18.102/1021 QUESTIONS FOR FINAL EXAM

No notes, books, reference material or communication equipment will be permitted!

Unless otherwise defined, H is an infinite-dimensional separable Hilbert space.

Problem 1

Let A be a compact, self-adjoint operator on a H which is positive in the sense that $\langle Au, u\rangle > 0$ for all $0 \neq u \in H$. Show that the range $R(A)$ is a dense subspace of H and that

$$\|v\|_A = \|u\|_H,\ v \in R(A),\ v = Au,\ u \in H$$

defines a norm on $R(A)$ with respect to which it is a Hilbert space.

Problem 2

Let a be a continuous function on the square $[0, 2\pi]^2$. Show that $[0, 2\pi] \ni x \longmapsto a(x, \cdot) \in \mathcal{C}^0([0, 2\pi])$ is a continuous map into the continuous functions with supremum norm. Using this, or otherwise, show that the Fourier coefficients with respect to the second variable

$$c_k(x) = \frac{1}{2\pi}\int_0^{2\pi} a(x,t)e^{ikt}dt \tag{1}$$

are continuous functions on $[0, 2\pi]$.

Problem 3

Let H_i, $i = 1, 2$ be two Hilbert spaces with inner products $(\cdot,\cdot)_i$ and suppose that $I : H_1 \longrightarrow H_2$ is a continuous linear map between them.

(1) Show that there is a continuous linear map $Q : H_2 \longrightarrow H_1$ such that $(u, If)_2 = (Qu, f)_1\ \forall\ f \in H_1$.
(2) Show that as a map from H_1 to itself, $Q \circ I$ is bounded and self-adjoint
(3) Show that the spectrum of $Q \circ I$ is contained in $[0, \|I\|^2]$.

Problem 4

Let $u_n : [0, 2\pi] \longrightarrow \mathbb{C}$ be a sequence of continuously differentiable functions which is uniformly bounded, with bounded derivatives i.e. $\sup_n \sup_{x\in[0,2\pi]} |u_n(x)| < \infty$ and $\sup_n \sup_{x\in[0,2\pi]} |u_n'(x)| < \infty$. Show that u_n has a subsequence which converges in $L^2([0, 2\pi])$.

Problem 5

Consider the subspace $H \subset \mathcal{C}[0, 2\pi]$ consisting of those continuous functions on $[0, 2\pi]$ which satisfy

$$u(x) = \int_0^x U,\ \forall\ x \in [0, 2\pi] \tag{2}$$

for some $U \in L^2(0, 2\pi)$ (depending on u of course).

(1) Show that the function U is determined by u (given that it exists).
(2) Show that

$$\|u\|_H^2 = \int_{(0,2\pi)} |U|^2 \tag{3}$$

turns H into a Hilbert space.
(3) If $\int_0^{2\pi} U = 0$, determine the Fourier series of u in terms of that of U.

Problem 6

Consider the space of those complex-valued functions on $[0, 1]$ for which there is a constant C (depending on the function) such that

$$|u(x) - u(y)| \le C|x - y|^{\frac{1}{2}} \ \forall\ x, y \in [0, 1]. \tag{4}$$

Show that this is a Banach space with norm

$$\|u\|_{\frac{1}{2}} = \sup_{[0,1]} |u(x)| + \inf_{(4) \text{ holds}} C. \tag{5}$$

Problem 7

Let $A_j \subset \mathbb{R}$ be a sequence of subsets with the property that the characteristic function, χ_j of A_j, is integrable for each j. Show that the characteristic function of $\mathbb{R} \setminus A$, where $A = \bigcup_j A_j$ is locally integrable.

Problem 8

A bounded operator $A \in \mathcal{B}(H)$ on a separable Hilbert space is a Hilbert-Schmidt operator if for some orthonormal basis $\{e_i\}$

$$\sum_i \|Ae_i\|^2 < \infty. \tag{6}$$

Show that if A and B are Hilbert-Schmidt operators then the sum

$$\mathrm{Tr}(AB) = \sum_i \langle Ae_i, B^* e_i \rangle \tag{7}$$

exists and is independent of the orthonormal basis used to define it.

Problem 9

Let B_n be a sequence of bounded linear operators on a Hilbert space H such that for each u and $v \in H$ the sequence $(B_n u, v)$ converges in $\mathbb{C}$. Show that there is a uniquely defined bounded operator B on H such that

$$(Bu, v) = \lim_{n\to\infty} (B_n u, v) \ \forall\ u, v \in H.$$

Problem 10

Suppose $P \subset H$ is a closed linear subspace of a Hilbert space, with $\pi_P : H \longrightarrow P$ the orthogonal projection onto P. If H is separable and A is a compact self-adjoint operator on H, show that there is a complete orthonormal basis of H each element of which satisfies $\pi_P A \pi_P e_i = t_i e_i$ for some $t_i \in \mathbb{R}$.

Problem 11

Let $e_j = c_j C^j e^{-x^2/2}$, $c_j > 0$, where $j = 1, 2, \dots,$ and $C = -\frac{d}{dx} + x$ is the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. You may assume completeness in $L^2(\mathbb{R})$ and use the facts established in class that $-\frac{d^2 e_j}{dx^2} + x^2 e_j = (2j+1)e_j$, that $c_j = 2^{-j/2}(j!)^{-\frac{1}{2}}\pi^{-\frac{1}{4}}$ and that $e_j = p_j(x)e_0$ for a polynomial of degree j. Compute Ce_j and Ae_j in terms of the basis and hence arrive at formulae for de_j/dx and xe_j. Conclude that if

$$H^1_{\text{iso}} = \{u \in L^2(\mathbb{R}); \sum_{j\ge 1} j|(u, e_j)|^2 < \infty\} \tag{8}$$

then there are uniquely defined operators x and $D : H^1_{\text{iso}} \longrightarrow L^2(\mathbb{R})$ defined correctly on the basis, so $De_j = \frac{de_j}{dx}$ for each j.

