PROBLEM SET 8 FOR 18.102
DUE FRIDAY 14TH APRIL, 2017

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Here are Ethan's comments:-

Problem 1) The first bit of problem 1 was handled well, although several students made heavy work of it instead of appealing directly to the spectral theorem. I saw two common mistakes in the second part: assuming that F was spanned by some subcollection of an eigenbasis for E, and assuming that $E : F \to F$, so that one could apply the spectral theorem again.

Problem 2) This was handled well.

Problem 3) Most students had the right idea, and managed to use it to varying degrees of effectiveness. The most common stumbling block was with what happens to the dimension of $D(F)$? It could jump down, and needs to be dealt with. There was also a bit of subtlety with "changing variables" $Du = v$, and how to pass from $||u|| = 1$ to $||v|| = 1$, and I didn't award full points to anyone who elided the subtlety.

Problem 4) This was handled well.

Problem 5) This was generally handled well. There was some confusion about what exactly was meant to be proven about the a_{jk}, and since part of the question is finding the correct thing to prove, anyone who didn't show that they were square-summable lost a point or two.

For problem 8.5 below you can use the fact which will be proved next week, namely that the trigonometric functions

$$\frac{e^{ikx}}{\sqrt{2\pi}},\ k \in \mathbb{Z} \tag{1}$$

form an orthonormal basis of $L^2(0, 2\pi)$ – it is the completeness which is not obvious. The Fourier coefficients of a function $a \in L^2(0, 2\pi)$ are normalized below to be

$$a_j = \int_{(0,2\pi)} a(x)e^{-ijx} \tag{2}$$

so there are some factors of $\sqrt{2\pi}$ to take care of.

Problem 8.1

Suppose that $E \in \mathcal{B}(H)$ is a compact self-adjoint operator on a separable Hilbert space and that E is non-negative in the sense that

$$(Eu, u) \geq 0\ \forall\ u \in H.$$

Show that E has no negative eigenvalues and that the positive eigenvalues can be arranged in a (weakly) decreasing sequence

$$s_1 \geq s_2 \geq \cdots \to 0$$

either finite, or decreasing to zero, such that if $F \subset H$ has dimension N then

$$\min_{u\in F,\|u\|=1}(Eu,u) \le s_N,\ \forall\ N.$$

NB. The s_j have to be repeated corresponding to the dimension of the associated eigenspace.

Problem 8.2

Extend this further to show that under the same conditions on E the eigenvalues are give by the minimax formula:-

$$s_j(E) = \max_{F\subset H;\dim F=j}\left(\min_{u\in F;\|u\|=1}(Eu,u)\right).$$

Problem 8.3

With E as above, suppose that $D \in \mathcal{B}(H)$ is a bounded self-adjoint operator. Show that

$$s_j(DED) \le \|D\|^2 s_j(E)\ \forall\ j.$$

NB. Be a bit careful about the minimax argument.

Problem 8.4

Let A be a self-adjoint Hilbert-Schmidt operator (see the preceding problem set). Explain why the eigenspaces for non-zero eigenvalues, λ_j, of A are finite dimensional and show that

$$\sum_j \lambda_j^2 < \infty.$$

Problem 8.5

Suppose $a \in \mathcal{C}^0([0,2\pi]^2)$ is a continuous function of two variables. Show that the Fourier coefficients of a in the second variable are continuous functions of the first variable and hence that the double Fourier coefficients

$$a_{jk} = \int_0^{2\pi}\int_0^{2\pi} a(x,y)e^{-ijx-iky}dydx$$

are well-defined. If A is the integral operator 'with kernel a', so

$$(Af)(x) = \int_0^{2\pi} a(x,y)f(y)dy,\ f \in L^2(0,2\pi)$$

show that

$$\sum_{k\in\mathbb{Z}} \|Ae^{iky}\|^2_{L^2(0,2\pi)} < \infty$$

and so conclude that A is a Hilbert-Schmidt operator. What does this imply about the coefficients a_{jk}?

Hint: Think about

$$\sum_{k=1}^N |c_k(x)|^2$$

where x is fixed and $c_k(x) = A(e^{ik\cdot})$. From the definition of A you can think of this as an inner product and so it can be bounded by an integral using Bessel's inequality. Integrate both sides in x and deduce that the integrated sum has a bound independent of N.

Problem 8.6 – extra

Consider the notion of an *unbounded* self-adjoint operator (since so far an operator is bounded, you should think of this as unbounded-self-adjoint-operator, a new notion which does include bounded self-adjoint operators). Namely, if H is a separable Hilbert space and $D \subset H$ is a *dense* linear subspace then a linear map $A : D \longrightarrow H$ is an unbounded self-adjoint operator if

(1) For all v, $w \in D$, $\langle Av, w\rangle_H = \langle v, Aw\rangle_H$.
(2) $\{u \in H; D \ni v \longmapsto \langle Av, u\rangle \in \mathbb{C}$ extends to a continuous map on $H\} = D$.

Show that

$$\mathrm{Gr}(A) = \{(u, Au) \in H \times H; u \in D\} \tag{3}$$

is a closed subspace of $H \times H$ and that $A + i\,\mathrm{Id} : D \longrightarrow H$ is surjective with a bounded two-sided inverse $B : H \longrightarrow H$ (with range D of course).

Problem 8.7 – extra

Suppose A is a compact self-adjoint operator on a separable Hilbert space and that $\mathrm{Nul}(A) = \{0\}$. Define a dense subspace $D \subset H$ in such a way that $A^{-1} : D \longrightarrow H$ is an unbounded self-adjoint operator which is a two-sided inverse of A.

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