag{4.278}$$

so proving (4.276).

So, we have found an orthonormal basis for $L^2(\mathbb{R})$ with elements which are all in $L^1(\mathbb{R})$ and which are also eigenfunctions for $\mathcal{F}$.

Theorem 4.5. *The Fourier transform maps $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ and extends by continuity to an isomorphism of $L^2(\mathbb{R})$ such that $\frac{1}{\sqrt{2\pi}}\mathcal{F}$ is unitary with the inverse of $\mathcal{F}$ the continuous extension from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ of*

$$\mathcal{F}(f)(x) = \frac{1}{2\pi} \int e^{ix\xi} f(\xi). \tag{4.279}$$

Proof. This really is what we have already proved. The elements of the Hermite basis e_j are all in both $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ so if $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ its image under $\mathcal{F}$ is in $L^2(\mathbb{R})$ because we can compute the L^2 inner products and see that

$$(\mathcal{F}(u), e_j) = \int_{\mathbb{R}^2} e_j(\xi) e^{ix\xi} u(x) dx d\xi = \int \mathcal{F}(e_j)(x) u(x) = \sqrt{2\pi} i^j (u, e_j). \tag{4.280}$$

Now Bessel's inequality shows that $\mathcal{F}(u) \in L^2(\mathbb{R})$ (it is of course locally integrable since it is continuous).

Everything else now follows easily. □

Notice in particular that we have also proved Parseval's and Plancherel's identities for the Fourier transform:-

$$\|\mathcal{F}(u)\|_{L^2} = \sqrt{2\pi}\|u\|_{L^2},\ (\mathcal{F}(u), \mathcal{F}(v)) = 2\pi(u, v),\ \forall\ u, v \in L^2(\mathbb{R}). \tag{4.281}$$

Now there are lots of applications of the Fourier transform which we do not have the time to get into. However, let me just indicate the definitions of Sobolev spaces and Schwartz space and how they are related to the Fourier transform.

First Sobolev spaces. We now see that $\mathcal{F}$ maps $L^2(\mathbb{R})$ isomorphically onto $L^2(\mathbb{R})$ and we can see from (4.277) for instance that it 'turns differentiations by x into multiplication by ξ'. Of course we do not know how to differentiate L^2 functions so we have some problems making sense of this. One way, the usual mathematicians trick, is to turn what we want into a definition.

Definition 4.4. The Sobolev spaces of order s, for any $s \in (0, \infty)$, are defined as subspaces of $L^2(\mathbb{R})$:

$$H^s(\mathbb{R}) = \{u \in L^2(\mathbb{R}); (1 + |\xi|^2)^s \hat{u} \in L^2(\mathbb{R})\}. \tag{4.282}$$

It is natural to identify $H^0(\mathbb{R}) = L^2(\mathbb{R})$.

These Sobolev spaces, for each positive order s, are Hilbert spaces with the inner product and norm

$$(u, v)_{H^s} = \int (1 + |\xi|^2)^s \hat{u}(\xi)\overline{\hat{v}(\xi)},\ \|u\|_s = \|(1 + |\xi|^2)^{\frac{s}{2}} \hat{u}\|_{L^2}. \tag{4.283}$$

That they are pre-Hilbert spaces is clear enough. Completeness is also easy, given that we know the completeness of $L^2(\mathbb{R})$. Namely, if u_n is Cauchy in $H^s(\mathbb{R})$ then it follows from the fact that

$$\|v\|_{L^2} \le C\|v\|_s\ \forall\ v \in H^s(\mathbb{R}) \tag{4.284}$$

that u_n is Cauchy in L^2 and also that $(1 + |\xi|^2)^{\frac{s}{2}} \hat{u}_n(\xi)$ is Cauchy in L^2. Both therefore converge to a limit u in L^2 and the continuity of the Fourier transform shows that $u \in H^s(\mathbb{R})$ and that $u_n \to u$ in H^s.

