THE PROBLEMS FOR THE SECOND TEST FOR 18.102 WILL BE SELECTED FROM THIS LIST

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Question 1

Show that if $A \in \mathcal{B}(H)$, where H is a separable Hilbert space, has the property that $u_n \rightharpoonup u$ (converges weakly) implies that Au_n is Cauchy. Show that $\overline{A(\{u \in H; \|u\| \le 1\})}$ is compact.

Question 2

A sequence of bounded operators $A_n \in \mathcal{B}(H)$ is said to *converge strongly* if for each $u \in H$, $A_n u$ converges in H. Show that $Au = \lim_n A_n u$ is necessarily a bounded linear operator on H (called the strong limit of the sequence).

Question 3

A sequence of bounded operators $A_n \in \mathcal{B}(H)$ is said to *converge weakly* if for each pair of elements u, $v \in H$, $\langle A_n u, v\rangle$ converges in $\mathbb{C}$. Show that there exists a bounded linear operator $A \in \mathcal{B}(H)$ (called the weak limit of the sequence) such that $\langle Au, v\rangle = \lim_n \langle A_n u, v\rangle$.

Question 4

If H is a separable, infinite dimensional, Hilbert space set

$$l^2(H) = \{u : \mathbb{N} \longrightarrow H; \|u\|^2_{l^2(H)} = \sum_i \|u_i\|^2_H < \infty\}. \tag{1}$$

Show that $l^2(H)$ has a Hilbert space structure and construct an explicit isometric (norm-preserving) isomorphism (bijection) from $l^2(H)$ to H.

Question 5

Starting from the definition of weak convergence of a sequence in a separable Hilbert space, $v_n \rightharpoonup v$, that $\langle v_n, w\rangle \to \langle v, w\rangle$ in $\mathbb{C}$ for each $w \in H$, show that a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if

$$\|v\|_H = \lim_{n\to\infty} \|v_n\|_H. \tag{2}$$

Question 6

Let e_k, $k \in \mathbb{N}$, be an orthonormal basis in a separable Hilbert space, H. Show that there is a unique bounded linear operator $T : H \longrightarrow H$ satisfying

$$Te_j = e_{j-1} \ \forall \ j \geq 2, \ Te_1 = 0, \tag{3}$$

and that if $B \in \mathcal{B}(H)$ has $\|B\| < 1$ then $T + B$ has one-dimensional null space.

Question 7

Suppose H is an infinite dimensional separable Hilbert space with an orthonormal basis $\{e_k\}_{k=1}^{\infty}$. Show that a continuous function $K : [0,1] \longrightarrow H$ has the property that the Fourier-Bessel series of $K(x) \in H$, for $x \in [0,1]$, converges uniformly in the sense that if $K_n(x) = \sum\limits_{k \leq n} \langle K(x), e_k \rangle e_k$ then $K_n : [0,1] \longrightarrow H$ is also continuous and

$$\sup_{x \in [0,1]} \|K(x) - K_n(x)\|_H \to 0. \tag{4}$$

Question 8

– I decided this was too hard/long so will not be on the test – If $A \in \mathcal{B}(H)$ is a bounded operator on a separable, infinite-dimensional, Hilbert space H, explain why $|A| = (A^*A)^{1/2} \in \mathcal{B}(H)$ is well-defined. For $p \in [1, \infty)$ let $\mathcal{L}^p(H) \subset \mathcal{B}(H)$ consist of those operators such that $|A|$ is compact and for any orthonormal basis of H consisting of eigenvectors of $|A|$,

$$\|A\|_{\mathcal{L}^p} = \left(\sum_i |\langle Ae_i, Ae_i \rangle|^{p/2} \right)^{1/p} < \infty. \tag{5}$$

Show that this is a norm making $\mathcal{L}^p(H)$ into a Banach space.

Question 9

Show that a separable Hilbert space H with the property that every bounded operator from H to itself is compact is necessarily finite dimensional.

Question 10

Show that if B is a compact operator on a separable Hilbert space H and A is an invertible operator then

$$\{u \in H; Bu = Au\} \tag{6}$$

is finite dimensional.

Question 11

Assuming the Stone-Weierstrass theorem, show that there is a complete orthonormal basis of $L^2([0, 2\pi])$ consisting of polynomials.

Question 12

Let H be a separable Hilbert space and let $\mathcal{C}_{\mathrm{c}}(\mathbb{R};H)$ be the linear space of continuous maps from $\mathbb{R}$ to H which vanish outside some interval $[-R,R]$ depending on the function Show that

$$\|u\|^2 = \int_{\mathbb{R}} \|u(x)\|_H^2 \tag{7}$$

defines a norm which comes from a pre-Hilbert structure on $\mathcal{C}_{\mathrm{c}}(\mathbb{R};H)$. Show that if u_n is a Cauchy sequence in this pre-Hilbert space and $h \in H$ then $\langle u_n(x), h\rangle_H$ converges in $L^2(\mathbb{R})$.

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