CHAPTER 5

Problems

1. Problems for Chapter 1

Missing or badly referenced:-

Norm from seminorm.
Norm on quotient and completeness.
Completness of the completion.
Subspace of functions vanishing at infinity.
Completeness of the space of k times differentiable functions.
Direct proof of open mapping.

PROBLEM 1.0. Show from first principles that if V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) then for any set X the space of all maps

$$\mathcal{F}(X;V) = \{u : X \longrightarrow V\} \tag{5.1}$$

is a linear space over the same field, with 'pointwise operations' (which you should write down carefully).

PROBLEM 1.1. Show that if V is a vector space and $S \subset V$ is a subset which is closed under addition and scalar multiplication:

$$v_1, v_2 \in S,\ \lambda \in \mathbb{K} \Longrightarrow v_1 + v_2 \in S \text{ and } \lambda v_1 \in S \tag{5.2}$$

then S is a vector space as well with operations 'inherited from V' (and called, of course, a subspace of V).

PROBLEM 1.2. If $S \subset V$ be a linear subspace of a vector space show that the relation on V

$$v_1 \sim v_2 \Longleftrightarrow v_1 - v_2 \in S \tag{5.3}$$

is an equivalence relation and that the set of equivalence classes, denoted usually V/S, is a vector space in a natural way.

PROBLEM 1.3. In case you do not know it, go through the basic theory of finite-dimensional vector spaces. Define a vector space V to be finite-dimensional if there is an integer N such that any N elements of V are linearly dependent – if $v_i \in V$ for $i = 1, \dots N$, then there exist $a_i \in \mathbb{K}$, not *all* zero, such that

$$\sum_{i=1}^{N} a_i v_i = 0 \text{ in } V. \tag{5.4}$$

Call the smallest such integer the dimension of V and show that a finite dimensional vector space always has a basis, $e_i \in V$, $i = 1, \dots, \dim V$ such that any element of

V can be written uniquely as a linear combination

$$v = \sum_{i=1}^{\dim V} b_i e_i, \; b_i \in \mathbb{K}. \tag{5.5}$$

PROBLEM 1.5. Recall that a map between vector spaces $L : V \longrightarrow W$ is linear if $L(v_1 + v_2) = Lv_1 + Lv_2$ and $L\lambda v = \lambda Lv$ for all elements $v_1,\ v_2,\ v \in V$ and all scalars λ. Show that given two finite dimensional vector spaces V and W over the same field

(1) If $\dim V \le \dim W$ then there is an injective linear map $L : V \longrightarrow W$.
(2) If $\dim V \ge W$ then there is a surjective linear map $L : V \longrightarrow W$.
(3) if $\dim V = \dim W$ then there is a linear isomorphism $L : V \longrightarrow W$, i.e. an injective and surjective linear map.

PROBLEM 1.5. Show that any two norms on a finite dimensional vector space are equivalent.

PROBLEM 1.5. Show that if two norms on a vector space are equivalent then the topologies induced are the same – the sets open with respect to the distance from one are open with respect to the distance coming from the other. The converse is also true, you can use another result from this section to prove it.

PROBLEM 1.5. Write out a proof for each p with $1 \le p < \infty$ that

$$l^p = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_{j=1}^{\infty} |a_j|^p < \infty,\ a_j = a(j)\}$$

is a normed space with the norm

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{\frac{1}{p}}.$$

This means writing out the proof that this is a linear space and that the three conditions required of a norm hold.

PROBLEM 1.5. Prove directly that each l^p as defined in Problem 1.5 is complete, i.e. it is a Banach space.

PROBLEM 1.5. The space l^∞ consists of the bounded sequences

$$l^\infty = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sup_n |a_n| < \infty\},\ \|a\|_\infty = \sup_n |a_n|. \tag{5.6}$$

Show that it is a Banach space.

PROBLEM 1.6. Another closely related space consists of the sequences converging to 0 :

$$c_0 = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \lim_{n\to\infty} a_n = 0\},\ \|a\|_\infty = \sup_n |a_n|. \tag{5.7}$$

Check that this is a Banach space and that it is a closed subspace of l^∞ (perhaps in the opposite order).

PROBLEM 1.7. Consider the 'unit sphere' in l^p. This is the set of vectors of length 1 :

$$S = \{a \in l^p; \|a\|_p = 1\}.$$

(1) Show that S is closed.
(2) Recall the sequential (so not the open covering definition) characterization of compactness of a set in a metric space (e.g. by checking in Rudin's book).
(3) Show that S is not compact by considering the sequence in l^p with kth element the sequence which is all zeros except for a 1 in the kth slot. Note that the main problem is not to get yourself confused about sequences of sequences!

PROBLEM 1.7. Show that the norm on any normed space is continuous.

PROBLEM 1.7. Finish the proof of the completeness of the space B constructed in the second proof of Theorem 1.1.

1.1. Hints for some problems.

HINT 1 (Problem 1.5). *You need to show that each Cauchy sequence converges. The problem here is to find the limit of a given Cauchy sequence. Show that for each N the sequence in $\mathbb{C}^N$ obtained by truncating each of the elements at point N is Cauchy with respect to the norm in Problem 1.1 on $\mathbb{C}^N$. Show that this is the same as being Cauchy in $\mathbb{C}^N$ in the usual sense (if you are doing $p = 2$ it is already the usual sense) and hence, this cut-off sequence converges. Use this to find a putative limit of the Cauchy sequence and then check that it works.*

1.2. Solutions to some problems.

SOLUTION 1 (1.0). *If V is a vector space (over $\mathbb{K}$ which is $\mathbb{R}$ or $\mathbb{C}$) then for any set X consider*

$$\mathcal{F}(X;V) = \{u : X \longrightarrow V\}. \tag{5.8}$$

Addition and scalar multiplication are defined 'pointwise':

$$(u+v)(x) = u(x) + v(x),\ (cu)(x) = cu(x),\ u, v \in \mathcal{F}(X;V),\ c \in \mathbb{K}. \tag{5.9}$$

These are well-defined functions since addition and multiplication are defined in $\mathbb{K}$.

So, one needs to check all the axioms of a vector space. Since an equality of functions is just equality at all points, these all follow from the corresponding identities for $\mathbb{K}$.

SOLUTION 2 (1.1). *If $S \subset V$ is a (non-empty) subset of a vector space and $S \subset V$ which is closed under addition and scalar multiplication:*

$$v_1,\ v_2 \in S,\ \lambda \in \mathbb{K} \Longrightarrow v_1 + v_2 \in S \text{ and } \lambda v_1 \in S \tag{5.10}$$

then $0 \in S$, since $0 \in \mathbb{K}$ and for any $v \in S$, $0v = 0 \in S$. Similarly, if $v \in S$ then $-v = (-1)v \in S$. Then all the axioms of a vector space follow from the corresponding identities in V.

SOLUTION 3. *If $S \subset V$ be a linear subspace of a vector space consider the relation on V*

$$v_1 \sim v_2 \Longleftrightarrow v_1 - v_2 \in S. \tag{5.11}$$

To say that this is an equivalence relation means that symmetry and transitivity hold. Since S is a subspace, $v \in S$ implies $-v \in S$ so

$$v_1 \sim v_2 \Longrightarrow v_1 - v_2 \in S \Longrightarrow v_2 - v_1 \in S \Longrightarrow v_2 \sim v_1.$$

Similarly, since it is also possible to add and remain in S

$$v_1 \sim v_2,\ v_2 \sim v_3 \Longrightarrow v_1 - v_2,\ v_2 - v_3 \in S \Longrightarrow v_1 - v_3 \in S \Longrightarrow v_1 \sim v_3.$$

So this is an equivalence relation and the quotient $V/\sim = V/S$ is well-defined – where the latter is notation. That is, and element of V/S is an equivalence class of elements of V which can be written $v + S$:

$$v + S = w + S \Longleftrightarrow v - w \in S. \tag{5.12}$$

Now, we can check the axioms of a linear space once we define addition and scalar multiplication. Notice that

$$(v + S) + (w + S) = (v + w) + S,\ \lambda(v + S) = \lambda v + S$$

are well-defined elements, independent of the choice of representatives, since adding an lement of S to v or w does not change the right sides.

Now, to the axioms. These amount to showing that S is a zero element for addition, $-v + S$ is the additive inverse of $v + S$ and that the other axioms follow directly from the fact that the hold as identities in V.

Solution 4 (1.3). *In case you do not know it, go through the basic theory of finite-dimensional vector spaces. Define a vector space V to be* finite-dimensional *if there is an integer N such that any $N + 1$ elements of V are linearly dependent in the sense that the satisfy a non-trivial dependence relation – if $v_i \in V$ for $i = 1, \dots N + 1$, then there exist $a_i \in \mathbb{K}$, not* all *zero, such that*

$$\sum_{i=1}^{N+1} a_i v_i = 0 \text{ in } V. \tag{5.13}$$

Call the smallest such integer the dimension of V – it is also the largest integer such that there are N linearly independent vectors – and show that a finite dimensional vector space always has a basis, $e_i \in V$, $i = 1, \dots, \dim V$ which are not *linearly dependent and such that any element of V can be written as a linear combination*

$$v = \sum_{i=1}^{\dim V} b_i e_i,\ b_i \in \mathbb{K}. \tag{5.14}$$

Solution 5 (1.5). *Show that any two norms on a finite dimensional vector space are equivalent.*

Solution 6 (1.5). *Show that if two norms on a vector space are equivalent then the topologies induced are the same – the sets open with respect to the distance from one are open with respect to the distance coming from the other. The converse is also true, you can use another result from this section to prove it.*

Solution 7 (1.5). *Write out a proof (you can steal it from one of many places but at least write it out in your own hand) either for $p = 2$ or for each p with $1 \le p < \infty$ that*

$$l^p = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_{j=1}^{\infty} |a_j|^p < \infty,\ a_j = a(j)\}$$

is a normed space with the norm

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}}.$$

This means writing out the proof that this is a linear space and that the three conditions required of a norm hold.

SOLUTION 8 (). *The 'tricky' part in Problem 1.0 is the triangle inequality. Suppose you knew – meaning I tell you – that for each N*

$$\left(\sum_{j=1}^{N} |a_j|^p\right)^{\frac{1}{p}} \text{ is a norm on } \mathbb{C}^N$$

would that help?

SOLUTION 9 (1.5). *Prove directly that each l^p as defined in Problem 1.0 is complete, i.e. it is a Banach space. At the risk of offending some, let me say that this means showing that each Cauchy sequence converges. The problem here is to find the limit of a given Cauchy sequence. Show that for each N the sequence in $\mathbb{C}^N$ obtained by truncating each of the elements at point N is Cauchy with respect to the norm in Problem 1.1 on $\mathbb{C}^N$. Show that this is the same as being Cauchy in $\mathbb{C}^N$ in the usual sense (if you are doing $p = 2$ it is already the usual sense) and hence, this cut-off sequence converges. Use this to find a putative limit of the Cauchy sequence and then check that it works.*

SOLUTION 10 (1.5). *The space l^∞ consists of the bounded sequences*

$$l^\infty = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sup_n |a_n| < \infty\}, \ \|a\|_\infty = \sup_n |a_n|. \tag{5.15}$$

Show that it is a Banach space.

SOLUTION 11 (1.6). *Another closely related space consists of the sequences converging to* 0 :

$$c_0 = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \lim_{n\to\infty} a_n = 0\}, \ \|a\|_\infty = \sup_n |a_n|. \tag{5.16}$$

Check that this is a Banach space and that it is a closed subspace of l^∞ (perhaps in the opposite order).

SOLUTION 12 (1.7). *Consider the 'unit sphere' in l^p. This is the set of vectors of length* 1 :

$$S = \{a \in l^p; \|a\|_p = 1\}.$$

(1) *Show that S is closed.*
(2) *Recall the sequential (so not the open covering definition) characterization of compactness of a set in a metric space (e .g . by checking in Rudin).*
(3) *Show that S is not compact by considering the sequence in l^p with kth element the sequence which is all zeros except for a* 1 *in the kth slot. Note that the main problem is not to get yourself confused about sequences of sequences!*

Solution 13 (1.7). *Since the distance between two points is $\|x-y\|$ the continuity of the norm follows directly from the 'reverse triangle inequality'*

$$|\|x\|-\|y\||\le\|x-y\| \tag{5.17}$$

which in turn follows from the triangle inequality applied twice:-

$$\|x\|\le\|x-y\|+\|y\|,\ \|y\|\le\|x-y\|+\|x\|. \tag{5.18}$$

2. Problems for Chapter 2

Missing

Problem 1.18. Let's consider an example of an absolutely summable sequence of step functions. For the interval $[0,1)$ (remember there is a strong preference for left-closed but right-open intervals for the moment) consider a variant of the construction of the standard Cantor subset based on 3 proceeding in steps. Thus, remove the 'central interval $[1/3,2/3)$. This leave $C_1=[0,1/3)\cup[2/3,1)$. Then remove the central interval from each of the remaining two intervals to get $C_2=[0,1/9)\cup[2/9,1/3)\cup[2/3,7/9)\cup[8/9,1)$. Carry on in this way to define successive sets $C_k\subset C_{k-1}$, each consisting of a finite union of semi-open intervals. Now, consider the *series* of step functions f_k where $f_k(x)=1$ on C_k and 0 otherwise.

(1) Check that this is an absolutely summable series.
(2) For which $x\in[0,1)$ does $\sum\limits_k|f_k(x)|$ converge?
(3) Describe a function on $[0,1)$ which is shown to be Lebesgue integrable (as defined in Lecture 4) by the existence of this series and compute its Lebesgue integral.
(4) Is this function Riemann integrable (this is easy, not hard, if you check the definition of Riemann integrability)?
(5) Finally consider the function g which is equal to one on the union of all the intervals which are *removed* in the construction and zero elsewhere. Show that g is Lebesgue integrable and compute its integral.

