No notes, books, reference material or communication equipment will be permitted!

Unless otherwise defined, $H$ is an infinite-dimensional separable Hilbert space.

**Problem 1**

Let $A$ be a compact, self-adjoint operator on a $H$ which is positive in the sense that $\langle Au, u \rangle > 0$ for all $0 \neq u \in H$. Show that the range $R(A)$ is a dense subspace of $H$ and that

$$\|v\|_A = \|u\|_H, \ v \in R(A), \ v = Au, \ u \in H$$

defines a norm on $R(A)$ with respect to which it is a Hilbert space.

**Problem 2**

Let $a$ be a continuous function on the square $[0, 2\pi]^2$. Show that $[0, 2\pi] \ni x \mapsto a(x, \cdot) \in C^0([0, 2\pi])$ is a continuous map into the continuous functions with supremum norm. Using this, or otherwise, show that the Fourier coefficients with respect to the second variable

$$c_h(x) = \frac{1}{2\pi} \int_0^{2\pi} a(x, t) e^{ikt} dt$$

are continuous functions on $[0, 2\pi]$.

**Problem 3**

Let $H_i, \ i = 1, 2$ be two Hilbert spaces with inner products $(\cdot, \cdot)_i$ and suppose that $I : H_1 \rightarrow H_2$ is a continuous linear map between them.

1. Show that there is a continuous linear map $Q : H_2 \rightarrow H_1$ such that $(u, If)_2 = (Qu, f)_1 \ \forall \ f \in H_1$.
2. Show that as a map from $H_1$ to itself, $Q \circ I$ is bounded and self-adjoint.
3. Show that the spectrum of $Q \circ I$ is contained in $[0, \|I\|^2]$.

**Problem 4**

Let $u_n : [0, 2\pi] \rightarrow \mathbb{C}$ be a sequence of continuously differentiable functions which is uniformly bounded, with bounded derivatives i.e. $\sup_n \sup_{x \in [0, 2\pi]} |u_n(x)| < \infty$ and $\sup_n \sup_{x \in [0, 2\pi]} |u'_n(x)| < \infty$. Show that $u_n$ has a subsequence which converges in $L^2([0, 2\pi])$.

**Problem 5**

Consider the subspace $H \subset C[0, 2\pi]$ consisting of those continuous functions on $[0, 2\pi]$ which satisfy

$$u(x) = \int_0^x U, \ \forall \ x \in [0, 2\pi]$$

for some $U \in L^2(0, 2\pi)$ (depending on $u$ of course).
(1) Show that the function $U$ is determined by $u$ (given that it exists).
(2) Show that
\[ \|u\|_H^2 = \int_{(0,2\pi)} |U|^2 \]
turns $H$ into a Hilbert space.
(3) If $\int_0^{2\pi} U = 0$, determine the Fourier series of $u$ in terms of that of $U$.

**Problem 6**

Consider the space of those complex-valued functions on $[0,1]$ for which there is a constant $C$ (depending on the function) such that
\[ |u(x) - u(y)| \leq C|x - y|^{\frac{1}{2}} \quad \forall x, y \in [0,1]. \]
Show that this is a Banach space with norm
\[ \|u\|_J = \sup_{[0,1]} |u(x)| + \inf_{(4) \text{ holds}} C. \]

**Problem 7**

Let $A_j \subset \mathbb{R}$ be a sequence of subsets with the property that the characteristic function, $\chi_j$ of $A_j$, is integrable for each $j$. Show that the characteristic function of $\mathbb{R} \setminus A$, where $A = \bigcup_j A_j$ is locally integrable.

**Problem 8**

A bounded operator $A \in B(H)$ on a separable Hilbert space is a Hilbert-Schmidt operator if for some orthonormal basis $\{e_i\}$
\[ \sum_i \|Ae_i\|^2 < \infty. \]
Show that if $A$ and $B$ are Hilbert-Schmidt operators then the sum
\[ \text{Tr}(AB) = \sum_i \langle Ae_i, B^* e_i \rangle \]
exists and is independent of the orthonormal basis used to define it.

