PROBLEM SET 7 FOR 18.102
DUE FRIDAY 7 APRIL, 2017.

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Here are Ethan's comments:-

Problem 1 was handled well. Pretty much everyone got it right, and gave some variation of the unit sphere or an orthonormal set as their examples. A few students tried to use the fact that componentwise convergence is equivalent to weak convergence for bounded sequences, but forgot to mention that the sequence needed to be bounded, so lost some points there.

Problem 2 was handled well.

There were lots of subtle mistakes in Problem 3. Many people didn't show that the inner product was well-defined, let alone didn't depend on the choice of basis. The question was a little unclear, and a lot of students proved that $\mathrm{HS}(H)$ was an ideal in the (non-unital) ring $\mathcal{K}(H)$, rather than in $\mathcal{B}(H)$. I didn't deduct points for this. I saw three variations of the completeness proof. A lot of students showed that $A_n e_i$ converged to some vector Ae_i, but assumed that this always had an extension to an operator on all of H. Students still aren't proving everything required to show completeness, too. Often they would get almost all the way, but omit the proof that there A was actually the HS limit, without saying anything else.

Problem 4 was done well.

Problem 5 was also done well, but many forgot to say why this means that $\mathrm{GL}(H)$ wasn't dense. Since this is more or less the point of this easy problem, I only awarded half points if someone didn't give me at least a one-line argument for why.

Problem 7.1

Give an example of a closed subset in an infinite-dimensional Hilbert space which is not weakly closed.

Problem 7.2

Let $A \in \mathcal{B}(H)$, H a separable Hilbert space, be such that for some orthonormal basis $\{e_i\}$

$$\sum_i \|Ae_i\|_H^2 < \infty. \tag{7.1}$$

Show that the same inequality is true for any other orthonormal basis and that A^* satisfies the same inequality.

Hint: For another basis, expand each norm $\|Ae_i\|^2$ using Bessel's identity and then use the adjoint identity and undo the double sum the opposite way.

Problem 7.3

The elements of $A \in \mathcal{B}(H)$ as in Problem 7.2 are called 'Hilbert-Schmidt operators'. Show that these form a 2-sided $*$-closed ideal $\mathrm{HS}(H)$, inside the compact operators and that

$$\langle A, B\rangle = \sum_i \langle Ae_i, Be_i\rangle, \tag{7.2}$$

for any choice of orthonormal basis, makes this into a Hilbert space.

Problem 7.4

Consider the 'shift' operators $S : l^2 \longrightarrow l^2$ and $T : l^2 \longrightarrow l^2$ defined by

$$S(\{a_k\}_{k=1}^{\infty}) = \{b_k\}_{k=1}^{\infty},\ b_k = a_{k+1}, k \geq 1,$$
$$T(\{a_k\}_{k=1}^{\infty}) = \{c_k\}_{k=1}^{\infty},\ c_1 = 0,\ c_k = a_{k-1},\ k \geq 2.$$

Compute the norms of these operators and show that $TS = \mathrm{Id} - \Pi_1$ and $ST = \mathrm{Id}$ where $\Pi_1(\{a_k\}_{k=1}^{\infty}) = \{d_k\}$, $d_1 = a_1$, $d_k = 0$, $k \geq 2$.

Problem 7.5

With the operators as defined in the preceding problem, show that for any $B \in \mathcal{B}(l^2)$ with $\|B\| < 1$, $S + B$ is not invertible (and so conclude that the invertible operators are not dense in $\mathcal{H}(l^2)$).

Hint: Show that $(S+B)T = \mathrm{Id} + BT$ is invertible and so if $S+B$ was invertible, with inverse G then so is $G(S+B)T = T$, as the product of invertibles.

Problem 7.6-extra

An operator T on a separable Hilbert space is said to be 'of trace class' (where this is just old-fashioned language) if it can be written as a finite sum

$$T = \sum_{i=1}^{N} A_i B_i$$

where all the A_i, B_i are Hilbert-Schmidt as discussed above. Show that these trace class operators form a 2-sided ideal in the bounded operators, closed under passage to adjoints and that

$$\sup_{\{e_i\},\{f_i\}} \sum_i |\langle Te_i, f_i\rangle| < \infty$$

where the sup is over all pairs of orthonormal bases. Show that the trace functonal

$$\mathrm{Tr}(T) = \sum_i \langle Te_i e_i\rangle$$

is well-defined on trace class operators, independent of the orthonormal basis $\{e_i\}$ used to compute it.

Problem 7.7-extra

Suppose $K : L^2(0,1) \longrightarrow L^2(0,1)$ is an integral operator (as considered in an earlier problem set) with a continuous kernel $k \in \mathcal{C}([0,1]^2)$,

$$Kf(x) = \int_{(0,1)} K(x,y)f(y)dy. \tag{7.3}$$

Show that K is Hilbert-Schmidt.

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