Problem 1.18. The covering lemma for $\mathbb{R}^2$. By a rectangle we will mean a set of the form $[a_1,b_1)\times[a_2,b_2)$ in $\mathbb{R}^2$. The area of a rectangle is $(b_1-a_1)\times(b_2-a_2)$.

(1) We may subdivide a rectangle by subdividing either of the intervals – replacing $[a_1,b_1)$ by $[a_1,c_1)\cup[c_1,b_1)$. Show that the sum of the areas of rectangles made by any repeated subdivision is always the same as that of the original.
(2) Suppose that a finite collection of disjoint rectangles has union a rectangle (always in this same half-open sense). Show, and I really mean prove, that the sum of the areas is the area of the whole rectange. Hint:- proceed by subdivision.
(3) Now show that for any countable collection of disjoint rectangles contained in a given rectange the sum of the areas is less than or equal to that of the containing rectangle.
(4) Show that if a finite collection of rectangles has union *containing* a given rectange then the sum of the areas of the rectangles is at least as large of that of the rectangle contained in the union.
(5) Prove the extension of the preceeding result to a countable collection of rectangles with union containing a given rectangle.

PROBLEM 1.18. (1) Show that any continuous function on $[0,1]$ is the *uniform limit* on $[0,1)$ of a sequence of step functions. Hint:- Reduce to the real case, divide the interval into 2^n equal pieces and define the step functions to take infimim of the continuous function on the corresponding interval. Then use uniform convergence.

(2) By using the 'telescoping trick' show that any continuous function on $[0,1)$ can be written as the sum

$$\sum_i f_j(x) \;\forall\; x \in [0,1) \tag{5.19}$$

where the f_j are step functions and $\sum_j |f_j(x)| < \infty$ for all $x \in [0,1)$.

(3) Conclude that any continuous function on $[0,1]$, extended to be 0 outside this interval, is a Lebesgue integrable function on $\mathbb{R}$ and show that the Lebesgue integral is equal to the Riemann integral.

PROBLEM 1.19. If f and $g \in \mathcal{L}^1(\mathbb{R})$ are Lebesgue integrable functions on the line show that

(1) If $f(x) \geq 0$ a.e. then $\int f \geq 0$.
(2) If $f(x) \leq g(x)$ a.e. then $\int f \leq \int g$.
(3) If f is complex valued then its real part, $\operatorname{Re} f$, is Lebesgue integrable and $|\int \operatorname{Re} f| \leq \int |f|$.
(4) For a general complex-valued Lebesgue integrable function

$$|\int f| \leq \int |f|. \tag{5.20}$$

Hint: You can look up a proof of this easily enough, but the usual trick is to choose $\theta \in [0,2\pi)$ so that $e^{i\theta}\int f = \int (e^{i\theta} f) \geq 0$. Then apply the preceeding estimate to $g = e^{i\theta} f$.

(5) Show that the integral is a continuous linear functional

$$\int : L^1(\mathbb{R}) \longrightarrow \mathbb{C}. \tag{5.21}$$

PROBLEM 1.21. If $I \subset \mathbb{R}$ is an interval, including possibly $(-\infty, a)$ or (a, ∞), we *define* Lebesgue integrability of a function $f : I \longrightarrow \mathbb{C}$ to mean the Lebesgue integrability of

$$\tilde{f} : \mathbb{R} \longrightarrow \mathbb{C},\; \tilde{f}(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I. \end{cases} \tag{5.22}$$

The integral of f on I is then defined to be

$$\int_I f = \int \tilde{f}. \tag{5.23}$$

(1) Show that the space of such integrable functions on I is linear, denote it $\mathcal{L}^1(I)$.
(2) Show that is f is integrable on I then so is $|f|$.
(3) Show that if f is integrable on I and $\int_I |f| = 0$ then $f = 0$ a.e. in the sense that $f(x) = 0$ for all $x \in I \setminus E$ where $E \subset I$ is of measure zero as a subset of $\mathbb{R}$.
(4) Show that the set of null functions as in the preceeding question is a linear space, denote it $\mathcal{N}(I)$.

(5) Show that $\int_I |f|$ defines a norm on $L^1(I) = \mathcal{L}^1(I)/\mathcal{N}(I)$.

(6) Show that if $f \in \mathcal{L}^1(\mathbb{R})$ then

$$g : I \longrightarrow \mathbb{C},\ g(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I \end{cases} \tag{5.24}$$

is integrable on I.

(7) Show that the preceeding construction gives a *surjective and continuous* linear map 'restriction to I'

$$L^1(\mathbb{R}) \longrightarrow L^1(I). \tag{5.25}$$

(Notice that these are the quotient spaces of integrable functions modulo equality a.e.)

Problem 1.25. Really continuing the previous one.

(1) Show that if $I = [a, b)$ and $f \in L^1(I)$ then the restriction of f to $I_x = [x, b)$ is an element of $L^1(I_x)$ for all $a \le x < b$.

(2) Show that the function

$$F(x) = \int_{I_x} f : [a, b) \longrightarrow \mathbb{C} \tag{5.26}$$

is continuous.

(3) Prove that the function $x^{-1}\cos(1/x)$ is not Lebesgue integrable on the interval $(0, 1]$. Hint: Think about it a bit and use what you have shown above.

Problem 1.26. [Harder but still doable] Suppose $f \in \mathcal{L}^1(\mathbb{R})$.

(1) Show that for each $t \in \mathbb{R}$ the translates

$$f_t(x) = f(x - t) : \mathbb{R} \longrightarrow \mathbb{C} \tag{5.27}$$

are elements of $\mathcal{L}^1(\mathbb{R})$.

(2) Show that

$$\lim_{t \to 0} \int |f_t - f| = 0. \tag{5.28}$$

This is called 'Continuity in the mean for integrable functions'. Hint: I will add one!

(3) Conclude that for each $f \in \mathcal{L}^1(\mathbb{R})$ the map (it is a 'curve')

$$\mathbb{R} \ni t \longmapsto [f_t] \in L^1(\mathbb{R}) \tag{5.29}$$

is continuous.

Problem 1.29. In the last problem set you showed that a continuous function on a compact interval, extended to be zero outside, is Lebesgue integrable. Using this, and the fact that step functions are dense in $L^1(\mathbb{R})$ show that the linear space of continuous functions on $\mathbb{R}$ each of which vanishes outside a compact set (which depends on the function) form a dense subset of $L^1(\mathbb{R})$.

Problem 1.29. (1) If $g : \mathbb{R} \longrightarrow \mathbb{C}$ is bounded and continuous and $f \in \mathcal{L}^1(\mathbb{R})$ show that $gf \in \mathcal{L}^1(\mathbb{R})$ and that

$$\int |gf| \le \sup_{\mathbb{R}} |g| \cdot \int |f|. \tag{5.30}$$

(2) Suppose now that $G \in \mathcal{C}([0,1]\times[0,1])$ is a continuous function (I use $\mathcal{C}(K)$ to denote the continuous functions on a compact metric space). Recall from the preceeding discussion that we have defined $L^1([0,1])$. Now, using the first part show that if $f \in L^1([0,1])$ then

$$F(x) = \int_{[0,1]} G(x,\cdot)f(\cdot) \in \mathbb{C} \tag{5.31}$$

(where $\cdot$ is the variable in which the integral is taken) is well-defined for each $x \in [0,1]$.

(3) Show that for each $f \in L^1([0,1])$, F is a continuous function on $[0,1]$.

(4) Show that

$$L^1([0,1]) \ni f \longmapsto F \in \mathcal{C}([0,1]) \tag{5.32}$$

is a bounded (i.e. continuous) linear map into the Banach space of continuous functions, with supremum norm, on $[0,1]$.

Problem 1.32. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be an element of $\mathcal{L}^1(\mathbb{R})$. Define

$$f_L(x) = \begin{cases} f(x) & x \in [-L,L] \\ 0 & \text{otherwise.} \end{cases} \tag{5.33}$$

Show that $f_L \in \mathcal{L}^1(\mathbb{R})$ and that $\int |f_L - f| \to 0$ as $L \to \infty$.

Problem 1.33. Consider a real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is locally integrable in the sense that

$$g_L(x) = \begin{cases} f(x) & x \in [-L,L] \\ 0 & x \in \mathbb{R} \setminus [-L,L] \end{cases} \tag{5.34}$$

is Lebesgue integrable of each $L \in \mathbb{N}$.

(1) Show that for each fixed L the function

$$g_L^{(N)}(x) = \begin{cases} g_L(x) & \text{if } g_L(x) \in [-N,N] \\ N & \text{if } g_L(x) > N \\ -N & \text{if } g_L(x) < -N \end{cases} \tag{5.35}$$

is Lebesgue integrable.

(2) Show that $\int |g_L^{(N)} - g_L| \to 0$ as $N \to \infty$.

(3) Show that there is a sequence, h_n, of step functions such that

$$h_n(x) \to f(x) \text{ a.e. in } \mathbb{R}. \tag{5.36}$$

(4) Defining

$$h_{n,L}^{(N)} = \begin{cases} 0 & x \notin [-L,L] \\ h_n(x) & \text{if } h_n(x) \in [-N,N],\ x \in [-L,L] \\ N & \text{if } h_n(x) > N,\ x \in [-L,L] \\ -N & \text{if } h_n(x) < -N,\ x \in [-L,L] \end{cases}. \tag{5.37}$$

Show that $\int |h_{n,L}^{(N)} - g_L^{(N)}| \to 0$ as $n \to \infty$.

Problem 1.37. Show that $\mathcal{L}^2(\mathbb{R})$ is a Hilbert space.

First working with real functions, define $\mathcal{L}^2(\mathbb{R})$ as the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which are locally integrable and such that $|f|^2$ is integrable.

(1) For such f choose h_n and define g_L, $g_L^{(N)}$ and $h_n^{(N)}$ by (5.34), (5.35) and (5.37).
(2) Show using the sequence $h_{n,L}^{(N)}$ for fixed N and L that $g_L^{(N)}$ and $(g_L^{(N)})^2$ are in $\mathcal{L}^1(\mathbb{R})$ and that $\int |(h_{n,L}^{(N)})^2 - (g_L^{(N)})^2| \to 0$ as $n \to \infty$.
(3) Show that $(g_L)^2 \in \mathcal{L}^1(\mathbb{R})$ and that $\int |(g_L^{(N)})^2 - (g_L)^2| \to 0$ as $N \to \infty$.
(4) Show that $\int |(g_L)^2 - f^2| \to 0$ as $L \to \infty$.
(5) Show that $f,\, g \in \mathcal{L}^2(\mathbb{R})$ then $fg \in \mathcal{L}^1(\mathbb{R})$ and that

$$|\int fg| \le \int |fg| \le \|f\|_{L^2}\|g\|_{L^2},\ \|f\|_{L^2}^2 = \int |f|^2. \tag{5.38}$$

(6) Use these constructions to show that $\mathcal{L}^2(\mathbb{R})$ is a linear space.
(7) Conclude that the quotient space $L^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R})/\mathcal{N}$, where $\mathcal{N}$ is the space of null functions, is a real Hilbert space.
(8) Extend the arguments to the case of complex-valued functions.

Problem 1.38. Consider the sequence space

$$h^{2,1} = \left\{ c : \mathbb{N} \ni j \longmapsto c_j \in \mathbb{C}; \sum_j (1+j^2)|c_j|^2 < \infty \right\}. \tag{5.39}$$

(1) Show that

$$h^{2,1} \times h^{2,1} \ni (c,d) \longmapsto \langle c, d\rangle = \sum_j (1+j^2) c_j \overline{d_j} \tag{5.40}$$

is an Hermitian inner form which turns $h^{2,1}$ into a Hilbert space.
(2) Denoting the norm on this space by $\|\cdot\|_{2,1}$ and the norm on l^2 by $\|\cdot\|_2$, show that

$$h^{2,1} \subset l^2,\ \|c\|_2 \le \|c\|_{2,1}\ \forall\ c \in h^{2,1}. \tag{5.41}$$

Problem 1.41. In the separable case, prove Riesz Representation Theorem directly.

Choose an orthonormal basis $\{e_i\}$ of the separable Hilbert space H. Suppose $T : H \longrightarrow \mathbb{C}$ is a bounded linear functional. Define a sequence

$$w_i = \overline{T(e_i)},\ i \in \mathbb{N}. \tag{5.42}$$

(1) Now, recall that $|Tu| \le C\|u\|_H$ for some constant C. Show that for every finite N,

$$\sum_{j=1}^{N} |w_i|^2 \le C^2. \tag{5.43}$$

(2) Conclude that $\{w_i\} \in l^2$ and that

$$w = \sum_i w_i e_i \in H. \tag{5.44}$$

(3) Show that

$$T(u) = \langle u, w\rangle_H\ \forall\ u \in H \text{ and } \|T\| = \|w\|_H. \tag{5.45}$$

PROBLEM 1.45. If $f \in L^1(\mathbb{R}^k \times \mathbb{R}^p)$ show that there exists a set of measure zero $E \subset \mathbb{R}^k$ such that

$$x \in \mathbb{R}^k \setminus E \Longrightarrow g_x(y) = f(x, y) \text{ defines } g_x \in L^1(\mathbb{R}^p), \tag{5.46}$$

that $F(x) = \int g_x$ defines an element $F \in L^1(\mathbb{R}^k)$ and that

$$\int_{\mathbb{R}^k} F = \int_{\mathbb{R}^k \times \mathbb{R}^p} f. \tag{5.47}$$

Note: These identities are usually written out as an equality of an iterated integral and a 'regular' integral:

$$\int_{\mathbb{R}^k} \int_{\mathbb{R}^p} f(x, y) = \int f. \tag{5.48}$$

It is often used to 'exchange the order of integration' since the hypotheses are the same if we exchange the variables.