**Problem 9**

Let $B_n$ be a sequence of bounded linear operators on a Hilbert space $H$ such that for each $u$ and $v \in H$ the sequence $(B_n u, v)$ converges in $C$. Show that there is a uniquely defined bounded operator $B$ on $H$ such that
\[ (Bu, v) = \lim_{n \to \infty} (B_n u, v) \quad \forall u, v \in H. \]

**Problem 10**

Suppose $P \subset H$ is a closed linear subspace of a Hilbert space, with $\pi_P : H \longrightarrow P$ the orthogonal projection onto $P$. If $H$ is separable and $A$ is a compact self-adjoint operator on $H$, show that there is a complete orthonormal basis of $H$ each element of which satisfies $\pi_P A \pi_P e_i = t_i e_i$ for some $t_i \in \mathbb{R}$. 
Problem 11

Let \( e_j = c_j C e^{-x^2/2} \), \( c_j > 0 \), where \( j = 1, 2, \ldots \), and \( C = -\frac{d}{dx} + x \) is the creation operator, be the orthonormal basis of \( L^2(\mathbb{R}) \) consisting of the eigenfunctions of the harmonic oscillator discussed in class. You may assume completeness in \( L^2(\mathbb{R}) \) and use the facts established in class that

\[
-\frac{d^2}{dx^2} e_j + x^2 e_j = (2j + 1) e_j,
\]

that \( c_j = 2^{-j/2}(\frac{j}{2})! - \frac{1}{4} \pi^{-1/4} \) and that \( e_j = p_j(x)e_0 \) for a polynomial of degree \( j \). Compute \( Ce_j \) and \( Ae_j \) in terms of the basis and hence arrive at formulæ for \( de_j/dx \) and \( xe_j \).

Conclude that if

\[
H_{\text{iso}}^1 = \{ u \in L^2(\mathbb{R}); \sum_{j \geq 1} |j(u, e_j)|^2 < \infty \}
\]

then there are uniquely defined operators \( x \) and \( D : H_{\text{iso}}^1 \rightarrow L^2(\mathbb{R}) \) defined correctly on the basis, so \( De_j = \frac{de_j}{dx} \) for each \( j \).

Problem 12

Suppose that \( f \in L^2(0, 2\pi) \) is such that there exists a function \( v \in L^2(\mathbb{R}) \) satisfying

\[
\int_{\mathbb{R}} f \hat{\phi} = \int_{\mathbb{R}} v \hat{\phi} \quad \forall \, \phi \in S(\mathbb{R})
\]

where \( \hat{\phi} \) is the Fourier transform of \( \phi \). Using if desired the density of \( S(\mathbb{R}) \) in \( L^2(\mathbb{R}) \) show that there is a bounded continuous function \( g \) such that \( f = g \) a.e.

Hint: There was really a typo here in that I meant \( f \in L^2(\mathbb{R}) \), the question is okay as it is but potentially confusing. Anyway, you can take \( \hat{\phi} = \psi \) then \( \phi(x) = \frac{1}{2\pi}(\hat{\psi})(-x) \) so

\[
\int_{\mathbb{R}} f(-x) i\xi \psi(x) = \int_{\mathbb{R}} v \psi \forall \, \psi \in S(\mathbb{R}).
\]

So

\[
\int_{\mathbb{R}} f(-x) i\xi \psi(\xi) = \int_{\mathbb{R}} v \psi
\]

from which it follows that \( \xi f \in L^2(\mathbb{R}) \). This means \( \hat{f} \in L^1(\mathbb{R}) \) so \( f \) is continuous.

Problem 13

Suppose \( A \in \mathcal{B}(H) \) has closed range. Show that \( A \) has a generalized inverse, \( B \in \mathcal{B}(H) \) such that

\[
AB = \Pi_R, \quad BA = \text{Id} - \Pi_N
\]

where \( \Pi_N \) and \( \Pi_R \) are the orthonormal projections onto the null space and range of \( A \) respectively.

