PROBLEM SET 2 FOR 18.102, SPRING 2017
BRIEF SOLUTIONS.

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1. Problem 2.1

Show that if $K \in \mathcal{C}([0,1]^2)$ is a continuous function of two variables, then the integral operator

$$Au(x) = \int_0^1 K(x,y)u(y)dy \tag{1}$$

(given by a Riemann integral) is a bounded operator, i.e. a continous linear map, from $\mathcal{C}([0,1])$ to itself with respect to the supremum norm.

Solution: A continuous function on a compact set, such as $[0,1]^2$, is uniformly continuous, so given ϵ there exists $\delta > 0$ such that

$$|x - x'| + |y - y'| < \delta \Longrightarrow |K(x,y) - K(x',y')| < \epsilon. \tag{2}$$

If $u \in \mathcal{C}([0,1])$ is fixed then the integrand in (1) is continuous for each fixed $x \in [0,1]$ so $Au : [0,1] \longrightarrow \mathbb{C}$ is well-defined as a Riemann integral. Moreover

$$|Au(x) - Au(x')| = |\int_0^1 (K(x,y) - K(x',y)u(y)dy| \le \sup_y |K(x,y) - K(x',y)| \sup |u|$$

by standard properties of the Riemann integral. Using (2) it follows that

$$|x - x'| < \delta \Longrightarrow |Au(x) - Au(x')| \le \sup |u| \epsilon$$

so Au is continous on $[0,1]$ and (1) defines a map

$$A : \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1]). \tag{3}$$

The linearity of this map follows from the linearity of the Riemann integral and

$$|u(x)| \le \sup |K| \sup |u| \ \forall \ x \in [0,1] \tag{4}$$

shows that it is bounded, i.e. continuous.

2. Problem 2.2

(1) Show that the 'Dirac delta function at $y \in [0,1]$' is well-defined as a continuous linear map

$$\delta_y : \mathcal{C}([0,1]) \ni u \longmapsto u(y) \in \mathbb{C} \tag{1}$$

with respect to the supremum norm on $\mathcal{C}([0,1])$.

(2) Show that δ_y is *not* continuous with respect to the L^1 norm $\int_0^1 |u|$.

Solution

(1) The map (1) is clearly linear since

(2) $$\delta_y(c_1u_1 + c_2u_2) = (c_1u_1 + c_2u_2)(y) = c_1\delta_y(u_1) + c_2\delta_u(u_2)$$

and it is bounded

$$|\delta_u(u)| \le \sup|u|$$

so continuous.

(2) It suffices to show that there is a sequence u_n in $\mathcal{C}([0,1])$ such that $\delta_y(u_n) = 1$ but $\|u_n\|_{L^1} \to 0$ since then a bound

$$|\delta_y(u)| \le C\|u\|_{L^1}$$

is impossible. Such a sequence is given by the 'triangle functions'

$$u_n(x) = \begin{cases} 0 & x \le y - 1/n \\ 1 - n|y-x| & y - 1/n \le x \le y + 1/n \\ 0 & x \ge y + 1/n \end{cases}$$

restricted to $[0,1]$. Indeed u_n is continuous at each point and

(3) $$u_n(y) = 1, \ \int_0^1 u_n(y) \le 1/n.$$

3. Problem 2.3

Suppose $a < b$ are real, show that the step function

(1) $$\chi_{(a,b]} = \begin{cases} 0 & \text{if } x \le a \\ 1 & \text{if } a < x \le b \\ 0 & \text{if } b < x \end{cases}$$

is an element of $\mathcal{L}^1(\mathbb{R})$. [Note that the definition requires you to find an absolutely summable series of continuous functions with appropriate properties.]

Addendum: Oops, Ethan points out to me that I should read the question before trying to answer it, and he has a point! The characteristic function is for $(a,b]$ not $[a,b]$ for which I give the proof below (it is in the notes anyway). So, to get something closer to full marks I would have done one of two things

(1) Noted that in class we showed that a point is a set of measure zero. So the construction below gives an absolutely summable series of continuous functions of compact support such that the partial sums converge $f_n(x) \longrightarrow \chi_{(a,b]}$ almost everywhere. From a Proposition in class or the notes this implies $\chi_{(a,b]} \in \mathcal{L}^1(\mathbb{R})$.