These spaces are examples of what is discussed above where we have a dense inclusion of one Hilbert space in another, $H^s(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$. In this case the inclusion in *not* compact but it does give rise to a bounded self-adjoint operator on $L^2(\mathbb{R})$, $E_s : L^2(\mathbb{R}) \longrightarrow H^s(\mathbb{R}) \subset L^2(\mathbb{R})$ such that

$$(u, v)_{L^2} = (E_s u, E_s v)_{H^s}. \tag{4.285}$$

It is reasonable to denote this as $E_s = (1 + |D_x|^2)^{-\frac{s}{2}}$ since

$$u \in L^2(\mathbb{R}^n) \Longrightarrow \widehat{E_s u}(\xi) = (1 + |\xi|^2)^{-\frac{s}{2}} \hat{u}(\xi). \tag{4.286}$$

It is a form of 'fractional integration' which turns any $u \in L^2(\mathbb{R})$ into $E_s u \in H^s(\mathbb{R})$.

Having defined these spaces, which get smaller as s increases it can be shown for instance that if $n \ge s$ is an integer then the set of n times continuously differentiable functions on $\mathbb{R}$ which vanish outside a compact set are dense in H^s. This allows us to justify, by continuity, the following statement:-

Proposition 4.12. *The bounded linear map*

$$\frac{d}{dx} : H^s(\mathbb{R}) \longrightarrow H^{s-1}(\mathbb{R}),\ s \ge 1,\ v(x) = \frac{du}{dx} \Longleftrightarrow \hat{v}(\xi) = i\xi\hat{u}(\xi) \tag{4.287}$$

is consistent with differentiation on n times continuously differentiable functions of compact support, for any integer $n \ge s$.

In fact one can even get a 'strong form' of differentiation. The condition that $u \in H^1(\mathbb{R})$, that $u \in L^2$ 'has one derivative in L^2' is actually equivalent, for $u \in L^2(\mathbb{R})$ to the existence of the limit

$$\lim_{t\to 0} \frac{u(x+t)u(x)}{t} = v, \text{ in } L^2(\mathbb{R}) \tag{4.288}$$

and then $\hat{v} = i\xi\hat{u}$. Another way of looking at this is

$$\begin{aligned} &u \in H^1(\mathbb{R}) \Longrightarrow u : \mathbb{R} \longrightarrow \mathbb{C} \text{ is continuous and} \\ &u(x) - u(y) = \int_y^x v(t)dt, \ v \in L^2. \end{aligned} \tag{4.289}$$

If such a $v \in L^2(\mathbb{R})$ exists then it is unique – since the difference of two such functions would have to have integral zero over any finite interval and we know (from one of the exercises) that this implies that the function vansishes a.e.

One of the more important results about Sobolev spaces – of which there are many – is the relationship between these 'L^2 derivatives' and 'true derivatives'.

THEOREM 4.6 (Sobolev embedding). *If n is an integer and $s > n + \frac{1}{2}$ then*

$$H^s(\mathbb{R}) \subset \mathcal{C}^n_\infty(\mathbb{R}) \tag{4.290}$$

consists of n times continuosly differentiable functions with bounded derivatives to order n (which also vanish at infinity).

This is actually not so hard to prove, there are some hints in the exercises below.

These are not the only sort of spaces with 'more regularity' one can define and use. For instance one can try to treat x and ξ more symmetrically and define smaller spaces than the H^s above by setting

$$H^s_{\text{iso}}(\mathbb{R}) = \{u \in L^2(\mathbb{R}); (1+|\xi|^2)^{\frac{s}{2}}\hat{u} \in L^2(\mathbb{R}),\ (1+|x|^2)^{\frac{s}{2}}u \in L^2(\mathbb{R})\}. \tag{4.291}$$