Problem 14

For \( u \in L^2(0, 1) \) show that

\[
Iu(x) = \int_0^x u(t) dt, \quad x \in (0, 1)
\]

is a bounded linear operator on \( L^2(0, 1) \). If \( V \in C([0, 1]) \), is real-valued and \( V \geq 0 \), show that there is a bounded linear operator \( B \) on \( L^2(0, 1) \) such that

\[
B^2u = u + I^* M_V Iu \quad \forall \, u \in L^2(0, 1)
\]
where $M_V$ denotes multiplication by $V$.

**Problem 15**

Let $A \in \mathcal{B}(H)$ be such that

\[ \sup \sum |\langle Ae_i, f_j \rangle| < \infty \]

where the supremum is over orthonormal sequence $e_i$ and $f_j$. Use the polar decomposition to show that $A = B_1 B_2$ where the $B_k \in \mathcal{B}(H)$ are Hilbert-Schmidt operators, i.e. $\sum \|B_k e_i\|^2 < \infty$, $k = 1, 2$.

**Problem 16**

Suppose $f \in L^1(\mathbb{R})$ and the Fourier transform $\hat{f} \in L^2(\mathbb{R})$ show that $f \in L^2(\mathbb{R})$.

Hint: For $f \in L^1(\mathbb{R})$, $\hat{f}$ is defined by a Lebesgue integral. Use approximation (for instance the definition of $L^1$) to show that for all $\phi \in S(\mathbb{R})$

\[ \int \hat{f} \phi = \int f \hat{\phi}. \]

We showed that there is an $L^2$ function $g$ with $\hat{g} = \hat{f}$ (defined by continuous extension) with this same property. Use this to show that $g = f$ a.e.

**Problem 17**

Show that the functions $c_k \cos(ke^x)$, $k = 0, 1, \ldots$, for appropriate constants $c_k$, form an orthonormal basis of $L^2(0, \pi)$. Using these, or otherwise, show that for each function $f \in C([0, \pi])$ the Neumann problem

\[ -\frac{d^2u}{dx^2} + u = f \text{ on } (0, \pi), \quad \frac{du}{dx}(0) = \frac{du}{dx}(\pi) = 0 \]

has a unique twice continuously differentiable solution and the resulting map $f \mapsto u$ extends by continuity to a compact self-adjoint operator on $L^2(0, \pi)$.

**Problem 18**

Let $a \in C([0, 1])$ be a real-valued continuous function. Show that multiplication by $a$ defines a self-adjoint operator $A_a$ on $L^2(0, 1)$ which has spectrum exactly the range $a([0, 1]) \subset \mathbb{R}$. If $h \in C(\mathbb{R})$ is real-valued, describe the operator $h(A_a)$ given by the functional calculus.

**Problem 19**

Explain carefully (but briefly ..) why the Schwartz space $S(\mathbb{R})$ gives a well-defined subspace of $L^2(\mathbb{R})$. Show that if $u \in L^2(\mathbb{R})$ has a weak derivative in $S(\mathbb{R})$ in the sense that there exists $v \in S(\mathbb{R})$ such that

\[ \int u \frac{d\phi}{dx} = -\int v \phi \forall \phi \in S(\mathbb{R}) \]

then $u \in S(\mathbb{R})$.

Hint: You can take $\phi = \hat{\psi}$ where $\psi \in S(\mathbb{R})$. Using properties of the Fourier transform it follows that $\hat{\phi}(\xi) - i\xi \hat{\psi}(\xi)$ ‘pairs’ to zero with all Schwartz functions, so vanishes a.e. Hence $\xi \hat{u} = i \hat{v}$ ‘is’ Schwartz. Now $\hat{u}$ is an $L^2$ function and the only
way $w(x)/x$ can be in $L^2$, where $w$ is Schwartz, is if $w(0) = 0$ so it follows that $\hat{u}$ is Schwartz and hence so is $u$. 