This is the last problem set. You may freely use the fact that the Fourier transform extends from an isomorphism on Schwartz space $\mathcal{S}(\mathbb{R})$, which is a dense subspace of $L^2(\mathbb{R})$, to an isomorphism of $L^2(\mathbb{R})$.

Define $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$ by the condition

$$u \in H^2(\mathbb{R}) \iff u \in L^2(\mathbb{R}) \text{ and } \xi^2 \hat{u}(\xi) \in L^2(\mathbb{R}).$$

Remark: This is a Sobolev space. You can do the same thing for any integer, or even non-negative real number, $s$ in place of 2 above by setting

$$H^s(\mathbb{R}) = \{ u \in L^2(\mathbb{R}) : |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}) \}$$

the point being that this space really is well-defined. It is the space of $L^2$ functions ‘with derivatives up to order $s$ in $L^2(\mathbb{R})$.’ Amuse yourself by showing that this too is a Hilbert space. You could even try to prove the Sobolev embedding theorem, that $H^s(\mathbb{R}) \subset C^\infty(\mathbb{R})$ if and only if $s > \frac{1}{2}$.

P10.1 Show that $H^2(\mathbb{R})$ is a Hilbert space with the norm

$$\|u\|_H^2 = (\|u\|_{L^2}^2 + \|D^2u\|_{L^2}^2)^{\frac{1}{2}}$$

where $\overline{D^2}u(\xi) = \xi^2 \hat{u}(\xi)$.

Hint: For a Cauchy sequence in $H^2(\mathbb{R})$ both $u_n \rightarrow u$ and $D^2u_n \rightarrow v$ converge in $L^2$ so you only need show that $\hat{v} = \xi^2 \hat{u}$ and this follows from Monotonicity/LDC.

P10.2 Show that if $u \in H^2(\mathbb{R})$ then $u$ ‘is’ continuously differentiable (meaning, since we value precision, has a representative which is a continuously differentiable function on $\mathbb{R}$).

Hint: Since $\hat{u}$ and $\xi^2 \hat{u}$ are bounded it follows that $\hat{u}$ and $\xi \hat{u}$ are in $L^1$ by Cauchy-Schwartz, so they have bounded continuous inverse FTs, $u$, $v$. Apply LDC to the integral for the IFT giving $u$ to see that the difference quotient converges to $v$.

P10.3 Show that $D^2 + 1$ is an isomorphism from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$

Hint: The inverse is $\times (1 + \xi^2)^{-1}$ on the FT side.

P10.4 Show that $(D^2 + 1)^{-1}$ is a self-adjoint operator on $L^2(\mathbb{R})$ and that it has spectrum precisely the interval $[0,1]$.

P10.5 Prove that if $V \geq 0$ is a bounded continuous function on $\mathbb{R}$ then

$$(2) \quad (D^2 + 1 + V) : H^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$$

is a topological isomorphism, i.e. a bijection with a bounded inverse.

Hint: Recall the discussion of the Dirichlet problem. If $A^2 = (1 + D^2)^{-1}$ is given by the functional calculus then $\text{Id} + AVA$ is invertible on $L^2$ and is of the form $\text{Id} + AE A$ with $E$ bounded; the inverse to (2) is $A(\text{Id} + AVA)^{-1} A$ and you need to check that this maps $L^2$ to $H^2$ and is indeed the inverse.
P10.6 – extra Show that if $f \in \mathcal{C}_c(\mathbb{R})$ is a continuous function of compact support then (under the same hypotheses as above)

\begin{equation}
\frac{d^2 u}{dx^2} + u + Vu = f
\end{equation}

has a unique twice continuously differentiable solution which is in $L^2(\mathbb{R})$.

Hint: Show by integration (making sure of the behaviour at infinity) that the equation has a unique solution $u$ which is $C^2$ and in $L^2$ for each $f \in \mathcal{C}_c(\mathbb{R})$.

P10.7 – extra Show that this solution to (3) defines a self-adjoint operator on $L^2(\mathbb{R})$ which has spectrum contained in $[0, 1]$.

Hint: Then show that the solution in the previous question is is actually $(D^2 + 1)^{-1} f$ and using the $L^2$ inverse and a regularity argument.