PROBLEM SET 5 FOR 18.102, SPRING 2017
DUE FRIDAY 17 MARCH IN THE USUAL SENSE.

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Problem 5.1
Let $H$ be a normed space (over $\mathbb{C}$) in which the norm satisfies the parallelogram law:
\[(1) \quad \|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2) \quad \forall \ u, v \in H.\]
Show that
\[(2) \quad \langle u, v \rangle = \frac{1}{4} (\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2)\]
is a positive-definite Hermitian form which induces the given norm.
Hint: Linearity is a pain. Try to get something, say for a mid-point, first.

Problem 5.2
Let $H$ be a finite dimensional (pre)Hilbert space. So, by definition $H$ has a basis
\[\{v_i\}_{i=1}^n,\]
meaning that any element of $H$ can be written
\[(3) \quad v = \sum_i c_i v_i\]
and there is no dependence relation between the $v_i$’s – the presentation of $v = 0$ in the form (3) is unique. Show that $H$ has an orthonormal basis, $\{e_i\}_{i=1}^n$ satisfying $(e_i, e_j) = \delta_{ij}$ (= 1 if $i = j$ and 0 otherwise). Check that for the orthonormal basis the coefficients in (3) are given by the inner products $c_i = \langle v, e_i \rangle$ and that the map
\[(4) \quad T : H \ni v \mapsto (\langle v, e_1 \rangle, \langle v, e_2 \rangle, \ldots, \langle v, e_n \rangle) \in \mathbb{C}^n\]
is a linear isomorphism with the properties
\[(5) \quad \langle u, v \rangle = \sum_i (Tu)_i \overline{(Tv)_i}, \quad \|u\|_H = \|Tu\|_{\mathbb{C}^n} \quad \forall \ u, v \in H.\]
Why is a finite dimensional pre-Hilbert space a Hilbert space?

Problem 5.3
Let $e_i, \ i \in \mathbb{N},$ be an orthonormal sequence in a separable Hilbert space $H.$ Suppose that for each element $u$ in a dense subset $D \subset H$
\[(6) \quad \sum_i |\langle u, e_i \rangle|^2 = \|u\|^2.\]
Conclude that $e_i$ is an orthonormal basis, i.e. is complete.
Consider the sequence space
\[ h^{2,1} = \left\{ c : \mathbb{N} \ni j \mapsto c_j \in \mathbb{C}; \sum_j (1 + j^2)|c_j|^2 < \infty \right\}. \]

(1) Show that
\[ h^{2,1} \times h^{2,1} \ni (c, d) \mapsto \langle c, d \rangle_{2,1} = \sum_j (1 + j^2)c_j d_j \]
is an Hermitian inner form which turns \( h^{2,1} \) into a Hilbert space.

(2) Denoting the norm on this space by \( \| \cdot \|_{2,1} \) and the norm on \( l^2 \) by \( \| \cdot \|_2 \), show that
\[ h^{2,1} \subset l^2, \quad \| c \|_2 \leq \| c \|_{2,1} \quad \forall \ c \in h^{2,1}. \]

Problem 5.5

Suppose that \( H_1 \) and \( H_2 \) are two different Hilbert spaces and \( A : H_1 \to H_2 \) is a bounded linear operator. Show that there is a unique bounded linear operator (the adjoint) \( A^* : H_2 \to H_1 \) with the property
\[ \langle Au_1, u_2 \rangle_{H_2} = \langle u_1, A^* u_2 \rangle_{H_1} \quad \forall \ u_1 \in H_1, \ u_2 \in H_2. \]

Problem 5.6 – Extra

Show that the complex finite linear combinations of the functions \( \sin ax, a \in \mathbb{R} \) form a pre-Hilbert space with respect to the norm given by
\[ \| f \|^2 = \lim_{R \to \infty} \frac{1}{R} \int_{[-R,R]} |f|^2. \]

Show that the completion is a non-separable Hilbert space. Can you give a concrete description of the elements of the completion?

Problem 5.7 – Extra

Consider the subspace of \( L^2(\mathbb{R}) \) which consists of continuous functions \( u \) with the additional property that there exists \( v \in L^2(\mathbb{R}) \) such that
\[ u(x) = \begin{cases} u(0) + \int_{(0,x)} v(t) & \text{if } x > 0 \\ u(0) - \int_{(x,0)} v(t) & \text{if } x < 0. \end{cases} \]

Show that for a given \( u \) if there are two such functions \( v \) then they differ by a null function. Prove that the set of pairs \( (u, [v]) \) where \([v] \in L^2(\mathbb{R}) \) is a Hilbert space with respect to the inner product
\[ \langle (u_1, [v_1]), (u_2, [v_2]) \rangle = \int u_1 \overline{u_2} + \int v_1 \overline{v_2}. \]