3. Solutions to problems

PROBLEM 1.48. Suppose that $f \in \mathcal{L}^1(0, 2\pi)$ is such that the constants

$$c_k = \int_{(0,2\pi)} f(x)e^{-ikx},\ k \in \mathbb{Z},$$

satisfy

$$\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty.$$

Show that $f \in \mathcal{L}^2(0, 2\pi)$.

Solution. So, this was a good bit harder than I meant it to be – but still in principle solvable (even though no one quite got to the end).

First, (for half marks in fact!) we know that the c_k exists, since $f \in \mathcal{L}^1(0, 2\pi)$ and e^{-ikx} is continuous so $fe^{-ikx} \in \mathcal{L}^1(0, 2\pi)$ and then the condition $\sum\limits_k |c_k|^2 < \infty$ implies that the Fourier series does converge in $L^2(0, 2\pi)$ so there is a function

$$g = \frac{1}{2\pi} \sum_{k \in \mathbb{C}} c_k e^{ikx}. \tag{5.49}$$

Now, what we want to show is that $f = g$ a .e . since then $f \in \mathcal{L}^2(0, 2\pi)$.

Set $h = f - g \in \mathcal{L}^1(0, 2\pi)$ since $\mathcal{L}^2(0, 2\pi) \subset \mathcal{L}^1(0, 2\pi)$. It follows from (5.49) that f and g have the same Fourier coefficients, and hence that

$$\int_{(0,2\pi)} h(x)e^{ikx} = 0 \ \forall\ k \in \mathbb{Z}. \tag{5.50}$$

So, we need to show that this implies that $h = 0$ a .e . Now, we can recall from class that we showed (in the proof of the completeness of the Fourier basis of L^2) that these exponentials are dense, in the supremum norm, in continuous functions which vanish near the ends of the interval. Thus, by continuity of the integral we know that

$$\int_{(0,2\pi)} hg = 0 \tag{5.51}$$

for all such continuous functions g. We also showed at some point that we can find such a sequence of continuous functions g_n to approximate the characteristic function of any interval χ_I. It is not true that $g_n \to \chi_I$ uniformly, but for any integrable function h, $hg_n \to h\chi_I$ in $\mathcal{L}^1$. So, the upshot of this is that we know a bit more than (5.51), namely we know that

$$\int_{(0,2\pi)} hg = 0 \;\forall \text{ step functions } g. \tag{5.52}$$

So, now the trick is to show that (5.52) implies that $h = 0$ almost everywhere. Well, this would follow if we know that $\int_{(0,2\pi)} |h| = 0$, so let's aim for that. Here is the trick. Since $g \in \mathcal{L}^1$ we know that there is a sequence (the partial sums of an absolutely convergent series) of step functions h_n such that $h_n \to g$ both in $L^1(0, 2\pi)$ and almost everywhere and also $|h_n| \to |h|$ in both these senses. Now, consider the functions

$$s_n(x) = \begin{cases} 0 & \text{if } h_n(x) = 0 \\ \frac{\overline{h_n(x)}}{|h_n(x)|} & \text{otherwise.} \end{cases} \tag{5.53}$$

Clearly s_n is a sequence of step functions, bounded (in absolute value by 1 in fact) and such that $s_n h_n = |h_n|$. Now, write out the wonderful identity

$$|h(x)| = |h(x)| - |h_n(x)| + s_n(x)(h_n(x) - h(x)) + s_n(x)h(x). \tag{5.54}$$

Integrate this identity and then apply the triangle inequality to conclude that

$$\begin{aligned} \int_{(0,2\pi)} |h| &= \int_{(0,2\pi)} (|h(x)| - |h_n(x)| + \int_{(0,2\pi)} s_n(x)(h_n - h) \\ &\le \int_{(0,2\pi)} (||h(x)| - |h_n(x)|| + \int_{(0,2\pi)} |h_n - h| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{5.55}$$

Here on the first line we have used (5.52) to see that the third term on the right in (5.54) integrates to zero. Then the fact that $|s_n| \le 1$ and the convergence properties.

Thus in fact $h = 0$ a .e . so indeed $f = g$ and $f \in \mathcal{L}^2(0, 2\pi)$. Piece of cake, right! Mia culpa.

4. Problems – Chapter 3

Problem 1.55. Let H be a normed space in which the norm satisfies the parallelogram law:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2) \;\forall\; u, v \in H. \tag{5.56}$$

Show that the norm comes from a positive definite sesquilinear (i.e. ermitian) inner product. Big Hint:- Try

$$(u, v) = \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2\right)! \tag{5.57}$$

Problem 1.57. Let H be a finite dimensional (pre)Hilbert space. So, by definition H has a basis $\{v_i\}_{i=1}^n$, meaning that any element of H can be written

$$v = \sum_i c_i v_i \tag{5.58}$$

and there is no dependence relation between the v_i's – the presentation of $v = 0$ in the form (5.58) is unique. Show that H has an orthonormal basis, $\{e_i\}_{i=1}^n$ satisfying

$(e_i, e_j) = \delta_{ij}$ ($= 1$ if $i = j$ and 0 otherwise). Check that for the orthonormal basis the coefficients in (5.58) are $c_i = (v, e_i)$ and that the map

$$T : H \ni v \longmapsto ((v, e_i)) \in \mathbb{C}^n \tag{5.59}$$

is a linear isomorphism with the properties

$$(u, v) = \sum_i (Tu)_i \overline{(Tv)_i}, \ \|u\|_H = \|Tu\|_{\mathbb{C}^n} \ \forall \ u, v \in H. \tag{5.60}$$

Why is a finite dimensional preHilbert space a Hilbert space?

Problem 1.60. : Prove (3.171). The important step is actually the fact that $\operatorname{Spec}(A) \subset [-\|A\|, \|A\|]$ if A is self-adjoint, which is proved somewhere above. Now, if f is a real polynomial, we can assume the leading constant, c, in (3.170) is 1. If $\lambda \notin f([-\|A\|, \|A\|])$ then $f(A)$ is self-adjoint and $\lambda - f(A)$ is invertible – it is enough to check this for each factor in (3.170). Thus $\operatorname{Spec}(f(A)) \subset f([-\|A\|, \|A\|])$ which means that

$$\|f(A)\| \le \sup\{z \in f([-\|A\|, \|A\|])\} \tag{5.61}$$

which is in fact (3.170).

Problem 1.61. Let H be a separable Hilbert space. Show that $K \subset H$ is compact if and only if it is closed, bounded and has the property that any sequence in K which is weakly convergent sequence in H is (strongly) convergent.

Hint (Problem 1.61) In one direction use the result from class that any bounded sequence has a weakly convergent subsequence.

Problem 1.61. Show that, in a separable Hilbert space, a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if the weak limit, v satisfies

$$\|v\|_H = \lim_{n\to\infty} \|v_n\|_H. \tag{5.62}$$

Hint (Problem 1.61) To show that this condition is sufficient, expand

$$(v_n - v, v_n - v) = \|v_n\|^2 - 2\operatorname{Re}(v_n, v) + \|v\|^2. \tag{5.63}$$

Problem 1.63. Show that a subset of a separable Hilbert space is compact if and only if it is closed and bounded and has the property of 'finite dimensional approximation' meaning that for any $\epsilon > 0$ there exists a linear subspace $D_N \subset H$ of finite dimension such that

$$d(K, D_N) = \sup_{u\in K} \inf_{v\in D_N} \{d(u, v)\} \le \epsilon. \tag{5.64}$$

See Hint 4

Hint (Problem 1.63) To prove necessity of this condition use the 'equi-small tails' property of compact sets with respect to an orthonormal basis. To use the finite dimensional approximation condition to show that any weakly convergent sequence in K is strongly convergent, use the convexity result from class to define the sequence $\{v_n'\}$ in D_N where v_n' is the closest point in D_N to v_n. Show that v_n' is weakly, hence strongly, convergent and hence deduce that $\{v_n\}$ is Cauchy.

Problem 1.64. Suppose that $A : H \longrightarrow H$ is a bounded linear operator with the property that $A(H) \subset H$ is finite dimensional. Show that if v_n is weakly convergent in H then Av_n is strongly convergent in H.

PROBLEM 1.64. Suppose that H_1 and H_2 are two different Hilbert spaces and $A : H_1 \longrightarrow H_2$ is a bounded linear operator. Show that there is a unique bounded linear operator (the adjoint) $A^* : H_2 \longrightarrow H_1$ with the property

$$(Au_1, u_2)_{H_2} = (u_1, A^*u_2)_{H_1} \ \forall \ u_1 \in H_1, \ u_2 \in H_2. \tag{5.65}$$

PROBLEM 1.65. Question:- Is it possible to show the completeness of the Fourier basis

$$\exp(ikx)/\sqrt{2\pi}$$

by computation? Maybe, see what you think. These questions are also intended to get you to say things clearly.

(1) Work out the Fourier coefficients $c_k(t) = \int_{(0,2\pi)} f_t e^{-ikx}$ of the step function

$$f_t(x) = \begin{cases} 1 & 0 \le x < t \\ 0 & t \le x \le 2\pi \end{cases} \tag{5.66}$$

for each fixed $t \in (0, 2\pi)$.

(2) Explain why this Fourier series converges to f_t in $L^2(0, 2\pi)$ if and only if

$$2\sum_{k>0} |c_k(t)|^2 = 2\pi t - t^2, \ t \in (0, 2\pi). \tag{5.67}$$

(3) Write this condition out as a Fourier series and apply the argument again to show that the completeness of the Fourier basis implies identities for the sum of k^{-2} and k^{-4}.

(4) Can you explain how reversing the argument, that knowledge of the sums of these two series should imply the completeness of the Fourier basis? There is a serious subtlety in this argument, and you get full marks for spotting it, without going ahead a using it to prove completeness.

PROBLEM 1.67. Prove that for appropriate choice of constants d_k, the functions $d_k \sin(kx/2)$, $k \in \mathbb{N}$, form an orthonormal basis for $L^2(0, 2\pi)$.

See Hint 4

Hint (Problem 1.67 The usual method is to use the basic result from class plus translation and rescaling to show that $d'_k \exp(ikx/2)$ $k \in \mathbb{Z}$ form an orthonormal basis of $L^2(-2\pi, 2\pi)$. Then extend functions as odd from $(0, 2\pi)$ to $(-2\pi, 2\pi)$.

PROBLEM 1.67. Let e_k, $k \in \mathbb{N}$, be an orthonormal basis in a separable Hilbert space, H. Show that there is a uniquely defined bounded linear operator $S : H \longrightarrow H$, satisfying

$$Se_j = e_{j+1} \ \forall \ j \in \mathbb{N}. \tag{5.68}$$

Show that if $B : H \longrightarrow H$ is a bounded linear operator then $S + \epsilon B$ is *not* invertible if $\epsilon < \epsilon_0$ for some $\epsilon_0 > 0$.

Hint (Problem 1.67)- Consider the linear functional $L : H \longrightarrow \mathbb{C}$, $Lu = (Bu, e_1)$. Show that $B'u = Bu - (Lu)e_1$ is a bounded linear operator from H to the Hilbert space $H_1 = \{u \in H; (u, e_1) = 0\}$. Conclude that $S + \epsilon B'$ is invertible as a linear map from H to H_1 for small ϵ. Use this to argue that $S + \epsilon B$ cannot be an isomorphism from H to H by showing that either e_1 is not in the range or else there is a non-trivial element in the null space.

PROBLEM 1.68. Show that the product of bounded operators on a Hilbert space is strong continuous, in the sense that if A_n and B_n are strong convergent sequences of bounded operators on H with limits A and B then the product A_nB_n is strongly convergent with limit AB.

Hint (Problem 1.68) Be careful! Use the result in class which was deduced from the Uniform Boundedness Theorem.

PROBLEM 1.68. Show that a continuous function $K : [0,1] \longrightarrow L^2(0,2\pi)$ has the property that the Fourier series of $K(x) \in L^2(0,2\pi)$, for $x \in [0,1]$, converges uniformly in the sense that if $K_n(x)$ is the sum of the Fourier series over $|k| \le n$ then $K_n : [0,1] \longrightarrow L^2(0,2\pi)$ is also continuous and

$$\sup_{x\in[0,1]} \|K(x) - K_n(x)\|_{L^2(0,2\pi)} \to 0. \tag{5.69}$$

Hint (Problem 1.68) Use one of the properties of compactness in a Hilbert space that you proved earlier.

PROBLEM 1.69. Consider an integral operator acting on $L^2(0,1)$ with a kernel which is continuous – $K \in \mathcal{C}([0,1]^2)$. Thus, the operator is

$$Tu(x) = \int_{(0,1)} K(x,y)u(y). \tag{5.70}$$

Show that T is bounded on L^2 (I think we did this before) and that it is in the norm closure of the finite rank operators.

Hint (Problem 1.68) Use the previous problem! Show that a continuous function such as K in this Problem defines a continuous map $[0,1] \ni x \longmapsto K(x,\cdot) \in \mathcal{C}([0,1])$ and hence a continuous function $K : [0,1] \longrightarrow L^2(0,1)$ then apply the previous problem with the interval rescaled.

Here is an even more expanded version of the hint: You can think of $K(x,y)$ as a continuous function of x with values in $L^2(0,1)$. Let $K_n(x,y)$ be the continuous function of x and y given by the previous problem, by truncating the Fourier series (in y) at some point n. Check that this defines a finite rank operator on $L^2(0,1)$ – yes it maps into continuous functions but that is fine, they are Lebesgue square integrable. Now, the idea is the difference $K - K_n$ defines a bounded operator with small norm as n becomes large. It might actually be clearer to do this the other way round, exchanging the roles of x and y.