The 'obvious' inner product with respect to which these 'isotropic' Sobolev spaces $H^s_{\text{iso}}(\mathbb{R})$ are indeed Hilbert spaces is

$$(u,v)_{s,\text{iso}} = \int_{\mathbb{R}} u\overline{v} + \int_{\mathbb{R}} |x|^{2s} u\overline{v} + \int_{\mathbb{R}} |\xi|^{2s}\hat{u}\overline{\hat{v}} \tag{4.292}$$

which makes them look rather symmetric between u and $\hat{u}$ and indeed

$$\mathcal{F} : H^s_{\text{iso}}(\mathbb{R}) \longrightarrow H^s_{\text{iso}}(\mathbb{R}) \text{ is an isomorphism } \forall\ s \ge 0. \tag{4.293}$$

At this point, by dint of a little, only moderately hard, work, it is possible to show that the harmonic oscillator extends by continuity to an isomorphism

$$H : H^{s+2}_{\text{iso}}(\mathbb{R}) \longrightarrow H^s_{\text{iso}}(\mathbb{R}) \ \forall\ s \ge 2. \tag{4.294}$$

Finally in this general vein, I wanted to point out that Hilbert, and even Banach, spaces are not the end of the road! One very important space in relation to a direct treatment of the Fourier transform, is the Schwartz space. The definition is reasonably simple. Namely we denote Schwartz space by $\mathcal{S}(\mathbb{R})$ and say

$$\begin{gathered} u \in \mathcal{S}(\mathbb{R}) \Longleftrightarrow u : \mathbb{R} \longrightarrow \mathbb{C} \\ \text{is continuously differentiable of all orders and for every } n, \\ \|u\|_n = \sum_{k+p\le n} \sup_{x\in\mathbb{R}} (1+|x|)^k \Big|\frac{d^p u}{dx^p}\Big| < \infty. \end{gathered} \tag{4.295}$$

All these inequalities just mean that all the derivatives of u are 'rapidly decreasing at ∞' in the sense that they stay bounded when multiplied by any polynomial.

So in fact we know already that $\mathcal{S}(\mathbb{R})$ is not empty since the elements of the Hermite basis, $e_j \in \mathcal{S}(\mathbb{R})$ for all j. In fact it follows immediately from this that

$$\mathcal{S}(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \text{ is dense.} \tag{4.296}$$

If you want to try your hand at something a little challenging, see if you can check that

$$\mathcal{S}(\mathbb{R}) = \bigcap_{s>0} H^s_{\mathrm{iso}}(\mathbb{R}) \tag{4.297}$$

which uses the Sobolev embedding theorem above.

As you can see from the definition in (4.295), $\mathcal{S}(\mathbb{R})$ is not likely to be a Banach space. Each of the $\|\cdot\|_n$ is a norm. However, $\mathcal{S}(\mathbb{R})$ is pretty clearly not going to be complete with respect to any one of these. However it is complete with respect to all, countably many, norms. What does this mean? In fact $\mathcal{S}(\mathbb{R})$ is a *metric space* with the metric

$$d(u,v) = \sum_n 2^{-n} \frac{\|u-v\|_n}{1+\|u-v\|_n} \tag{4.298}$$

as you can check. So the claim is that $\mathcal{S}(\mathbb{R})$ *is* complete as a metric space – such a thing is called a Fréchet space.

What has this got to do with the Fourier transform? The point is that

$$\mathcal{F} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}) \text{ is an isomorphism and } \mathcal{F}(\frac{du}{dx}) = i\xi\mathcal{F}(u),\ \mathcal{F}(xu) = -i\frac{d\mathcal{F}(u)}{d\xi} \tag{4.299}$$

where this now makes sense. The dual space of $\mathcal{S}(\mathbb{R})$ – the space of continuous linear functionals on it, is the space, denoted $\mathcal{S}'(\mathbb{R})$, of tempered distributions on $\mathbb{R}$.

