PROBLEM SET 9 FOR 18.102, SPRING 2017
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Define $H^2_0([0,\pi]) \subset L^2(0,\pi)$ as consisting of those functions for which the (unnormalized) Fourier-Bessel coefficients (for the basis introduced to solve the Dirichlet problem) satisfy

$$\sum_{k\in\mathbb{N}_0} |k^2 c_k|^2 < \infty \text{ where } c_k = \int_0^\pi \sin kx f(x)dx. \tag{1}$$

This is a *Sobolev space*. I'm not sure what the H stands for (Hilbert maybe) but the superscript '2' stands for two deriviatives (in L^2) and the subscript 0 means vanishing at the boundary – see below!

Problem 9.1

Show that if $u \in H^2_0([0,\pi])$ and u_N is the sum of the first N terms in the Fourier-Bessel series for u (which is in $L^2(0,\pi)$) then

$$u_N \to u,\ \frac{du_N}{dx} \to F_1,\ \frac{d^2u_N}{dx^2} \to F_2 \tag{2}$$

where in the first two cases we have convergence in supremum norm and in the third, convergence in $L^2(0,\pi)$. Deduce that $u \in \mathcal{C}^0[0,\pi]$, $u(0) = u(\pi) = 0$ and $F_1 \in \mathcal{C}^0[0,\pi]$ whereas $F_2 \in L^2(0,\pi)$.

Problem 9.2

Let $\mathcal{C}^2_0([0,\pi])$ be the space of twice continuously differentiable functions on $[0,\pi]$ (one-sided derivatives at the ends) which vanish at 0 and π – this is the space considered in lecture for the Dirichlet problem. Show that $H^2_0([0,\pi])$ is a Hilbert space with respect to the norm

$$\|u\|^2 = \sum_k (1+k^4)|c_k|^2 \tag{3}$$

and that $\mathcal{C}^2_0([0,\pi]) \subset H^2_0([0,\pi])$ is a dense subspace.

Hint: Try not to belabour the proof of completeness since you have done so many – but really do it nevertheless! If $\phi \in \mathcal{C}^2_0([0,\pi])$ compute the integrals in (1) above and ingtegrate by parts to show the rest of (1). Think about $\sin kx$ (maybe write down a related orthonormal basis of $H^2_0([0,\pi])$) to prove density.

Problem 9.3

With F_1 and F_2 as in (2) for $u \in H^2_0([0,\pi])$ show that

$$\int_{0,\pi} u\phi' = -\int_{0,\pi} F_1\phi,\ \int_{0,\pi} u\phi'' = \int_{0,\pi} F_2\phi,\ \forall\ \phi \in \mathcal{C}^2_0([0,\pi]) \tag{4}$$

and show that if $u \in \mathcal{C}^2_0([0,\pi]) \subset H^2_0([0,\pi])$ then $F_1 = u'$, $F_2 = u''$.

Hint: Try Cauchy-Schwartz inequality on the sum of the first N terms in the Fourier-Bessel series using the inequalities from (1).

Problem 9.4

Show that if $V \in \mathcal{C}^0[0,\pi]$ then the linear map

$$Q_V : H_0^2([0,\pi]) \ni u \longmapsto -F_2 + Vu \in L^2(0,\pi) \tag{5}$$

is bounded and reduces to

$$u \longmapsto -u'' + Vu \text{ on } \mathcal{C}_0^2([0,\pi]). \tag{6}$$

Problem 9.5

Show (it is really a matter of recalling) that the inverse $Q_0^{-1} = A^2$ is the square of a compact self-adjoint non-negative operator on $L^2(0,\pi)$ and that

$$Q_V^{-1} = A(\mathrm{Id} + AVA)^{-1}A \tag{7}$$

(where we are assuming that $0 \le V \in \mathcal{C}^0[0,\pi]$). Using results from class or the notes on the Dirichlet problem (or otherwise ..) show that if $V \ge 0$ then Q_V is an isomorphism (meaning just a bounded bijection with bounded inverse) of $H_0^2([0,\pi])$ to $L^2(0,\pi)$.

Hint: For the boundedness of the inverse of Q_V use the formula

$$Q_V^{-1} = A(\mathrm{Id} + AVA)^{-1}A$$

from class where $\mathrm{Id} + AVA$ is invertible using the spectral theorem. By expanding out the definition of the inverse of this operator, show that it is of the form $\mathrm{Id} + AFA$ where F is bounded on L^2. Substitute this into the formula for Q_V^{-1} and see that there is a factor of A^{-2} on the left. What is this?

Problem 9.6 – extra

Use the minimax princple from an earlier problem set to show that the eigenvalues of Q_V, repeated with multiplicity, and arranged as an increasing sequence, are such that

$$\sup_j |\lambda_j - j^2| < \infty. \tag{8}$$

Problem 9.7 – extra

Show that if $V \in \mathcal{C}^2[0,\pi]$ is twice continuously differentiable and real-valued, then all the eigenfunctions of the Dirichlet problem for $-d^2/dx^2 + V$ are four times continously differentiable on $[0,\pi]$.

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