PROBLEM 1.70. Although we have concentrated on the Lebesgue integral in one variable, you proved at some point the covering lemma in dimension 2 and that is pretty much all that was needed to extend the discussion to 2 dimensions. Let's just assume you have assiduously checked everything and so you know that $L^2((0,2\pi)^2)$ is a Hilbert space. Sketch a proof – noting anything that you are not sure of – that the functions $\exp(ikx+ily)/2\pi$, $k,l \in \mathbb{Z}$, form a complete orthonormal basis.

PROBLEM 1.70. Let H be a separable (partly because that is mostly what I have been talking about) Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Say that a sequence u_n in H converges weakly if (u_n,v) is Cauchy in $\mathbb{C}$ for each $v \in H$.

(1) Explain why the sequence $\|u_n\|_H$ is bounded.

Solution: Each u_n defines a continuous linear functional on H by

$$T_n(v) = (v, u_n),\ \|T_n\| = \|u_n\|, T_n : H \longrightarrow \mathbb{C}. \tag{5.71}$$

For fixed v the sequence $T_n(v)$ is Cauchy, and hence bounded, in $\mathbb{C}$ so by the 'Uniform Boundedness Principle' the $\|T_n\|$ are bounded, hence $\|u_n\|$ is bounded in $\mathbb{R}$.

(2) Show that there exists an element $u \in H$ such that $(u_n, v) \to (u, v)$ for each $v \in H$.

Solution: Since (v, u_n) is Cauchy in $\mathbb{C}$ for each fixed $v \in H$ it is convergent. Set

$$Tv = \lim_{n\to\infty} (v, u_n) \text{ in } \mathbb{C}. \tag{5.72}$$

This is a linear map, since

$$T(c_1v_1 + c_2v_2) = \lim_{n\to\infty} c_1(v_1, u_n) + c_2(v_2, u) = c_1Tv_1 + c_2Tv_2 \tag{5.73}$$

and is bounded since $|Tv| \le C\|v\|$, $C = \sup_n \|u_n\|$. Thus, by Riesz' theorem there exists $u \in H$ such that $Tv = (v, u)$. Then, by definition of T,

$$(u_n, v) \to (u, v)\ \forall\ v \in H. \tag{5.74}$$

(3) If e_i, $i \in \mathbb{N}$, is an orthonormal sequence, give, with justification, an example of a sequence u_n which is *not* weakly convergent in H but is such that (u_n, e_j) converges for each j.

Solution: One such example is $u_n = ne_n$. Certainly $(u_n, e_i) = 0$ for all $i > n$, so converges to 0. However, $\|u_n\|$ is not bounded, so the sequence cannot be weakly convergent by the first part above.

(4) Show that if the e_i form an orthonormal basis, $\|u_n\|$ is bounded and (u_n, e_j) converges for each j then u_n converges weakly.

Solution: By the assumption that (u_n, e_j) converges for all j it follows that (u_n, v) converges as $n \to \infty$ for all v which is a finite linear combination of the e_i. For general $v \in H$ the convergence of the Fourier-Bessell series for v with respect to the orthonormal basis e_j

$$v = \sum_k (v, e_k)e_k \tag{5.75}$$

shows that there is a sequence $v_k \to v$ where each v_k is in the finite span of the e_j. Now, by Cauchy's inequality

$$|(u_n, v) - (u_m, v)| \le |(u_nv_k) - (u_m, v_k)| + |(u_n, v - v_k)| + |(u_m, v - v_k)|. \tag{5.76}$$

Given $\epsilon > 0$ the boundedness of $\|u_n\|$ means that the last two terms can be arranged to be each less than $\epsilon/4$ by choosing k sufficiently large. Having chosen k the first term is less than $\epsilon/4$ if $n, m > N$ by the fact that (u_n, v_k) converges as $n \to \infty$. Thus the sequence (u_n, v) is Cauchy in $\mathbb{C}$ and hence convergent.

Problem 1.76. Consider the two spaces of sequences

$$h_{\pm 2} = \{c : \mathbb{N} \longmapsto \mathbb{C}; \sum_{j=1}^{\infty} j^{\pm 4}|c_j|^2 < \infty\}.$$

Show that both $h_{\pm 2}$ are Hilbert spaces and that any linear functional satisfying

$$T : h_2 \longrightarrow \mathbb{C},\ |Tc| \le C\|c\|_{h_2}$$

for some constant C is of the form

$$Tc = \sum_{j=1}^{\infty} c_i d_i$$

where $d : \mathbb{N} \longrightarrow \mathbb{C}$ is an element of h_{-2}.

Solution: Many of you hammered this out by parallel with l^2. This is fine, but to prove that $h_{\pm 2}$ are Hilbert spaces we can actually use l^2 itself. Thus, consider the maps on complex sequences

$$(T^{\pm}c)_j = c_j j^{\pm 2}. \tag{5.77}$$

Without knowing anything about $h_{\pm 2}$ this is a bijection between the sequences in $h_{\pm 2}$ and those in l^2 which takes the norm

$$\|c\|_{h_{\pm 2}} = \|Tc\|_{l^2}. \tag{5.78}$$

It is also a linear map, so it follows that $h_{\pm}$ are linear, and that they are indeed Hilbert spaces with $T^{\pm}$ isometric isomorphisms onto l^2; The inner products on $h_{\pm 2}$ are then

$$(c, d)_{h_{\pm 2}} = \sum_{j=1}^{\infty} j^{\pm 4} c_j \overline{d_j}. \tag{5.79}$$

Don't feel bad if you wrote it all out, it is good for you!

Now, once we know that h_2 is a Hilbert space we can apply Riesz' theorem to see that any continuous linear functional $T : h_2 \longrightarrow \mathbb{C}$, $|Tc| \le C\|c\|_{h_2}$ is of the form

$$Tc = (c, d')_{h_2} = \sum_{j=1}^{\infty} j^4 c_j \overline{d'_j},\ d' \in h_2. \tag{5.80}$$

Now, if $d' \in h_2$ then $d_j = j^4 d'_j$ defines a sequence in h_{-2}. Namely,

$$\sum_j j^{-4}|d_j|^2 = \sum_j j^4 |d'_j|^2 < \infty. \tag{5.81}$$

Inserting this in (5.80) we find that

$$Tc = \sum_{j=1}^{\infty} c_j d_j,\ d \in h_{-2}. \tag{5.82}$$

(1) In P9.2 (2), and elsewhere, $\mathcal{C}^\infty(\mathbb{S})$ should be $\mathcal{C}^0(\mathbb{S})$, the space of continuous functions on the circle – with supremum norm.
(2) In (5.95) it should be $u = Fv$, not $u = Sv$.
(3) Similarly, before (5.96) it should be $u = Fv$.
(4) Discussion around (5.98) clarified.
(5) Last part of P10.2 clarified.

This week I want you to go through the invertibility theory for the operator

$$Qu = (-\frac{d^2}{dx^2} + V(x))u(x) \tag{5.83}$$

acting on periodic functions. Since we have not developed the theory to handle this directly we need to approach it through integral operators.

Problem 1.83. Let $\mathbb{S}$ be the circle of radius 1 in the complex plane, centered at the origin, $\mathbb{S} = \{z; |z| = 1\}$.

(1) Show that there is a 1-1 correspondence

$$\begin{aligned}(5.84)\quad \mathcal{C}^0(\mathbb{S}) = \{u : \mathbb{S} \longrightarrow \mathbb{C},\ \text{continuous}\} \longrightarrow \\ \{u : \mathbb{R} \longrightarrow \mathbb{C};\ \text{continuous and satisfying } u(x + 2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}.\end{aligned}$$

(2) Show that there is a 1-1 correspondence

$$\begin{aligned}(5.85)\quad L^2(0, 2\pi) \longleftrightarrow \{u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}); u\big|_{(0,2\pi)} \in \mathcal{L}^2(0, 2\pi) \\ \text{and } u(x + 2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}/\mathcal{N}_P\end{aligned}$$

where $\mathcal{N}_P$ is the space of null functions on $\mathbb{R}$ satisfying $u(x + 2\pi) = u(x)$ for all $x \in \mathbb{R}$.

(3) If we denote by $L^2(\mathbb{S})$ the space on the left in (5.85) show that there is a dense inclusion

$$\mathcal{C}^0(\mathbb{S}) \longrightarrow L^2(\mathbb{S}). \tag{5.86}$$

So, the idea is that we can think of functions on $\mathbb{S}$ as 2π-periodic functions on $\mathbb{R}$.

Next are some problems dealing with Schrödinger's equation, or at least it is an example thereof:

$$-\frac{d^2u(x)}{dx^2} + V(x)u(x) = f(x),\ x \in \mathbb{R}, \tag{5.87}$$

(1) First we will consider the special case $V = 1$. Why not $V = 0$? – Don't try to answer this until the end!
(2) Recall how to solve the differential equation

$$-\frac{d^2u(x)}{dx^2} + u(x) = f(x),\ x \in \mathbb{R}, \tag{5.88}$$

where $f(x) \in \mathcal{C}^0(\mathbb{S})$ is a continuous, 2π-periodic function on the line. Show that there is a unique 2π-periodic and twice continuously differentiable function, u, on $\mathbb{R}$ satisfying (5.88) and that this solution can be written in the form

$$u(x) = (Sf)(x) = \int_{0,2\pi} A(x, y)f(y) \tag{5.89}$$

where $A(x, y) \in \mathcal{C}^0(\mathbb{R}^2)$ satisfies $A(x+2\pi, y+2\pi) = A(x, y)$ for all $(x, y) \in \mathbb{R}$.

Extended hint: In case you managed to avoid a course on differential equations! First try to find a solution, igonoring the periodicity issue. To do so one can (for example, there are other ways) factorize the differential operator involved, checking that

$$-\frac{d^2u(x)}{dx^2} + u(x) = -(\frac{dv}{dx} + v) \text{ if } v = \frac{du}{dx} - u \tag{5.90}$$

since the cross terms cancel. Then recall the idea of integrating factors to see that

$$\begin{aligned} &\frac{du}{dx} - u = e^{x}\frac{d\phi}{dx},\ \phi = e^{-x}u, \\ &\frac{dv}{dx} + v = e^{-x}\frac{d\psi}{dx},\ \psi = e^{x}v. \end{aligned} \tag{5.91}$$

Now, solve the problem by integrating twice from the origin (say) and hence get a solution to the differential equation (5.88). Write this out explicitly as a double integral, and then change the order of integration to write the solution as

$$u'(x) = \int_{0,2\pi} A'(x,y)f(y)dy \tag{5.92}$$

where A' is continuous on $\mathbb{R}\times[0,2\pi]$. Compute the difference $u'(2\pi)-u'(0)$ and $\frac{du'}{dx}(2\pi) - \frac{du'}{dx}(0)$ as integrals involving f. Now, add to u' as solution to the *homogeneous* equation, for $f=0$, namely $c_1e^x + c_2e^{-x}$, so that the new solution to (5.88) satisfies $u(2\pi) = u(0)$ and $\frac{du}{dx}(2\pi) = \frac{du}{dx}(0)$. Now, check that u is given by an integral of the form (5.89) with A as stated.

(3) Check, either directly or indirectly, that $A(y,x) = A(x,y)$ and that A is real.
(4) Conclude that the operator S extends by continuity to a bounded operator on $L^2(\mathbb{S})$.
(5) Check, probably indirectly rather than directly, that

$$S(e^{ikx}) = (k^2+1)^{-1}e^{ikx},\ k\in\mathbb{Z}. \tag{5.93}$$

(6) Conclude, either from the previous result or otherwise that S is a compact self-adjoint operator on $L^2(\mathbb{S})$.
(7) Show that if $g \in \mathcal{C}^0(\mathbb{S}))$ then Sg is twice continuously differentiable. Hint: Proceed directly by differentiating the integral.
(8) From (5.93) conclude that $S = F^2$ where F is also a compact self-adjoint operator on $L^2(\mathbb{S})$ with eigenvalues $(k^2+1)^{-\frac{1}{2}}$.
(9) Show that $F : L^2(\mathbb{S}) \longrightarrow \mathcal{C}^0(\mathbb{S})$.
(10) Now, going back to the real equation (5.87), we assume that V is continuous, real-valued and 2π-periodic. Show that if u is a twice-differentiable 2π-periodic function satisfying (5.87) for a given $f \in \mathcal{C}^0(\mathbb{S})$ then

$$u + S((V-1)u) = Sf \text{ and hence } u = -F^2((V-1)u) + F^2f \tag{5.94}$$

and hence conclude that

$$u = Fv \text{ where } v \in L^2(\mathbb{S}) \text{ satisfies } v + (F(V-1)F)v = Ff \tag{5.95}$$

where $V-1$ is the operator defined by multiplication by $V-1$.

(11) Show the converse, that if $v \in L^2(\mathbb{S})$ satisfies

$$v + (F(V-1)F)v = Ff,\ f\in\mathcal{C}^0(\mathbb{S}) \tag{5.96}$$

then $u = Fv$ is 2π-periodic and twice-differentiable on $\mathbb{R}$ and satisfies (5.87).

(12) Apply the Spectral theorem to $F(V-1)F$ (including why it applies) and show that there is a sequence λ_j in $\mathbb{R}\setminus\{0\}$ with $|\lambda_j|\to 0$ such that for all $\lambda\in\mathbb{C}\setminus\{0\}$, the equation

$$\lambda v + (F(V-1)F)v = g,\ g\in L^2(\mathbb{S}) \tag{5.97}$$

has a unique solution for every $g\in L^2(\mathbb{S})$ if and only if $\lambda\neq\lambda_j$ for any j.