15. Schwartz distributions

We do not have time in this course to really discuss distributions. Still, it is a good idea for you to know what they are and why they are useful. Of course to really appreciate their utility you need to read a bit more than I have here. First think a little about the Schwartz space $\mathcal{S}(\mathbb{R})$ introduced above. The metric in (4.298) might seem rather mysterious but it has the important property that *each* of the norms $\|\cdot\|_n$ defines a continuous function $\mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{R}$ with respect to this metric topology. In fact a linear map

$$T : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C} \text{ linear is continuous iff } \exists\ N, C \text{ s.t. } \|T\phi\| \le C\|\phi\|_N\ \forall\ \phi \in \mathcal{S}(\mathbb{R}). \tag{4.300}$$

So, the continuous linear functionals on $\mathcal{S}(\mathbb{R})$ are just those which are continous with respect to one of the norms.

These functionals are exactly the space of *tempered distributions*

$$\mathcal{S}'(\mathbb{R}) = \{T : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C} \text{ linear and continuous}\}. \tag{4.301}$$

The relationship to functions is that each $f \in L^2(\mathbb{R})$ (or more generally such that $(1+|x|)^{-N} \in L^1(\mathbb{R})$ for some N) defines an element of $\mathcal{S}'(\mathbb{R})$ by integration:

$$T_f : \mathcal{S}(\mathbb{R}) \ni \phi \longmapsto \int f(x)\phi(x) \in \mathbb{C} \Longrightarrow T_f \in \mathcal{S}'(\mathbb{R}). \tag{4.302}$$

Indeed, this amounts to showing that $\|\phi\|_{L^2}$ is a continuous norm on $\mathcal{S}(\mathbb{R})$ (so it must be bounded by a multiple of one of the $\|\phi\|_N$, which one?)

It is relatively straightforward to show that $L^2(\mathbb{R}) \ni f \longmapsto T_f \in \mathcal{S}'(\mathbb{R})$ is injective – nothing is 'lost'. So after a little more experience with distributions one comes to identify f and T_f. Notice that this is just an extension of the behaviour of $L^2(\mathbb{R})$ where (because we can drop the complex conjugate in the inner product) by Riesz' Theorem we can identify (linearly) $L^2(\mathbb{R})$ with it dual, exactly by the map $f \longmapsto T_f$.

Other elements of $\mathcal{S}'(\mathbb{R})$ include the delta 'function' at the origin and even its 'derivatives' for each j

$$\delta^j : \mathcal{S}(\mathbb{R}) \ni \phi \longmapsto (-1)^j \frac{d^j\phi}{dx^j}(0) \in \mathbb{C}. \tag{4.303}$$

In fact one of the main points about the space $\mathcal{S}'(\mathbb{R})$ is that differentiation and multiplication by polynomials is well defined

$$\frac{d}{dx} : \mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R}),\ \times x : \mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R}) \tag{4.304}$$

in a way that is consistent with their actions under the identification $\mathcal{S}(\mathbb{R}) : \phi \longmapsto T_\phi \in \mathcal{S}'(\mathbb{R})$. This property is enjoyed by other spaces of distributions but the fundamental fact that the Fourier transform extends to

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R}) \text{ as an isomorphism} \tag{4.305}$$

is more characteristic of $\mathcal{S}'(\mathbb{R})$.

16. Poisson summation formula

We have talked both about Fourier series and the Fourier transform. It is natural to ask: What is the connection between these? The Fourier series of a function in $L^2(0, 2\pi)$ we thought of as given by the Fourier-Bessel series with respect to the orthonormal basis

$$\frac{\exp(ikx)}{\sqrt{2\pi}},\ k \in \mathbb{Z}. \tag{4.306}$$

The interval here is just a particular choice – if the upper limit is changed to T then the corresponding orthonormal basis of $L^2(0, T)$ is

$$\frac{\exp(i2\pi kx/T)}{\sqrt{T}},\ k \in \mathbb{Z}. \tag{4.307}$$

Sometimes the Fourier transform is thought of as the limit of the Fourier series expansion when $T \to \infty$. This is actually not such a nice limit, so unless you *have* (or *want*) to do this I recommend against it!

