PROBLEM SET 10 FOR 18.102/18.1021
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RICHARD MELROSE

This is the last problem set. You may freely use the fact that the Fourier transform extends from an isomorphism on Schwartz space $\mathcal{S}(\mathbb{R})$, which is a dense subspace of $L^2(\mathbb{R})$, to an isomorphism of $L^2(\mathbb{R})$.

Define $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$ by the condition

$$u \in H^2(\mathbb{R}) \Longleftrightarrow u \in L^2(\mathbb{R}) \text{ and } \xi^2 \hat{u}(\xi) \in L^2(\mathbb{R}).$$

Remark: This is a *Sobolev space*. You can do the same thing for any integer, or even non-negative real number, s in place of 2 above by setting

$$H^s(\mathbb{R}) = \{u \in L^2(\mathbb{R}); |\xi|^s \hat{u} \in L^2(\mathbb{R})\} \tag{1}$$

the point being that this space really is well-defined. It is the space of L^2 functions 'with derivatives up to order s in $L^2(\mathbb{R})$.' Amuse yourself by showing that this too is a Hilbert space. You could even try to prove the Sobolev embedding theorem, that $H^s(\mathbb{R}) \subset \mathcal{C}_\infty(\mathbb{R})$ if and only if $s > \frac{1}{2}$.

P10.1 Show that $H^2(\mathbb{R})$ is a Hilbert space with the norm $(\|u\|_{L^2}^2 + \|D^2u\|_{L^2}^2)^{\frac{1}{2}}$ where $\widehat{D^2u}(\xi) = \xi^2 \hat{u}(\xi)$.

Hint: For a Cauchy sequence in $H^2(\mathbb{R})$ both $u_n \to u$ and $D^2 u_n \to v$ converge in L^2 so you only need show that $\hat{v} = \xi^2 \hat{u}$ and this follows from Monotonicity/LDC.

P10.2 Show that if $u \in H^2(\mathbb{R})$ then u 'is' continuously differentiable (meaning, since we value precision, has a representative which is a continuously differentiable function on $\mathbb{R}$).

Hint: Since $\hat{u}$ and $\xi^2 \hat{u} \in L^2$ it follows that $\hat{u}$ and $\xi \hat{u} \in L^1$ by Cauchy-Schwartz, so they have bounded continuous inverse FTs, u, v. Apply LDC to the integral for the IFT giving u to see that the difference quotient converges to v.

P10.3 Show that $D^2 + 1$ is an isomorphism from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$

Hint: The inverse is $\times(1 + \xi^2)^{-1}$ on the FT side.

P10.4 Show that $(D^2 + 1)^{-1}$ is a self-adjoint operator on $L^2(\mathbb{R})$ and that it has spectrum precisely the interval $[0, 1]$.

P10.5 Prove that if $V \geq 0$ is a bounded continuous function on $\mathbb{R}$ then

$$(D^2 + 1 + V) : H^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \tag{2}$$

is a topological isomorphism, i.e. a bijection with a bounded inverse.

Hint: Recall the discussion of the Dirichlet problem. If $A^2 = (1 + D^2)^{-1}$ is given by the functional calculus then $\mathrm{Id} + AVA$ is invertible on L^2 and is of the form $\mathrm{Id} + AEA$ with E bounded; the inverse to (2) is $A(\mathrm{Id} + AVA)^{-1}A$ and you need to check that this maps L^2 to H^2 and is indeed the inverse.

P10.6 – extra Show that if $f \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ is a continous function of compact support then (under the same hypotheses as above)

$$-\frac{d^2u}{dx^2} + u + Vu = f \tag{3}$$

has a unique twice continuously differentiable solution which is in $L^2(\mathbb{R})$.

Hint: Show by integration (making sure of the behaviour at infinity) that the equation has a unique solution u which is $\mathcal{C}^2$ and in L^2 for each $f \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$.

P10.7 – extra Show that this solution to (3) defines a self-adjoint operator on $L^2(\mathbb{R})$ which has spectrum contained in $[0, 1]$.

Hint: Then show that the solution in the previous question is is actually $(D^2 + 1)^{-1}f$ and using the L^2 inverse and a regularity argument.

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: rbm@math.mit.edu