(13) Show that for the λ_j the solutions of

$$\lambda_j v + (F(V-1)F)v = 0, v\in L^2(\mathbb{S}), \tag{5.98}$$

are all continuous 2π-periodic functions on $\mathbb{R}$.

(14) Show that the corresponding functions $u = Fv$ where v satisfies (5.98) are all twice continuously differentiable, 2π-periodic functions on $\mathbb{R}$ satisfying

$$-\frac{d^2u}{dx^2} + (1 - s_j + s_jV(x))u(x) = 0,\ s_j = 1/\lambda_j. \tag{5.99}$$

(15) Conversely, show that if u is a twice continuously differentiable and 2π-periodic function satisfying

$$-\frac{d^2u}{dx^2} + (1 - s + sV(x))u(x) = 0,\ s\in\mathbb{C}, \tag{5.100}$$

and u is not identically 0 then $s = s_j$ for some j.

(16) Finally, conclude that Fredholm's alternative holds for the equation (5.87)

THEOREM 5.1. *For a given real-valued, continuous 2π-periodic function V on $\mathbb{R}$, either* (5.87) *has a unique twice continuously differentiable, 2π-periodic, solution for each f which is continuous and 2π-periodic or else there exists a finite, but positive, dimensional space of twice continuously differentiable 2π-periodic solutions to the homogeneous equation*

$$-\frac{d^2w(x)}{dx^2} + V(x)w(x) = 0,\ x\in\mathbb{R}, \tag{5.101}$$

and (5.87) *has a solution if and only if $\int_{(0,2\pi)} fw = 0$ for every 2π-periodic solution, w, to* (5.101).

PROBLEM 1.101. Check that we really can understand all the 2π periodic eigenfunctions of the Schrödinger operator using the discussion above. First of all, there was nothing sacred about the addition of 1 to $-d^2/dx^2$, we could add any positive number and get a similar result – the problem with 0 is that the constants satisfy the homogeneous equation $d^2u/dx^2 = 0$. What we have shown is that the operator

$$u \longmapsto Qu = -\frac{d^2u}{dx^2}u + Vu \tag{5.102}$$

applied to twice continuously differentiable functions has at least a left inverse unless there is a non-trivial solution of

$$-\frac{d^2u}{dx^2}u + Vu = 0. \tag{5.103}$$

Namely, the left inverse is $R = F(\mathrm{Id} + F(V-1)F)^{-1}F$. This is a compact self-adjoint operator. Show – and there is still a bit of work to do – that (twice continuously differentiable) eigenfunctions of Q, meaning solutions of $Qu = \tau u$ are precisely the non-trivial solutions of $Ru = \tau^{-1}u$.

What to do in case (5.103) does have a non-trivial solution? Show that the space of these is finite dimensional and conclude that essentially the same result holds by working on the orthocomplement in $L^2(\mathbb{S})$.

By now you should have become reasonably comfortable with a separable Hilbert space such as l_2. However, it is worthwhile checking once again that it is rather large – if you like, let me try to make you uncomfortable for one last time. An important result in this direction is Kuiper's theorem, which I will *not* ask you to prove[1]. However, I want you to go through the closely related result sometimes known as *Eilenberg's swindle*. Perhaps you will appreciate the little bit of trickery. First some preliminary results. Note that everything below is a closed curve in the $x \in [0,1]$ variable – you might want to identify this with a circle instead, I just did it the primitive way.

Problem 1.103. Let H be a separable, infinite dimensional Hilbert space. Show that the direct sum of two copies of H is a Hilbert space with the norm

$$H \oplus H \ni (u_1, u_2) \longmapsto (\|u_1\|_H^2 + \|u_2\|_H^2)^{\frac{1}{2}} \tag{5.104}$$

either by constructing an isometric isomorphism

$$T : H \longrightarrow H \oplus H,\ \text{1-1 and onto},\ \|u\|_H = \|Tu\|_{H\oplus H} \tag{5.105}$$

or otherwise. In any case, construct a map as in (5.105).

Problem 1.105. One can repeat the preceding construction any finite number of times. Show that it can be done 'countably often' in the sense that if H is a separable, infinite dimensional, Hilbert space then

$$l_2(H) = \{u : \mathbb{N} \longrightarrow H; \|u\|_{l_2(H)}^2 = \sum_i \|u_i\|_H^2 < \infty\} \tag{5.106}$$

has a Hilbert space structure and construct an explicit isometric isomorphism from $l_2(H)$ to H.

Problem 1.106. Recall, or perhaps learn about, the winding number of a closed curve with values in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We take as given the following fact:[2] If $Q = [0,1]^N$ and $f : Q \longrightarrow \mathbb{C}^*$ is continuous then for each choice of $b \in \mathbb{C}$ satisfying $\exp(2\pi i b) = f(0)$, there exists a unique continuous function $F : Q \longrightarrow \mathbb{C}$ satisfying

$$\exp(2\pi i F(q)) = f(q),\ \forall\ q \in Q \text{ and } F(0) = b. \tag{5.107}$$

Of course, you are free to change b to $b + n$ for any $n \in \mathbb{Z}$ but then F changes to $F + n$, just shifting by the same integer.

(1) Now, suppose $c : [0,1] \longrightarrow \mathbb{C}^*$ is a closed curve – meaning it is continuous and $c(1) = c(0)$. Let $C : [0,1] \longrightarrow \mathbb{C}$ be a choice of F for $N = 1$ and $f = c$. Show that the winding number of the closed curve c may be defined *unambiguously* as

$$\operatorname{wn}(c) = C(1) - C(0) \in \mathbb{Z}. \tag{5.108}$$

---

[1]Kuiper's theorem says that for any (norm) continuous map, say from any compact metric space, $g : M \longrightarrow \mathrm{GL}(H)$ with values in the invertible operators on a separable infinite-dimensional Hilbert space there exists a continuous map, an homotopy, $h : M \times [0,1] \longrightarrow \mathrm{GL}(H)$ such that $h(m,0) = g(m)$ and $h(m,1) = \mathrm{Id}_H$ for all $m \in M$.

[2]Of course, you are free to give a proof – it is not hard.

(2) Show that $\mathrm{wn}(c)$ is *constant under homotopy.* That is if $c_i : [0,1] \longrightarrow \mathbb{C}^*$, $i = 1, 2$, are two closed curves so $c_i(1) = c_i(0)$, $i = 1, 2$, which are *homotopic* through closed curves in the sense that there exists $f : [0,1]^2 \longrightarrow \mathbb{C}^*$ continuous and such that $f(0,x) = c_1(x)$, $f(1,x) = c_2(x)$ for all $x \in [0,1]$ and $f(y,0) = f(y,1)$ for all $y \in [0,1]$, then $\mathrm{wn}(c_1) = \mathrm{wn}(c_2)$.

(3) Consider the closed curve $L_n : [0,1] \ni x \longmapsto e^{2\pi ix} \mathrm{Id}_{n\times n}$ of $n\times n$ matrices. Using the standard properties of the determinant, show that this curve is not homotopic to the identity through closed curves in the sense that there does not exist a continuous map $G : [0,1]^2 \longrightarrow \mathrm{GL}(n)$, with values in the invertible $n \times n$ matrices, such that $G(0,x) = L_n(x)$, $G(1,x) \equiv \mathrm{Id}_{n\times n}$ for all $x \in [0,1]$, $G(y,0) = G(y,1)$ for all $y \in [0,1]$.

Problem 1.108. Consider the closed curve corresponding to L_n above in the case of a separable but now infinite dimensional Hilbert space:

$$L : [0,1] \ni x \longmapsto e^{2\pi ix} \mathrm{Id}_H \in \mathrm{GL}(H) \subset \mathcal{B}(H) \tag{5.109}$$

taking values in the invertible operators on H. Show that after identifying H with $H \oplus H$ as above, there is a continuous map

$$M : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H) \tag{5.110}$$

with values in the invertible operators and satisfying

$$M(0,x) = L(x),\ M(1,x)(u_1,u_2) = (e^{4\pi ix}u_1, u_2),\ M(y,0) = M(y,1),\ \forall\, x,y \in [0,1]. \tag{5.111}$$

Hint: So, think of $H \oplus H$ as being 2-vectors (u_1,u_2) with entries in H. This allows one to think of 'rotation' between the two factors. Indeed, show that

$$U(y)(u_1,u_2) = (\cos(\pi y/2)u_1 + \sin(\pi y/2)u_2, -\sin(\pi y/2)u_1 + \cos(\pi y/2)u_2) \tag{5.112}$$

defines a continuous map $[0,1] \ni y \longmapsto U(y) \in \mathrm{GL}(H \oplus H)$ such that $U(0) = \mathrm{Id}$, $U(1)(u_1,u_2) = (u_2, -u_1)$. Now, consider the 2-parameter family of maps

$$U^{-1}(y)V_2(x)U(y)V_1(x) \tag{5.113}$$

where $V_1(x)$ and $V_2(x)$ are defined on $H \oplus H$ as multiplication by $\exp(2\pi ix)$ on the first and the second component respectively, leaving the other fixed.

Problem 1.113. Using a rotation similar to the one in the preceeding problem (or otherwise) show that there is a continuous map

$$G : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H) \tag{5.114}$$

such that

$$\begin{aligned} G(0,x)(u_1,u_2) &= (e^{2\pi ix}u_1, e^{-2\pi ix}u_2),\\ G(1,x)(u_1,u_2) &= (u_1,u_2),\ G(y,0) = G(y,1)\ \forall\, x,y \in [0,1]. \end{aligned} \tag{5.115}$$

Problem 1.115. Now, think about combining the various constructions above in the following way. Show that on $l_2(H)$ there is an homotopy like (5.114), $\tilde{G} : [0,1]^2 \longrightarrow \mathrm{GL}(l_2(H))$, (very like in fact) such that

$$\begin{aligned} \tilde{G}(0,x)\left\{u_k\right\}_{k=1}^{\infty} &= \left\{\exp((-1)^k 2\pi ix)u_k\right\}_{k=1}^{\infty},\\ \tilde{G}(1,x) &= \mathrm{Id},\ \tilde{G}(y,0) = \tilde{G}(y,1)\ \forall\, x,y \in [0,1]. \end{aligned} \tag{5.116}$$

PROBLEM 1.116. "Eilenberg's swindle" For an infinite dimenisonal separable Hilbert space, construct an homotopy – meaning a continuous map $G : [0,1]^2 \longrightarrow \mathrm{GL}(H)$ – with $G(0,x) = L(x)$ in (5.109) and $G(1,x) = \mathrm{Id}$ and of course $G(y,0) = G(y,1)$ for all $x, y \in [0,1]$.

Hint: Just put things together – of course you can rescale the interval at the end to make it all happen over $[0,1]$. First 'divide H into 2 copies of itself' and deform from L to $M(1,x)$ in (5.111). Now, 'divide the second H up into $l_2(H)$' and apply an argument just like the preceding problem to turn the identity on this factor into alternating terms multiplying by $\exp(\pm 4\pi i x)$ – starting with $-$. Now, you are on $H \oplus l_2(H)$, 'renumbering' allows you to regard this as $l_2(H)$ again and when you do so your curve has become alternate multiplication by $\exp(\pm 4\pi i x)$ (with $+$ first). Finally then, apply the preceding problem again, to deform to the identity (always of course through closed curves). Presto, Eilenberg's swindle!

PROBLEM 1.116. Check that we really can understand all the 2π periodic eigenfunctions of the Schrödinger operator using the discussion above. First of all, there was nothing sacred about the addition of 1 to $-d^2/dx^2$, we could add any positive number and get a similar result – the problem with 0 is that the constants satisfy the homogeneous equation $d^2u/dx^2 = 0$. What we have shown is that the operator

$$u \longmapsto Qu = -\frac{d^2u}{dx^2}u + Vu \tag{5.117}$$

applied to twice continuously differentiable functions has at least a left inverse unless there is a non-trivial solution of

$$-\frac{d^2u}{dx^2}u + Vu = 0. \tag{5.118}$$

Namely, the left inverse is $R = F(\mathrm{Id} + F(V-1)F)^{-1}F$. This is a compact self-adjoint operator. Show – and there is still a bit of work to do – that (twice continuously differentiable) eigenfunctions of Q, meaning solutions of $Qu = \tau u$ are precisely the non-trivial solutions of $Ru = \tau^{-1}u$.

What to do in case (5.118) does have a non-trivial solution? Show that the space of these is finite dimensional and conclude that essentially the same result holds by working on the orthocomplement in $L^2(\mathbb{S})$.

5. Exam Preparation Problems

EP.1 Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and suppose that

$$B : H \times H \longleftrightarrow \mathbb{C} \tag{5.119}$$

is a(nother) sesquilinear form – so for all c_1, $c_2 \in \mathbb{C}$, u, u_1, u_2 and $v \in H$,

$$B(c_1u_1 + c_2u_2, v) = c_1B(u_1, v) + c_2B(u_2, v),\ B(u, v) = \overline{B(v, u)}. \tag{5.120}$$

Show that B is continuous, with respect to the norm $\|(u, v)\| = \|u\|_H + \|v\|_H$ on $H \times H$ if and only if it is bounded, in the sense that for some $C > 0$,

$$|B(u, v)| \le C\|u\|_H\|v\|_H. \tag{5.121}$$

EP.2 A continuous linear map $T : H_1 \longrightarrow H_2$ between two, possibly different, Hilbert spaces is said to be *compact* if the image of the unit ball in H_1 under T is precompact in H_2. Suppose $A : H_1 \longrightarrow H_2$ is a continuous linear operator which is injective and surjective and $T : H_1 \longrightarrow H_2$ is compact. Show that there is a compact operator $K : H_2 \longrightarrow H_2$ such that $T = KA$.