A more fundamental relationship between the two comes about as follows. We can think of $L^2(0, 2\pi)$ as 'really' being the 2π-periodic functions restricted to this interval. Since the values at the end-points don't matter this does give a bijection – between 2π-periodic, locally square-integrable functions on the line and $L^2(0, 2\pi)$. On the other hand we can also think of the periodic functions as being defined on the circle, $|z| = 1$ in $\mathbb{C}$ or identified with the values of $\theta \in \mathbb{R}$ modulo repeats:

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \ni \theta \longmapsto e^{i\theta} \in \mathbb{C}. \tag{4.308}$$

Let us denote by $\mathcal{C}^\infty(\mathbb{T})$ the space of infinitely differentiable, 2π-periodic functions on the line; this is also the space of smooth functions on the circle, thought of as a manifold.

How can one construct such functions. There are plenty of examples, for instance the $\exp(ikx)$. Another way to construct examples is to sum over translations:-

LEMMA 4.21. *The map*

$$A : \mathcal{S}(\mathbb{R}) \ni f \longrightarrow \sum_{k\in\mathbb{Z}} f(\cdot - 2\pi k) \in \mathcal{C}^\infty(\mathbb{T}) \tag{4.309}$$

is surjective.

PROOF. That the series in (4.309) converges uniformly on $[0, 2\pi]$ (or any bounded interval) is easy enought to see, since the rapid decay of elements of $\mathcal{S}(\mathbb{R})$ shows that

$$|f(x)| \le C(1+|x|)^{-2},\ x \in \mathbb{R} \Longrightarrow |f(x-2\pi k)| \le C'(1+|k|)^{-2},\ x \in [0, 2\pi] \tag{4.310}$$

since if $k > 2$ $|x-2\pi k| \ge k$ if $x \in [0, 2\pi]$. Clearly (4.310) implies uniform convergence of the series. Since the derivatives of f are also in $\mathcal{S}(\mathbb{R})$ the series obtained by term-by-term differentiation also converges uniformly and by standard arguments the limit Ag is therefore infinitely differentiable, with

$$\frac{d^jAf}{dx^j} = A\frac{d^jf}{dx^j}. \tag{4.311}$$

This shows that the map A, clearly linear, is well-defined. Now, how to see that it is surjective? Let's first prove a special case. Indeed, look for a function $\psi \in \mathcal{C}^\infty_c(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ which is non-negative and such that $A\psi = 1$. We know that we can find $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$, $\phi \ge 0$ with $\phi > 0$ on $[0, 2\pi]$. Then consider $A\phi \in \mathcal{C}^\infty(\mathbb{T})$. It must be stricly positive, $A\phi \ge \epsilon > 0$ since it is larger that ϕ. So consider instead the function

$$\psi = \frac{\phi}{A\phi} \in \mathcal{C}^\infty_c(\mathbb{R}) \tag{4.312}$$

where we think of $A\phi$ as 2π-periodic on $\mathbb{R}$. In fact using this periodicity we see that

$$A\psi \equiv 1. \tag{4.313}$$

So this shows that the constant function 1 is in the range of A. In general, just take $g \in \mathcal{C}^\infty(\mathbb{T})$, thought of as 2π-periodic on the line, and it follows that

$$f = Bg = \psi g \in \mathcal{C}^\infty_c(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}) \text{ satsifies } Af = g. \tag{4.314}$$

Indeed,

$$Ag = \sum_k \psi(x-2\pi k)g(x-2\pi k) = g(x)\sum_k \psi(x-2\pi k) = g \tag{4.315}$$

using the periodicity of g. In fact B is a right inverse for A,

$$AB = \mathrm{Id} \text{ on } \mathcal{C}^\infty(\mathbb{T}). \tag{4.316}$$

□

QUESTION 2. What is the null space of A?

