# PROBLEM SET 3 FOR 18.102, SPRING 2017 DUE ELECTRONICALLY ON FRIDAY 24 FEBRUARY (IN THE USUAL SENSE)

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There is only one lecture this week but these problems do not depend on it – you should be able to tackle them now. The extra problems will reappear next week.

Problem 3.1

Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is such that there is a sequence $f_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ with real values, such that $f_n(x)$ is increasing for each $x$, $\int f_n$ is bounded and

$$\lim_n f_n(x) = f(x) \tag{1}$$

whenever the limit exists. Show that $f \in \mathcal{L}^1(\mathbb{R})$.

Problem 3.2

Suppose $E \subset \mathbb{R}$ has the following (well-known) property:-

$$\begin{gathered}\forall\ \epsilon > 0\ \exists \text{ a countable collection of intervals } (a_i, b_i) \text{ s.t.}\\ \sum_i (b_i - a_i) < \epsilon,\ E \subset \bigcup_i (a_i, b_i).\end{gathered} \tag{2}$$

Show that $E$ is a set of measure zero in the sense used in lectures and the notes.

Remark: I will ask you to prove the converse in the next problem set, using results from Lecture 4 or 5; the next question will then be rather easy.

Problem 3.3

Write out proof (I described one briefly in Lecture 2) that a non-trivial interval $[a, b] \subset \mathbb{R}$ were $b > a$, is not of measure zero.

Problem 3.4

Show that the function with $F(0) = 0$ and

$$F(x) = \begin{cases} 0 & x > 1 \\ \exp(i/x) & 0 < |x| \le 1 \\ 0 & x < -1, \end{cases}$$

is an element of $\mathcal{L}^1(\mathbb{R})$.

Problem 3.5

Suppose $f \in \mathcal{L}^1(\mathbb{R})$ is real-valued. Show that there is a sequence $f_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ and another element $F \in \mathcal{L}^1(\mathbb{R})$ such that

$$f_n(x) \to f(x) \text{ a.e. on } \mathbb{R},\ |f_n(x)| \le F(x) \text{ a.e.}$$

Hint: Take an approximating series $u_n$ as in the definition and think about $|u_n|$.

Remark: The converse of this, where the $f_n$ are allowed to be in $\mathcal{L}^1(\mathbb{R})$ is 'Lebesgue Dominated Convergence' which we will get to.

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