EP.3 Suppose $P \subset H$ is a (non-trivial, i.e. not $\{0\}$) closed linear subspace of a Hilbert space. Deduce from a result done in class that each u in H has a unique decomposition

$$u = v + v',\ v \in P,\ v' \perp P \tag{5.122}$$

and that the map $\pi_P : H \ni u \longmapsto v \in P$ has the properties

$$(\pi_P)^* = \pi_P,\ (\pi_P)^2 = \pi_P,\ \|\pi_P\|_{\mathcal{B}(H)} = 1. \tag{5.123}$$

EP.4 Show that for a sequence of non-negative step functions f_j, defined on $\mathbb{R}$, which is absolutely summable, meaning $\sum\limits_j \int f_j < \infty$, the series $\sum\limits_j f_j(x)$ cannot diverge for all $x \in (a, b)$, for any $a < b$.

EP.5 Let $A_j \subset [-N, N] \subset \mathbb{R}$ (for N fixed) be a sequence of subsets with the property that the characteristic function, χ_j of A_j, is integrable for each j. Show that the characteristic function of

$$A = \bigcup_j A_j \tag{5.124}$$

is integrable.

EP.6 Let $e_j = c_j C^j e^{-x^2/2}$, $c_j > 0$, $C = -\frac{d}{dx} + x$ the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. Define an operator on $L^2(\mathbb{R})$ by

$$Au = \sum_{j=0}^{\infty} (2j+1)^{-\frac{1}{2}} (u, e_j)_{L^2} e_j. \tag{5.125}$$

(1) Show that A is compact as an operator on $L^2(\mathbb{R})$.
(2) Suppose that $V \in \mathcal{C}^0_\infty(\mathbb{R})$ is a bounded, real-valued, continuous function on $\mathbb{R}$. What can you say about the eigenvalues τ_j, and eigenfunctions v_j, of $K = AVA$, where V is acting by multiplication on $L^2(\mathbb{R})$?
(3) Show that for $C > 0$ a large enough constant, $\operatorname{Id} + A(V + C)A$ is invertible (with bounded inverse on $L^2(\mathbb{R})$).
(4) Show that $L^2(\mathbb{R})$ has an orthonormal basis of eigenfunctions of $J = A(\operatorname{Id} + A(V + C)A)^{-1}A$.
(5) What would you need to show to conclude that these eigenfunctions of J satisfy

$$-\frac{d^2 v_j(x)}{dx^2} + x^2 v_j(x) + V(x) v_j(x) = \lambda_j v_j? \tag{5.126}$$

(6) What would you need to show to check that all the square-integrable, twice continuously differentiable, solutions of (5.126), for some $\lambda_j \in \mathbb{C}$, are eigenfunctions of K?

EP.7 Test 1 from last year (N.B. There may be some confusion between $\mathcal{L}^1$ and L^1 here, just choose the correct interpretation!):-

Q1. Recall Lebesgue's Dominated Convergence Theorem and use it to show that if $u \in \mathcal{L}^2(\mathbb{R})$ and $v \in \mathcal{L}^1(\mathbb{R})$ then

$$\begin{aligned}&\lim_{N\to\infty}\int_{|x|>N}|u|^2 = 0,\ \lim_{N\to\infty}\int |C_N u - u|^2 = 0,\\ &\lim_{N\to\infty}\int_{|x|>N}|v| = 0 \text{ and } \lim_{N\to\infty}\int |C_N v - v| = 0.\end{aligned} \tag{Eq1}$$

where

$$C_N f(x) = \begin{cases} N & \text{if } f(x) > N \\ -N & \text{if } f(x) < -N \\ f(x) & \text{otherwise.}\end{cases} \tag{Eq2}$$

Q2. Show that step functions are dense in $L^1(\mathbb{R})$ and in $L^2(\mathbb{R})$ (Hint:- Look at Q1 above and think about $f - f_N$, $f_N = C_N f \chi_{[-N,N]}$ and its square. So it suffices to show that f_N is the limit in L^2 of a sequence of step functions. Show that if g_n is a sequence of step functions converging to f_N in L^1 then $C_N g_n \chi_{[-N,N]}$ is converges to f_N in L^2.) and that if $f \in L^1(\mathbb{R})$ then there is a sequence of step functions u_n and an element $g \in L^1(\mathbb{R})$ such that $u_n \to f$ a.e. and $|u_n| \le g$.

Q3. Show that $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ are separable, meaning that each has a countable dense subset.

Q4. Show that the minimum and the maximum of two locally integrable functions is locally integrable.

Q5. A subset of $\mathbb{R}$ is said to be (Lebesgue) measurable if its characteristic function is locally integrable. Show that a countable union of measurable sets is measurable. Hint: Start with two!

Q6. Define $\mathcal{L}^\infty(\mathbb{R})$ as consisting of the locally integrable functions which are bounded, $\sup_{\mathbb{R}} |u| < \infty$. If $\mathcal{N}_\infty \subset L^\infty(\mathbb{R})$ consists of the bounded functions which vanish outside a set of measure zero show that

$$\|u + \mathcal{N}_\infty\|_{L^\infty} = \inf_{h\in\mathcal{N}_\infty}\sup_{x\in\mathbb{R}} |u(x) + h(x)| \tag{Eq3}$$

is a norm on $L^\infty(\mathbb{R}) = \mathcal{L}^\infty(\mathbb{R})/\mathcal{N}_\infty$.

Q7. Show that if $u \in L^\infty(\mathbb{R})$ and $v \in L^1(\mathbb{R})$ then $uv \in L^1(\mathbb{R})$ and that

$$|\int uv| \le \|u\|_{L^\infty}\|v\|_{L^1}. \tag{Eq4}$$

Q8. Show that each $u \in L^2(\mathbb{R})$ is continuous in the mean in the sense that $T_z u(x) = u(x - z) \in L^2(\mathbb{R})$ for all $z \in \mathbb{R}$ and that

$$\lim_{|z|\to 0}\int |T_z u - u|^2 = 0. \tag{Eq5}$$

Q9. If $\{u_j\}$ is a Cauchy sequence in $L^2(\mathbb{R})$ show that both (Eq5) and (Eq1) are uniform in j, so given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int |T_z u_j - u_j|^2 < \epsilon,\ \int_{|x|>1/\delta} |u_j|^2 < \epsilon\ \forall\ |z| < \delta \text{ and all } j. \tag{Eq6}$$

Q10. Construct a sequence in $L^2(\mathbb{R})$ for which the uniformity in (Eq6) does not hold.

EP.8 Test 2 from last year.

(1) Recall the discussion of the Dirichlet problem for d^2/dx^2 from class and carry out an analogous discussion for the Neumann problem to arrive at a complete orthonormal basis of $L^2([0,1])$ consisting of $\psi_n \in \mathcal{C}^2$ functions which are all eigenfunctions in the sense that

$$\text{(NeuEig)} \qquad \frac{d^2\psi_n(x)}{dx^2} = \gamma_n \psi_n(x)\ \forall\ x \in [0,1],\ \frac{d\psi_n}{dx}(0) = \frac{d\psi_n}{dx}(1) = 0.$$

This is actually a little harder than the Dirichlet problem which I did in class, because there is an eigenfunction of norm 1 with $\gamma = 0$. Here are some individual steps which may help you along the way!

What is the eigenfunction with eigenvalue 0 for (NeuEig)?

What is the operator of orthogonal projection onto this function?

What is the operator of orthogonal projection onto the orthocomplement of this function?

The crucual part. Find an integral operator $A_N = B - B_N$, where B is the operator from class,

$$\text{(B-Def)} \qquad (Bf)(x) = \int_0^x (x-s)f(s)ds$$

and B_N is of finite rank, such that if f is continuous then $u = A_N f$ is twice continuously differentiable, satisfies $\int_0^1 u(x)dx = 0$, $A_N 1 = 0$ (where 1 is the constant function) and

$$\text{(GI)} \qquad \begin{gathered} \int_0^1 f(x)dx = 0 \Longrightarrow \\ \frac{d^2u}{dx^2} = f(x)\ \forall\ x \in [0,1],\ \frac{du}{dx}(0) = \frac{du}{dx}(1) = 0. \end{gathered}$$

Show that A_N is compact and self-adjoint.

Work out what the spectrum of A_N is, including its null space.

Deduce the desired conclusion.

(2) Show that these two orthonormal bases of $L^2([0,1])$ (the one above and the one from class) can each be turned into an orthonormal basis of $L^2([0,\pi])$ by change of variable.

(3) Construct an orthonormal basis of $L^2([-\pi,\pi])$ by dividing each element into its odd and even parts, resticting these to $[0,\pi]$ and using the Neumann basis above on the even part and the Dirichlet basis from class on the odd part.

(4) Prove the basic theorem of Fourier series, namely that for any function $u \in L^2([-\pi,\pi])$ there exist unique constants $c_k \in \mathbb{C}$, $k \in \mathbb{Z}$ such that

$$\text{(FS)} \qquad u(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx} \text{ converges in } L^2([-\pi,\pi])$$

and give an integral formula for the constants.

EP.9 Let $B \in \mathcal{C}([0,1]^2)$ be a continuous function of two variables. Explain why the integral operator

$$Tu(x) = \int_{[0,1]} B(x,y)u(y)$$

defines a bounded linear map $L^1([0,1]) \longrightarrow \mathcal{C}([0,1])$ and hence a bounded operator on $L^2([0,1])$.

(a) Explain why T is not surjective as a bounded operator on $L^2([0,1])$.
(b) Explain why $\operatorname{Id} - T$ has finite dimensional null space $N \subset L^2([0,1])$ as an operator on $L^2([0,1])$
(c) Show that $N \subset \mathcal{C}([0,1])$.
(d) Show that $\operatorname{Id} - T$ has closed range $R \subset L^2([0,1]))$ as a bounded operator on $L^2([0,1])$.
(e) Show that the orthocomplement of R is a subspace of $\mathcal{C}([0,1])$.

EP.10 Let $c : \mathbb{N}^2 \longrightarrow \mathbb{C}$ be an 'infinite matrix' of complex numbers satisfying

$$\sum_{i,j=1}^{\infty} |c_{ij}|^2 < \infty. \tag{5.127}$$

If $\{e_i\}_{i=1}^{\infty}$ is an orthornomal basis of a (separable of course) Hilbert space $\mathcal{H}$, show that

$$Au = \sum_{i,j=1}^{\infty} c_{ij}(u, e_j)e_i \tag{5.128}$$

defines a compact operator on $\mathcal{H}$.

6. Solutions to problems

Solution 14 (Problem 1.0). *Write out a proof (you can steal it from one of many places but at least write it out in your own hand) either for $p = 2$ or for each p with $1 \le p < \infty$ that*

$$l^p = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_{j=1}^{\infty} |a_j|^p < \infty,\ a_j = a(j)\}$$

is a normed space with the norm

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}}.$$

This means writing out the proof that this is a linear space and that the three conditions required of a norm hold.

Solution:- We know that the functions from any set with values in a linear space form a linear space – under addition of values (don't feel bad if you wrote this out, it is a good thing to do once). So, to see that l^p is a linear space it suffices to see that it is closed under addition and scalar multiplication. For scalar multiples this is clear:-

$$|ta_i| = |t||a_i| \text{ so } \|ta\|_p = |t|\|a\|_p \tag{5.129}$$

which is part of what is needed for the proof that $\|\cdot\|_p$ is a norm anyway. The fact that $a,\, b \in l^p$ imples $a + b \in l^p$ follows once we show the triangle inequality or we can be a little cruder and observe that

$$\begin{gathered}|a_i + b_i|^p \le (2\max(|a|_i, |b_i|))^p = 2^p \max(|a|_i^p, |b_i|^p) \le 2^p(|a_i| + |b_i|) \\ \|a+b\|_p^p = \sum_j |a_i + b_i|^p \le 2^p(\|a\|^p + \|b\|^p),\end{gathered} \tag{5.130}$$

where we use the fact that t^p is an increasing function of $t \ge 0$.

Now, to see that l^p is a normed space we need to check that $\|a\|_p$ is indeed a norm. It is non-negative and $\|a\|_p = 0$ implies $a_i = 0$ for all i which is to say $a = 0$. So, only the triangle inequality remains. For $p = 1$ this is a direct consequence of the usual triangle inequality:

$$\|a+b\|_1 = \sum_i |a_i + b_i| \le \sum_i (|a_i| + |b_i|) = \|a\|_1 + \|b\|_1. \tag{5.131}$$

For $1 < p < \infty$ it is known as Minkowski's inequality. This in turn is deduced from Hölder's inequality – which follows from Young's inequality! The latter says if $1/p + 1/q = 1$, so $q = p/(p-1)$, then

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q} \ \forall\ \alpha, \beta \ge 0. \tag{5.132}$$

To check it, observe that as a function of $\alpha = x$,

$$f(x) = \frac{x^p}{p} - x\beta + \frac{\beta^q}{q} \tag{5.133}$$

if non-negative at $x = 0$ and clearly positive when $x >> 0$, since x^p grows faster than $x\beta$. Moreover, it is differentiable and the derivative only vanishes at $x^{p-1} = \beta$, where it must have a global minimum in $x > 0$. At this point $f(x) = 0$ so Young's inequality follows. Now, applying this with $\alpha = |a_i|/\|a\|_p$ and $\beta = |b_i|/\|b\|_q$ (assuming both are non-zero) and summing over i gives Hölder's inequality

$$\begin{gathered} |\sum_i a_i b_i|/\|a\|_p\|b\|_q \le \sum_i |a_i||b_i|/\|a\|_p\|b\|_q \le \sum_i \left(\frac{|a_i|^p}{\|a\|_p^p p} + \frac{|b_i|^q}{\|b\|_q^q q} \right) = 1 \\ \Longrightarrow |\sum_i a_i b_i| \le \|a\|_p\|b\|_q. \end{gathered} \tag{5.134}$$

Of course, if either $\|a\|_p = 0$ or $\|b\|_q = 0$ this inequality holds anyway.