Problem 12

Suppose that $f \in \mathcal{L}^2(0, 2\pi)$ is such that there exists a function $v \in L^2(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} f\phi' = \int v\hat{\phi} \ \forall\ \phi \in \mathcal{S}(\mathbb{R})$$

where $\hat{\phi}$ is the Fourier transform of ϕ. Using if desired the density of $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$ show that there is a bounded continuous function g such that $f = g$ a.e.

Hint: There was really a typo here in that I meant $f \in L^2(\mathbb{R})$, the question is okay as it is but potentially confusing. Anyway, you can take $\hat{\phi} = \psi$ then $\phi(x) = \frac{1}{2\pi}(\hat{\psi})(-x)$ so

$$\int_{\mathbb{R}} f(-x)i\hat{\xi\psi}(x) = \int v\psi \ \forall \psi \in \mathcal{S}(\mathbb{R}).$$

So

$$\int_{\mathbb{R}} f(\hat{-x})i\xi\psi(\xi) = \int v\psi$$

from which it follows that $\xi\hat{f} \in L^2(\mathbb{R})$. This means $\hat{f} \in L^1(\mathbb{R})$ so f is continuous.

Problem 13

Suppose $A \in \mathcal{B}(H)$ has closed range. Show that A has a generalized inverse, $B \in \mathcal{B}(H)$ such that

$$AB = \Pi_R,\ BA = \operatorname{Id} - \Pi_N$$

where Π_N and Π_R are the orthonormal projections onto the null space and range of A respectively.

Problem 14

For $u \in L^2(0, 1)$ show that

$$Iu(x) = \int_0^x u(t)dt,\ x \in (0, 1)$$

is a bounded linear operator on $L^2(0, 1)$. If $V \in \mathcal{C}([0, 1])$, is real-valued and $V \ge 0$, show that there is a bounded linear operator B on $L^2(0, 1)$ such that

$$B^2 u = u + I^* M_V I u \ \forall\ u \in L^2(0, 1) \tag{9}$$

where M_V denotes multiplication by V.

Problem 15

Let $A \in \mathcal{B}(H)$ be such that

$$\sup \sum_i |\langle Ae_i, f_i\rangle| < \infty \tag{10}$$

where the supremum is over orthonormal sequence e_i and f_j. Use the polar decomposition to show that $A = B_1 B_2$ where the $B_k \in \mathcal{B}(H)$ are Hilbert-Schmidt operators, i.e. $\sum\limits_i \|B_k e_i\|^2 < \infty$, $k = 1, 2$.

Problem 16

Suppose $f \in L^1(\mathbb{R})$ and the Fourier transform $\hat{f} \in L^2(\mathbb{R})$ show that $f \in L^2(\mathbb{R})$.

Hint: For $f \in L^1(\mathbb{R})$, $\hat{f}$ is defined by a Lebesgue integral. Use approximation (for instance the definition of L^1) to show that for all $\phi \in \mathcal{S}(\mathbb{R})$

$$\int \hat{f}\phi = \int f\hat{\phi}.$$

We showed that there is an L^2 function g with $\hat{g} = \hat{f}$ (defined by continuous extension) with this same property. Use this to show that $g = f$ a.e.

Problem 17

Show that the functions $c_k \cos(kx)$, $k = 0, 1, \dots,$ for appropriate constants c_k, form an orthonormal basis of $L^2(0, \pi)$. Using these, or otherwise, show that for each function $f \in \mathcal{C}([0, \pi])$ the Neumann problem

$$-\frac{d^2u}{dx^2} + u = f \text{ on } (0, \pi),\ \frac{du}{dx}(0) = \frac{du}{dx}(\pi) = 0 \tag{11}$$

has a unique twice continuously differentiable solution and the resulting map $f \longmapsto u$ extends by continuity to a compact self-adjoint operator on $L^2(0, \pi)$.

Problem 18

Let $a \in \mathcal{C}([0, 1])$ be a real-valued continouous function. Show that multiplication by a defines a self-adjoint operator A_a on $L^2(0, 1)$ which has spectrum exactly the range $a([0, 1]) \subset \mathbb{R}$. If $h \in \mathcal{C}(\mathbb{R})$ is real-valued, describe the operator $h(A_a)$ given by the functional calculus.

Problem 19

Explain carefully (but briefly ..) why the Schwartz space $\mathcal{S}(\mathbb{R})$ gives a well-defined subspace of $L^2(\mathbb{R})$. Show that if $u \in L^2(\mathbb{R})$ has a weak derivative in $\mathcal{S}(\mathbb{R})$ in the sense that there exists $v \in \mathcal{S}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} u\frac{d\phi}{dx} = -\int_{\mathbb{R}} v\phi \ \forall\ \phi \in \mathcal{S}(\mathbb{R})$$

then $u \in \mathcal{S}(\mathbb{R})$.

Hint: You can take $\phi = \hat{\psi}$ where $\psi \in \mathcal{S}(\mathbb{R})$. Using properties of the Fourier transform it follows that $\hat{v}(\xi) - i\xi\hat{u}(\xi)$ 'pairs' to zero with all Schwartz functions, so vanishes a.e. Hence $\xi\hat{u} = i\hat{v}$ 'is' Schwartz. Now $\hat{u}$ is an L^2 function and the only way $w(x)/x$ can be in L^2, where w is Schwartz, is if $w(0) = 0$ so it follows that $\hat{u}$ is Schwartz and hence so is u.