Problem 6.1
Recall the space $h^{2,1}$ (discussed in the preceding problem set) consisting of the complex valued sequences $c_i$ such that
\[
\|c\|^2 = \sum_i (1 + |i|^2)|c_i|^2 < \infty.
\]
Show that the unit ball in this space, considered as a subset of $l^2$, has compact closure.

Problem 6.2
Define the space $L^2(0, 1)$ as consisting of those elements of $L^2(\mathbb{R})$ which vanish outside $(0, 1)$ and show that the quotient $L^2(0, 1) = L^2(0, 1)/\mathcal{N}(0, 1)$ by the null functions in $L^2(0, 1)$ is a Hilbert space.

Remark: This is indeed easy, but make sure you do it properly (for instance identify $L^2(0, 1)$ with a closed subspace of $L^2(\mathbb{R})$ by treating the null functions properly). You can use the fact that $L^2(\mathbb{R})$ is a Hilbert space.

Problem 6.3
Identify $C[0, 1]$, the space of continuous functions on the closed interval, as a subspace of $L^2(0, 1)$. For each $n \in \mathbb{N}$ let $F_n \subset L^2(0, 1)$ be the subspace of functions which are constant on each interval $([m - 1]/n, m/n]$ for $m = 1, \ldots, n$. Show that if $f \in C[0, 1]$ there exists a sequence $g_n \in F_n$ such that
\[
\delta_n = \sup_{|t-s|\leq 1/n} |f(t) - f(s)| \implies \|f - g_n\|_{L^2} \leq \delta_n.
\]

Problem 6.4
Show that a bounded and equicontinuous subset of $C[0, 1]$ has compact closure in $L^2(0, 1)$. Note that equicontinuity means ‘uniform equicontinuity’ so for each $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all elements $f$ of the set.

Problem 6.5
If $K \in C([0, 1] \times [0, 1])$ is a continuous function of two variables, show that
\[
(6.1) \quad Af(x) = \int K(x, y)f(y)
\]
defines a compact linear operator on $L^2(0, 1)$.

Hint: Show that $A$ defines a bounded linear map from $L^2(0, 1)$ to $C[0, 1]$ and that the image of the unit ball is equicontinuous using the uniform continuity of $K$. 
Problem 6.6 – extra

Show that a closed and bounded subset of $L^2(\mathbb{R})$ is compact if and only if it is ‘uniformly equicontinuous in the mean’ and ‘uniformly small at infinity’ so that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathbb{R}\setminus[-1/\delta,1/\delta]} |f|^2 < \epsilon^2 \text{ and } |t| < \delta \implies \int \left| f(x) - f(x-t) \right|^2 < \epsilon^2$$

for all elements of the set.

Problem 6.7 – extra

Consider the space of continuous functions on $\mathbb{R}$ vanishing outside $(0, 1)$ which are of the form

$$u(x) = \int_0^x v, \ v \in L^2(0,1).$$

Show that these form a Hilbert space and that the unit ball of this space has compact closure in $L^2(0,1)$. 