Now, from this Minkowski's inequality follows. Namely from the ordinary triangle inequality and then Minkowski's inequality (with q power in the first factor)

$$\begin{aligned} \sum_i |a_i + b_i|^p &= \sum_i |a_i + b_i|^{(p-1)}|a_i + b_i| \\ &\le \sum_i |a_i + b_i|^{(p-1)}|a_i| + \sum_i |a_i + b_i|^{(p-1)}|b_i| \\ &\le \left(\sum_i |a_i + b_i|^p \right)^{1/q} (\|a\|_p + \|b\|_p) \end{aligned} \tag{5.135}$$

gives after division by the first factor on the right

$$\|a+b\|_p \le \|a\|_p + \|b\|_p. \tag{5.136}$$

Thus, l^p is indeed a normed space.

I did not necessarily expect you to go through the proof of Young-Hölder-Minkowksi, but I think you should do so at some point since I will not do it in class.

SOLUTION 15. *The 'tricky' part in Problem 1.1 is the triangle inequality. Suppose you knew – meaning I tell you – that for each N*

$$\left(\sum_{j=1}^{N}|a_j|^p\right)^{\frac{1}{p}} \quad \textit{is a norm on } \mathbb{C}^N$$

would that help?

SOLUTION. Yes indeed it helps. If we know that for each N

$$\left(\sum_{j=1}^{N}|a_j+b_j|^p\right)^{\frac{1}{p}} \le \left(\sum_{j=1}^{N}|a_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{N}|b_j|^p\right)^{\frac{1}{p}} \tag{5.137}$$

then for elements of l^p the norms always bounds the right side from above, meaning

$$\left(\sum_{j=1}^{N}|a_j+b_j|^p\right)^{\frac{1}{p}} \le \|a\|_p + \|b\|_p. \tag{5.138}$$

Since the left side is increasing with N it must converge and be bounded by the right, which is independent of N. That is, the triangle inequality follows. Really this just means it is enough to go through the discussion in the first problem for finite, but arbitrary, N. □

SOLUTION 16. *Prove directly that each l^p as defined in Problem 1.1 – or just l^2 – is complete, i.e. it is a Banach space. At the risk of offending some, let me say that this means showing that each Cauchy sequence converges. The problem here is to find the limit of a given Cauchy sequence. Show that for each N the sequence in $\mathbb{C}^N$ obtained by truncating each of the elements at point N is Cauchy with respect to the norm in Problem 1.2 on $\mathbb{C}^N$. Show that this is the same as being Cauchy in $\mathbb{C}^N$ in the usual sense (if you are doing $p = 2$ it is already the usual sense) and hence, this cut-off sequence converges. Use this to find a putative limit of the Cauchy sequence and then check that it works.*

SOLUTION. So, suppose we are given a Cauchy sequence $a^{(n)}$ in l^p. Thus, each element is a sequence $\{a_j^{(n)}\}_{j=1}^{\infty}$ in l^p. From the continuity of the norm in Problem 1.5 below, $\|a^{(n)}\|$ must be Cauchy in $\mathbb{R}$ and so converges. In particular the sequence is norm bounded, there exists A such that $\|a^{(n)}\|_p \le A$ for all n. The Cauchy condition itself is that given $\epsilon > 0$ there exists M such that for all $m, n > M$,

$$\|a^{(n)} - a^{(m)}\|_p = \left(\sum_i |a_i^{(n)} - a_i^{(m)}|^p\right)^{\frac{1}{p}} < \epsilon/2. \tag{5.139}$$

Now for each i, $|a_i^{(n)} - a_i^{(m)}| \le \|a^{(n)} - a^{(m)}\|_p$ so each of the sequences $a_i^{(n)}$ must be Cauchy in $\mathbb{C}$. Since $\mathbb{C}$ is complete

$$\lim_{n\to\infty} a_i^{(n)} = a_i \text{ exists for each } i = 1, 2, \ldots. \tag{5.140}$$

So, our putative limit is a, the sequence $\{a_i\}_{i=1}^{\infty}$. The boundedness of the norms shows that

$$\sum_{i=1}^{N}|a_i^{(n)}|^p \le A^p \tag{5.141}$$

and we can pass to the limit here as $n \to \infty$ since there are only finitely many terms. Thus

$$\sum_{i=1}^{N} |a_i|^p \leq A^p \ \forall\ N \Longrightarrow \|a\|_p \leq A. \tag{5.142}$$

Thus, $a \in l^p$ as we hoped. Similarly, we can pass to the limit as $m \to \infty$ in the finite inequality which follows from the Cauchy conditions

$$(\sum_{i=1}^{N} |a_i^{(n)} - a_i^{(m)}|^p)^{\frac{1}{p}} < \epsilon/2 \tag{5.143}$$

to see that for each N

$$(\sum_{i=1}^{N} |a_i^{(n)} - a_i|^p)^{\frac{1}{p}} \leq \epsilon/2 \tag{5.144}$$

and hence

$$\|a^{(n)} - a\| < \epsilon \ \forall\ n > M. \tag{5.145}$$

Thus indeed, $a^{(n)} \to a$ in l^p as we were trying to show.

Notice that the trick is to 'back off' to finite sums to avoid any issues of interchanging limits. □

Solution 17. *Consider the 'unit sphere' in l^p – where if you want you can set $p = 2$. This is the set of vectors of length 1 :*

$$S = \{a \in l^p; \|a\|_p = 1\}.$$

(1) *Show that S is closed.*
(2) *Recall the sequential (so not the open covering definition) characterization of compactness of a set in a metric space (e .g . by checking in Rudin).*
(3) *Show that S is not compact by considering the sequence in l^p with kth element the sequence which is all zeros except for a 1 in the kth slot. Note that the main problem is not to get yourself confused about sequences of sequences!*

Solution. By the next problem, the norm is continuous as a function, so

$$S = \{a; \|a\| = 1\} \tag{5.146}$$

is the inverse image of the closed subset $\{1\}$, hence closed.

Now, the standard result on metric spaces is that a subset is compact if and only if every sequence with values in the subset has a convergent subsequence with limit in the subset (if you drop the last condition then the closure is compact).

In this case we consider the sequence (of sequences)

$$a_i^{(n)} = \begin{cases} 0 & i \neq n \\ 1 & i = n \end{cases}. \tag{5.147}$$

This has the property that $\|a^{(n)} - a^{(m)}\|_p = 2^{\frac{1}{p}}$ whenever $n \neq m$. Thus, it cannot have any Cauchy subsequence, and hence cannot have a convergent subsequence, so S is not compact.

This is important. In fact it is a major difference between finite-dimensional and infinite-dimensional normed spaces. In the latter case the unit sphere cannot be compact whereas in the former it is. □

Solution 18. *Show that the norm on any normed space is continuous. Solution:-Right, so I should have put this problem earlier!*

The triangle inequality shows that for any u, v in a normed space

$$\|u\| \leq \|u-v\| + \|v\|, \ \|v\| \leq \|u-v\| + \|u\| \tag{5.148}$$

which implies that

$$|\|u\| - \|v\|| \leq \|u-v\|. \tag{5.149}$$

This shows that $\|\cdot\|$ is continuous, indeed it is Lipschitz continuous.

Solution 19. *Finish the proof of the completeness of the space B constructed in lecture on February 10. The description of that construction can be found in the notes to Lecture 3 as well as an indication of one way to proceed.*

Solution. The proof could be shorter than this, I have tried to be fairly complete.

To recap. We start of with a normed space V. From this normed space we construct the new linear space $\tilde{V}$ with points the absolutely summable series in V. Then we consider the subspace $S \subset \tilde{V}$ of those absolutely summable series which converge to 0 in V. We are interested in the quotient space

$$B = \tilde{V}/S. \tag{5.150}$$

What we know already is that this is a normed space where the norm of $b = \{v_n\}+S$ – where $\{v_n\}$ is an absolutely summable series in V is

$$\|b\|_B = \lim_{N\to\infty} \|\sum_{n=1}^{N} v_n\|_V. \tag{5.151}$$

This is independent of which series is used to represent b – i.e. is the same if an element of S is added to the series.

Now, what is an absolutely summable series in B? It is a sequence $\{b_n\}$, thought of a series, with the property that

$$\sum_n \|b_n\|_B < \infty. \tag{5.152}$$

We have to show that it converges in B. The first task is to guess what the limit should be. The idea is that it should be a series which adds up to 'the sum of the b_n's'. Each b_n is represented by an absolutely summable series $v_k^{(n)}$ in V. So, we can just look for the usual diagonal sum of the double series and set

$$w_j = \sum_{n+k=j} v_k^{(n)}. \tag{5.153}$$

The problem is that this will not in generall be absolutely summable as a series in V. What we want is the estimate

$$\sum_j \|w_j\| = \sum_j \|\sum_{j=n+k} v_k^{(n)}\| < \infty. \tag{5.154}$$

The only way we can really estimate this is to use the triangle inequality and conclude that

$$\sum_{j=1}^{\infty} \|w_j\| \leq \sum_{k,n} \|v_k^{(n)}\|_V. \tag{5.155}$$

Each of the sums over k on the right is finite, but we do not know that the sum over k is then finite. This is where the first suggestion comes in:-

We can *choose* the absolutely summable series $v_k^{(n)}$ representing b_n such that

$$\sum_k \|v_k^{(n)}\| \le \|b_n\|_B + 2^{-n}. \tag{5.156}$$

Suppose an initial choice of absolutely summable series representing b_n is u_k, so $\|b_n\| = \lim_{N\to\infty} \|\sum\limits_{k=1}^{N} u_k\|$ and $\sum\limits_k \|u_k\|_V < \infty$. Choosing M large it follows that

$$\sum_{k>M} \|u_k\|_V \le 2^{-n-1}. \tag{5.157}$$

With this choice of M set $v_1^{(n)} = \sum\limits_{k=1}^{M} u_k$ and $v_k^{(n)} = u_{M+k-1}$ for all $k \ge 2$. This does still represent b_n since the difference of the sums,

$$\sum_{k=1}^{N} v_k^{(n)} - \sum_{k=1}^{N} u_k = -\sum_{k=N}^{N+M-1} u_k \tag{5.158}$$

for all N. The sum on the right tends to 0 in V (since it is a fixed number of terms). Moreover, because of (5.157),

$$\sum_k \|v_k^{(n)}\|_V = \|\sum_{j=1}^{M} u_j\|_V + \sum_{k>M} \|u_k\| \le \|\sum_{j=1}^{N} u_j\| + 2\sum_{k>M} \|u_k\| \le \|\sum_{j=1}^{N} u_j\| + 2^{-n} \tag{5.159}$$

for all N. Passing to the limit as $N \to \infty$ gives (5.156).

Once we have chosen these 'nice' representatives of each of the b_n's if we define the w_j's by (5.153) then (5.154) means that

$$\sum_j \|w_j\|_V \le \sum_n \|b_n\|_B + \sum_n 2^{-n} < \infty \tag{5.160}$$

because the series b_n is absolutely summable. Thus $\{w_j\}$ defines an element of $\tilde{V}$ and hence $b \in B$.

Finally then we want to show that $\sum\limits_n b_n = b$ in B. This just means that we need to show

$$\lim_{N\to\infty} \|b - \sum_{n=1}^{N} b_n\|_B = 0. \tag{5.161}$$

The norm here is itself a limit – $b - \sum\limits_{n=1}^{N} b_n$ is represented by the summable series with nth term

$$w_k - \sum_{n=1}^{N} v_k^{(n)} \tag{5.162}$$

and the norm is then

$$\lim_{p\to\infty} \|\sum_{k=1}^{p} (w_k - \sum_{n=1}^{N} v_k^{(n)})\|_V. \tag{5.163}$$

Then we need to understand what happens as $N \to \infty$! Now, w_k is the diagonal sum of the $v_j^{(n)}$'s so sum over k gives the difference of the sum of the $v_j^{(n)}$ over the first p anti-diagonals minus the sum over a square with height N (in n) and width p. So, using the triangle inequality the norm of the difference can be estimated by the sum of the norms of all the 'missing terms' and then some so

$$\|\sum_{k=1}^{p}(w_k - \sum_{n=1}^{N} v_k^{(n)})\|_V \le \sum_{l+m\ge L} \|v_l^{(m)}\|_V \tag{5.164}$$

where $L = \min(p, N)$. This sum is finite and letting $p \to \infty$ is replaced by the sum over $l + m \ge N$. Then letting $N \to \infty$ it tends to zero by the absolute (double) summability. Thus

$$\lim_{N\to\infty} \|b - \sum_{n=1}^{N} b_n\|B = 0 \tag{5.165}$$

which is the statelent we wanted, that $\sum_n b_n = b$. □

Problem 1.165. Let's consider an example of an absolutely summable sequence of step functions. For the interval $[0, 1)$ (remember there is a strong preference for left-closed but right-open intervals for the moment) consider a variant of the construction of the standard Cantor subset based on 3 proceeding in steps. Thus, remove the 'central interval $[1/3, 2/3)$. This leave $C_1 = [0, 1/3) \cup [2/3, 1)$. Then remove the central interval from each of the remaining two intervals to get $C_2 = [0, 1/9)\cup[2/9, 1/3)\cup[2/3, 7/9)\cup[8/9, 1)$. Carry on in this way to define successive sets $C_k \subset C_{k-1}$, each consisting of a finite union of semi-open intervals. Now, consider the *series* of step functions f_k where $f_k(x) = 1$ on C_k and 0 otherwise.