Since $f \in \mathcal{S}(\mathbb{R})$ and $Af \in \mathcal{C}^\infty(\mathbb{T}) \subset L^2(0, 2\pi)$ with our identifications above, the question arises as to the relationship between the Fourier transform of f and the Fourier series of Af.

PROPOSITION 4.13 (Poisson summation formula). *If $g = Af$, $g \in \mathcal{C}^\infty(\mathbb{T})$ and $f \in \mathcal{S}(\mathbb{R})$ then the Fourier coefficients of g are*

$$c_k = \int_{[0,2\pi]} g e^{-ikx} = \hat{f}(k). \tag{4.317}$$

PROOF. Just substitute in the formula for g and, using uniform convergenc, check that the sum of the integrals gives after translation the Fourier transform of f. □

If we think of recovering g from its Fourier series,

$$g(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} c_k e^{ikx} = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikx} \tag{4.318}$$

then in terms of the Fourier transform on $\mathcal{S}'(\mathbb{R})$ alluded to above, this takes the rather elegant form

$$\frac{1}{2\pi}\mathcal{F}\left(\sum_{k\in\mathbb{Z}} \delta(\cdot - k)\right)(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{ikx} = \sum_{k\in\mathbb{Z}} \delta(x - 2\pi k). \tag{4.319}$$

The sums of translated Dirac deltas and oscillating exponentails all make sense in $\mathcal{S}'(\mathbb{R})$.

17. Dirichlet problem

As a final application, which I do not have time to do in full detail in lectures, I want to consider the Dirichlet problem again, but now in higher dimensions. Of course this is a small issue, since I have not really gone through the treatment of the Lebesgue integral etc in higher dimensions – still I hope it is clear that with a little more application we could do it and for the moment I will just pretend that we have.

So, what is the issue? Consider Laplace's equation on an open set in $\mathbb{R}^n$. That is, we want to find a solution of

$$-(\frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2^2} + \cdots + \frac{\partial^2 u(x)}{\partial x_n^2}) = f(x) \text{ in } \Omega \subset \mathbb{R}^n. \tag{4.320}$$

Now, maybe some of you have not had a rigorous treatment of partical derivatives either. Just add that to the heap of unresolved issues. In any case, partial derivatives are just one-dimensional derivatives in the variable concerned with the other variables held fixed. So, we are looking for a function u which has *all* partial derivatives up to order 2 existing everywhere and continous. So, f will have to be continuous too. Unfortunately this is *not* enough to guarantee the existence of a twice continuously differentiable solution – later we will just suppose that f itself is once continuously differentiable.

Now, we want a solution of (4.320) which satisfies the Dirichlet condition. For this we need to have a reasonable domain, which has a decent boundary. To short cut the work involved, let's just suppose that $0 \in \Omega$ and that it is given by an inequality of the sort

$$\Omega = \{z \in \mathbb{R}^n; |z| < \rho(z/|z|) \tag{4.321}$$

where ρ is another once continuously differentiable, and strictly positive, function on $\mathbb{R}^n$ (although we only care about its values on the unit vectors). So, this is no worse than what we are already dealing with.

Now, the Dirichlet condition can be stated as

$$u \in \mathcal{C}^0(\overline{\Omega}),\ u\big|z| = \rho(z/|z|) = 0. \tag{4.322}$$

Here we need the first condition to make much sense of the second.

So, what I want to approach is the following result – which can be improved a lot and which I will not quite manage to prove anyway.

THEOREM 4.7. *If* $0 < \rho \in \mathcal{C}^1(\mathbb{R}^n)$, *and* $f \in \mathcal{C}^1(\mathbb{R}^n)$ *then there exists a unique* $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ *satisfying* (4.320) *and* (4.322).