PREPARATION FOR TEST 1 FOR 18.102
TEST ON MARCH 9, 2017

The test will consist of some of the questions below. Hints on request.

No papers, books, electronic divices or such may be consulted during the test.

You may use the results we have shown in class up to this point including Monotonicity, Fatou, Lebesgue Dominated Convergence and the result that an absolutely summable sequence in $\mathcal{L}^1(\mathbb{R})$ converges almost everywhere.

Recall the definition

$$\mathcal{L}^2(\mathbb{R}) = \{u : \mathbb{R} \longrightarrow \mathbb{C}; \exists\, u_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R}),\ u_n(x) \to u(x) \text{ a.e. and } \exists\, F \in \mathcal{L}^1(\mathbb{R}) \text{ such that } |u_n(x)|^2 \le F \text{ a.e.}\}. \tag{1}$$

Then $L^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R})/\mathcal{N}(\mathbb{R})$ where $\mathcal{N}(\mathbb{R})$ is the space of null functions.

Question 1. Show that the function

$$u(x) = \begin{cases} 0 & x \le 0 \\ \min(x^{-\frac{1}{2}}, x^{-2}) & x > 0 \end{cases} \tag{2}$$

is in $\mathcal{L}^1(\mathbb{R})$.

Question 2. Prove from the definition above that the product of two functions in $\mathcal{L}^2(\mathbb{R})$ is in $\mathcal{L}^1(\mathbb{R})$.

Question 3. Show that the product of a bounded continuous function on $\mathbb{R}$ and an element of $\mathcal{L}^2(\mathbb{R})$ is in $\mathcal{L}^2(\mathbb{R})$.

Question 4. Give an example of a function which is in $\mathcal{L}^1(\mathbb{R})$ but not in $\mathcal{L}^2(\mathbb{R})$ and another example of a function which is in $\mathcal{L}^2(\mathbb{R})$ but not in $\mathcal{L}^1(\mathbb{R})$; justify both.

Question 5. Show that if $t \in \mathbb{R}$ and $f \in \mathcal{L}^1(\mathbb{R})$ then

$$f_t(x) = f(x - t) \tag{3}$$

is an element of $\mathcal{L}^1(\mathbb{R})$. Prove that $f \in \mathcal{L}^1(\mathbb{R})$ is *continuous-in-the-mean* in the sense that given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|t| < \delta \Longrightarrow \int |f_t - f| < \epsilon. \tag{4}$$

Question 6. Show that if $f : \mathbb{R} \longrightarrow [0, \infty)$ and $f^2 \in \mathcal{L}^1(\mathbb{R})$ then $f \in \mathcal{L}^2(\mathbb{R})$.

Question 7. Suppose that $B : L^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is a bounded linear operator. Show that if $B(\phi) = 0$ for all $\phi \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ then $B = 0$ as an operator.

Question 8. Find the real numbers s such that $(1 + |x|)^{s/2} \in L^2(\mathbb{R})$ and justify your conclusion.

Question 9. Suppose that $[f] \in L^2(\mathbb{R})$ and $\phi \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$, explain why $\int f\phi$ exists and show that if $\int f\phi = 0$ for all $\phi \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ then $[f] = 0$.

Question 10. Suppose that $[f_j] \in L^1(\mathbb{R})$ is a Cauchy sequence, show that f_j has a subsequence which converges almost everywhere.