(2) I could 'shift the left leg a little' defining, for n large enough

(2) $$f_n = \begin{cases} 0 & x \le a \\ n(x-a) & a < x \le a + 1/n \\ 1 & a + 1/n < x < b \\ 1 - n(x-b) \le b \le x \le b + 1/n & \\ 0 & x \ge b + 1/n. \end{cases}$$

Then a similar argument – breaking the difference $f_n - f_{n-1}$ into the sum of a positive and a negative piece supported near a and b (or just computing the integral of the absolute value directly) proves that this comes from an absolutely summable series and it converges to $\chi_{(a,b]}$ everywhere.

Solution. Define a sequence of contuous functions $f_n \in \mathcal{C}_c(\mathbb{R})$ much as above,

$$f_n(x) = \begin{cases} 0 & x < a - 1/n \\ 1 - n(a-x) & a - 1/n \le x < a \\ 1 & a \le x < b \\ 1 - n(x-b) \le b \le x \le b + 1/n & \\ 0 & x \ge b + 1/n. \end{cases} \tag{3}$$

Thus $f_n = \chi_{[a,b]}$ on $[a,b]$ and at all other points $f_n(x) \to 0$, so $f_n(x) \to \chi_{(a,b]}$ as $n \to \infty$ for all $x \in \mathbb{R}$. Morever

$$\int f_n \le 2 + b - a$$

since it is non-negative and bounded above by $\chi_{[a-1,b+1]}$. Define the terms of the series for which the f_n are the partial sums

$$u_1 = f_1,\ u_n = f_n - f_{n-1},\ n > 1 \tag{4}$$

as usual. Then $u_n \in \mathcal{C}_c(\mathbb{R})$ and the u_n are non-positive, for $n > 1$. Thus

$$\sum_n \int |u_n| = \int f_1 - \sum_{n>1} (f_n - f_{n-1}) \le 2 \int f_1 < \infty. \tag{5}$$

So this is an absolutely summable approximating series and hence $\chi_{[a,b]} \in \mathcal{L}^1(\mathbb{R})$. You can easily compute the integrals of course.

4. Problem 2.4

A subset $E \subset \mathbb{R}$ is said to be *of measure zero* if there exists an absolutely summable sequence $f_n \in \mathcal{C}_c(\mathbb{R})$ (so $\sum\limits_n \int |f_n| < \infty$) such that

$$E \subset \{x \in \mathbb{R}; \sum_n |f_n(x)| = +\infty\}. \tag{1}$$

Show that if E is of measure zero and $\epsilon > 0$ is given then there exists $f_n \in \mathcal{C}_c(\mathbb{R})$ satisfying (1) and in addition

$$\sum_n \int |f_n| < \epsilon. \tag{2}$$

Solution: Take such a series f_n with $\sum\limits_n \int |f_n(x)| = C$ and replace it by $\frac{\epsilon}{C+1} f_n$ or choose N so large that

$$\sum_{n \le N} \int |f_n(x)| > C - \epsilon$$

and consider the new series $u_n = f_{n+N}$ which has

$$\sum_n \int |u_n(x)| < \epsilon \tag{3}$$

and for which $\sum\limits_n |u_n(x)| C$ diverges wherever $\sum\limits_n |f_n(x)|$ diverges, so in particular on E.

5. Problem 2.5

Using the previous problem (or otherwise ...) show that a countable union of sets of measure zero is a set of measure zero.

Solution: Let E_j be the countable collection of sets of measure zero. Choose a summable series $f_{j,n}$ for each j which satisfies

$$\sum_n \int |f_{j,n}| < 2^{-j}, \ \sum_n |f_{j,n}(x)| = \infty \text{ for } x \in E_j. \tag{1}$$

Now, rearrange the countably many terms $f_{j,n}$ into a sequence $g_k \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ – using for instance a bijection from $\mathbb{N}^2$ to $\mathbb{N}$ applied to the indices. Then, standard rearrangement properties of absolutely summable series (look at Rudin if you need to, we will use this next week) show that

$$\begin{gathered}\sum_k \int |g_k| = \sum_j \sum_n \int |f_{j,n}| < \sum_j 2^{-j} = 2, \\ \sum_k |g_k(x)| \ge \sum_n |f_{j,n}(x)| = \infty \ \forall \ x \in E_j, \ \forall \ j.\end{gathered} \tag{2}$$

Thus $E = \sum\limits_j E_j$ has measure zero.

6. Problem 2.6 – Extra

Let's generalize the theorem about $\mathcal{B}(V, W)$ given last week to bilinear maps – this may seem hard but just take it step by step!

(1) Check that if U and V are normed spaces then $U \times V$ (the linear space of all pairs (u, v) where $u \in U$ and $v \in V$) is a normed space where addition and scalar multiplication is 'componentwise' and the norm is the sum

$$\|(u, v)\|_{U\times V} = \|u\|_U + \|v\|_V. \tag{1}$$

(2) Show that $U \times V$ is a Banach space if both U and V are Banach spaces.

(3) Consider three normed spaces U, V and W. Let

$$B : U \times V \longrightarrow W \tag{2}$$

be a *bilinear* map. This means that

$$\begin{gathered}B(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 B(u_1, v) + \lambda_2 B(u_2, v), \\ B(u, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 B(u, v_1) + \lambda_2 B(u, v_2)\end{gathered}$$

for all u, u_1, $u_2 \in U$, v, v_1, $v_2 \in V$ and λ_1, $\lambda_2 \in \mathbb{C}$. Show that B is continuous if and only if it satisfies

$$\|B(u, v)\|_W \le C\|u\|_U \|v\|_V \ \forall \ u \in U, \ v \in V. \tag{3}$$

(4) Let $\mathcal{M}(U, V; W)$ be the space of all such continuous bilinear maps. Show that this is a linear space and that

$$\|B\| = \sup_{\|u\|=1, \|v\|=1} \|B(u, v)\|_W \tag{4}$$

is a norm.

(5) Show that $\mathcal{M}(U, V; W)$ is a Banach space if W is a Banach space.

Solution: Third last part only and brief. An estimate (3) implies continuity, since if $u_n \to u$ and $v_n \to v$ then

(5)
$$\|B(u_n,v_n)-B(u,v)\|_W \le \|B(u_n,v_n)-B(u_n,v)\|_W + \|B(u_n,v)-B(u,v)\|_W \\ \le C(\|u_n\|\|v_n-v\| + \|u_n-u\|\|u\|) \to 0.$$

Conversely, if B is continuous then $B^{-1}(\{\|w\|<1\}) \ni 0$ is open, so

$$\|u\|+\|v\|<\epsilon \Longrightarrow \|B(u,v)\| \le 1$$

for some $\epsilon > 0$. If u and v are non-zero then

$$\|\epsilon/4(\frac{u}{\|u\|},\frac{v}{\|v\|}) < \epsilon \Longrightarrow \|B(u,v)\| \le \frac{4}{\epsilon}\|u\|\|v\|$$

using the bilinearity. If either vanishe then $B(u,v)$ vanishes so (3) is equivalent to continuity.

Everything else is very similar to the linear case.

7. Problem 2.7 – Extra

Consider the space $\mathcal{C}_c(\mathbb{R}^n)$ of continuous functions $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ which vanish outside a compact set, i.e. in $|x| > R$ for some R (depending on u). Check (quickly) that this is a linear space.

Show that if $y \in \mathbb{R}^{n-1}$ and $u \in \mathcal{C}_c(\mathbb{R}^n)$ then

$$U_y : \mathbb{R} \ni t \longmapsto u(y,t) \in \mathbb{C} \tag{1}$$

defines an element $U_y \in \mathcal{C}_c(\mathbb{R})$. Fix an overall 'rectangle' $[-R,R]^n$ and only consider functions $\mathcal{C}_{c,R}(\mathbb{R})$ vanishing outside this rectangle. With this restriction on supports show for each R that $\mathbb{R}^{n-1} \ni y \longmapsto U_y$ is a continuous map into $\mathcal{C}_{c,R}(\mathbb{R})$ *with respect to the supremum norm* which vanishes for $|y| > R$, i.e. has compact support. Conclude that 'integration in the last variable' gives a continuous linear map (with respect to supremum norms)

$$\mathcal{C}_{c,R}(\mathbb{R}^n) \ni u \longrightarrow v \in \mathcal{C}_{c,R}(\mathbb{R}^{n-1}),\ v(y) = \int U_y. \tag{2}$$

By iterating this statement show that the iterated Riemann integral is well defined

$$\int : \mathcal{C}_{c,R}(\mathbb{R}^n) \longrightarrow \mathbb{C} \tag{3}$$

and that $\int |u|$ is a norm which is independent of R – so defined on the whole of $\mathcal{C}_c(\mathbb{R}^n)$.