(1) Check that this is an absolutely summable series.
(2) For which $x \in [0, 1)$ does $\sum_k |f_k(x)|$ converge?
(3) Describe a function on $[0, 1)$ which is shown to be Lebesgue integrable (as defined in Lecture 4) by the existence of this series and compute its Lebesgue integral.
(4) Is this function Riemann integrable (this is easy, not hard, if you check the definition of Riemann integrability)?
(5) Finally consider the function g which is equal to one on the union of all the subintervals of $[0, 1)$ which are *removed* in the construction and zero elsewhere. Show that g is Lebesgue integrable and compute its integral.

Solution. (1) The total length of the intervals is being reduced by a factor of 1/3 each time. Thus $l(C_k) = \frac{2^k}{3^k}$. Thus the integral of f, which is non-negative, is actually

$$\int f_k = \frac{2^k}{3^k} \Longrightarrow \sum_k \int |f_k| = \sum_{k=1}^{\infty} \frac{2^k}{3^k} = 2 \tag{5.166}$$

Thus the series is absolutely summable.

(2) Since the C_k are decreasing, $C_k \supset C_{k+1}$, only if

$$x \in E = \bigcap_k C_k \tag{5.167}$$

does the series $\sum_k |f_k(x)|$ diverge (to $+\infty$) otherwise it converges.

(3) The function defined as the sum of the series where it converges and zero otherwise

$$f(x) = \begin{cases} \sum_k f_k(x) & x \in \mathbb{R} \setminus E \\ 0 & x \in E \end{cases} \tag{5.168}$$

is integrable by definition. Its integral is by definition

$$\int f = \sum_k \int f_k = 2 \tag{5.169}$$

from the discussion above.

(4) The function f is not Riemann integrable since it is not bounded – and this is part of the definition. In particular for $x \in C_k \setminus C_{k+1}$, which is not an empty set, $f(x) = k$.

(5) The set F, which is the union of the intervals removed is $[0, 1) \setminus E$. Taking step functions equal to 1 on each of the intervals removed gives an absolutely summable series, since they are non-negative and the kth one has integral $1/3 \times (2/3)^{k-1}$ for $k = 1, \dots$. This series converges to g on F so g is Lebesgue integrable and hence

$$\int g = 1. \tag{5.170}$$

□

PROBLEM 1.170. The covering lemma for $\mathbb{R}^2$. By a rectangle we will mean a set of the form $[a_1, b_1) \times [a_2, b_2)$ in $\mathbb{R}^2$. The area of a rectangle is $(b_1 - a_1) \times (b_2 - a_2)$.

(1) We may subdivide a rectangle by subdividing either of the intervals – replacing $[a_1, b_1)$ by $[a_1, c_1) \cup [c_1, b_1)$. Show that the sum of the areas of rectangles made by any repeated subdivision is always the same as that of the original.
(2) Suppose that a finite collection of disjoint rectangles has union a rectangle (always in this same half-open sense). Show, and I really mean prove, that the sum of the areas is the area of the whole rectange. Hint:- proceed by subdivision.
(3) Now show that for any countable collection of disjoint rectangles contained in a given rectange the sum of the areas is less than or equal to that of the containing rectangle.
(4) Show that if a finite collection of rectangles has union *containing* a given rectange then the sum of the areas of the rectangles is at least as large of that of the rectangle contained in the union.
(5) Prove the extension of the preceeding result to a countable collection of rectangles with union containing a given rectangle.

SOLUTION. (1) For the subdivision of one rectangle this is clear enough. Namely we either divide the first side in two or the second side in two at an intermediate point c. After subdivision the area of the two rectanges is either

$$\begin{gathered}(c - a_1)(b_2 - a_2) + (b_1 - c)(b_2 - a_2) = (b_1 - c_1)(b_2 - a_2) \text{ or} \\ (b_1 - a_1)(c - a_2) + (b_1 - a_1)(b_2 - c) = (b_1 - c_1)(b_2 - a_2).\end{gathered} \tag{5.171}$$

this shows by induction that the sum of the areas of any the rectangles made by repeated subdivision is always the same as the original.

(2) If a finite collection of disjoint rectangles has union a rectangle, say $[a_1, b_2) \times [a_2, b_2)$ then the same is true after any subdivision of any of the rectangles. Moreover, by the preceeding result, after such subdivision the sum of the areas is always the same. Look at all the points $C_1 \subset [a_1, b_1)$ which occur as an endpoint of the first interval of one of the rectangles. Similarly let C_2 be the corresponding set of end-points of the second intervals of the rectangles. Now divide each of the rectangles repeatedly using the finite number of points in C_1 and the finite number of points in C_2. The total area remains the same and now the rectangles covering $[a_1, b_1) \times [A_2, b_2)$ are precisely the $A_i \times B_j$ where the A_i are a set of disjoint intervals covering $[a_1, b_1)$ and the B_j are a similar set covering $[a_2, b_2)$. Applying the one-dimensional result from class we see that the sum of the areas of the rectangles with first interval A_i is the product

$$\text{length of } A_i \times (b_2 - a_2). \tag{5.172}$$

Then we can sum over i and use the same result again to prove what we want.

(3) For any finite collection of disjoint rectangles contained in $[a_1, b_1) \times [a_2, b_2)$ we can use the same division process to show that we can add more disjoint rectangles to cover the whole big rectangle. Thus, from the preceeding result the sum of the areas must be less than or equal to $(b_1 - a_1)(b_2 - a_2)$. For a countable collection of disjoint rectangles the sum of the areas is therefore bounded above by this constant.

(4) Let the rectangles be D_i, $i = 1, \dots, N$ the union of which contains the rectangle D. Subdivide D_1 using all the endpoints of the intervals of D. Each of the resulting rectangles is either contained in D or is disjoint from it. Replace D_1 by the (one in fact) subrectangle contained in D. Proceeding by induction we can suppose that the first $N - k$ of the rectangles are disjoint and all contained in D and together all the rectangles cover D. Now look at the next one, D_{N-k+1}. Subdivide it using all the endpoints of the intervals for the earlier rectangles $D_1, \dots, D_k$ and D. After subdivision of D_{N-k+1} each resulting rectangle is either contained in one of the D_j, $j \leq N - k$ or is *not* contained in D. All these can be discarded and the result is to decrease k by 1 (maybe increasing N but that is okay). So, by induction we can decompose and throw away rectangles until what is left are disjoint and individually contained in D but still cover. The sum of the areas of the remaining rectangles is precisely the area of D by the previous result, so the sum of the areas must originally have been at least this large.

(5) Now, for a countable collection of rectangles covering $D = [a_1, b_1) \times [a_2, b_2)$ we proceed as in the one-dimensional case. First, we can assume that there is a fixed upper bound C on the lengths of the sides. Make the kth rectangle a little larger by extending both the upper limits by $2^{-k}\delta$ where $\delta > 0$. The area increases, but by no more than $2C2^{-k}$. After extension the interiors of the countable collection cover the compact set $[a_1, b_1 - \delta] \times [a_2, b_1 - \delta]$. By compactness, a finite number of these open

rectangles cover, and hence there semi-closed version, with the same endpoints, covers $[a_1, b_1-\delta)\times[a_2, b_1-\delta)$. Applying the preceeding finite result we see that

$$\text{Sum of areas} + 2C\delta \geq \text{Area } D - 2C\delta. \tag{5.173}$$

Since this is true for all $\delta > 0$ the result follows.

□

I encourage you to go through the discussion of integrals of step functions – now based on rectangles instead of intervals – and see that everything we have done can be extended to the case of two dimensions. In fact if you want you can go ahead and see that everything works in $\mathbb{R}^n$!

Problem 2.4

(1) Show that any continuous function on $[0,1]$ is the *uniform limit* on $[0,1)$ of a sequence of step functions. Hint:- Reduce to the real case, divide the interval into 2^n equal pieces and define the step functions to take infimim of the continuous function on the corresponding interval. Then use uniform convergence.

(2) By using the 'telescoping trick' show that any continuous function on $[0,1)$ can be written as the sum

$$\sum_i f_j(x) \ \forall \ x \in [0,1) \tag{5.174}$$

where the f_j are step functions and $\sum\limits_j |f_j(x)| < \infty$ for all $x \in [0,1)$.

(3) Conclude that any continuous function on $[0,1]$, extended to be 0 outside this interval, is a Lebesgue integrable function on $\mathbb{R}$.

Solution. (1) Since the real and imaginary parts of a continuous function are continuous, it suffices to consider a real continous function f and then add afterwards. By the *uniform* continuity of a continuous function on a compact set, in this case $[0,1]$, given n there exists N such that $|x-y| \leq 2^{-N} \Longrightarrow |f(x)-f(y)| \leq 2^{-n}$. So, if we divide into 2^N equal intervals, where N depends on n and we insist that it be non-decreasing as a function of n and take the step function f_n on each interval which is equal to $\min f = \inf f$ on the closure of the interval then

$$|f(x) - F_n(x)| \leq 2^{-n} \ \forall \ x \in [0,1) \tag{5.175}$$

since this even works at the endpoints. Thus $F_n \to f$ uniformly on $[0,1)$.

(2) Now just define $f_1 = F_1$ and $f_k = F_k - F_{k-1}$ for all $k > 1$. It follows that these are step functions and that

$$\sum_{k=1}^{n} = f_n. \tag{5.176}$$

Moreover, each interval for F_{n+1} is a subinterval for F_n. Since f can varying by no more than 2^{-n} on each of the intervals for F_n it follows that

$$|f_n(x)| = |F_{n+1}(x) - F_n(x)| \leq 2^{-n} \ \forall \ n > 1. \tag{5.177}$$

Thus $\int |f_n| \leq 2^{-n}$ and so the series is absolutely summable. Moreover, it actually converges everywhere on $[0,1)$ and uniformly to f by (5.175).

(3) Hence f is Lebesgue integrable.
(4) For some reason I did not ask you to check that

$$\int f = \int_0^1 f(x)dx \tag{5.178}$$

where on the right is the Riemann integral. However this follows from the fact that

$$\int f = \lim_{n\to\infty} \int F_n \tag{5.179}$$

and the integral of the step function is between the Riemann upper and lower sums for the corresponding partition of $[0,1]$.

□

Solution 20. *If f and $g \in \mathcal{L}^1(\mathbb{R})$ are Lebesgue integrable functions on the line show that*
(1) *If $f(x) \geq 0$ a.e. then $\int f \geq 0$.*
(2) *If $f(x) \leq g(x)$ a.e. then $\int f \leq \int g$.*
(3) *If f is complex valued then its real part,* $\operatorname{Re} f$, *is Lebesgue integrable and* $|\int \operatorname{Re} f| \leq \int |f|$.
(4) *For a general complex-valued Lebesgue integrable function*

$$|\int f| \leq \int |f|. \tag{5.180}$$

Hint: You can look up a proof of this easily enough, but the usual trick is to choose $\theta \in [0,2\pi)$ so that $e^{i\theta}\int f = \int(e^{i\theta}f) \geq 0$. Then apply the preceeding estimate to $g = e^{i\theta}f$.
(5) *Show that the integral is a continuous linear functional*

$$\int : L^1(\mathbb{R}) \longrightarrow \mathbb{C}. \tag{5.181}$$

Solution. (1) If f is real and f_n is a real-valued absolutely summable series of step functions converging to f where it is absolutely convergent (if we only have a complex-valued sequence use part (3)). Then we know that

$$g_1 = |f_1|,\ g_j = |f_j| - |f_{j-1}|,\ f \geq 1 \tag{5.182}$$

is an absolutely convergent sequence converging to $|f|$ almost everywhere. It follows that $f_+ = \frac{1}{2}(|f|+f) = f$, if $f \geq 0$, is the limit almost everywhere of the series obtained by interlacing $\frac{1}{2}g_j$ and $\frac{1}{2}f_j$:

$$h_n = \begin{cases} \frac{1}{2}g_k & n = 2k-1 \\ f_k & n = 2k. \end{cases} \tag{5.183}$$

Thus f_+ is Lebesgue integrable. Moreover we know that

$$\int f_+ = \lim_{k\to\infty} \sum_{n\leq 2k} \int h_k = \lim_{k\to\infty} \int \left(|\sum_{j=1}^k f_j| + \sum_{j=1}^k f_j \right) \tag{5.184}$$

where each term is a non-negative step function, so $\int f_+ \geq 0$.

(2) Apply the preceeding result to $g - f$ which is integrable and satisfies

$$\int g - \int f = \int (g - f) \geq 0. \tag{5.185}$$

(3) Arguing from first principles again, if f_n is now complex valued and an absolutely summable series of step functions converging a .e . to f then define

$$h_n = \begin{cases} \operatorname{Re} f_k & n = 3k - 2 \\ \operatorname{Im} f_k & n = 3k - 1 \\ -\operatorname{Im} f_k & n = 3k. \end{cases} \tag{5.186}$$

This series of step functions is absolutely summable and

$$\sum_n |h_n(x)| < \infty \Longleftrightarrow \sum_n |f_n(x)| < \infty \Longrightarrow \sum_n h_n(x) = \operatorname{Re} f. \tag{5.187}$$

Thus $\operatorname{Re} f$ is integrable. Since $\pm \operatorname{Re} f \leq |f|$

$$\pm \int \operatorname{Re} f \leq \int |f| \Longrightarrow |\int \operatorname{Re} f| \leq \int |f|. \tag{5.188}$$

(4) For a complex-valued f proceed as suggested. Choose $z \in \mathbb{C}$ with $|z| = 1$ such that $z \int f \in [0, \infty)$ which is possible by the properties of complex numbers. Then by the linearity of the integral

$$z \int f = \int (zf) = \int \operatorname{Re}(zf) \leq \int |z \operatorname{Re} f| \leq \int |f| \Longrightarrow |\int f| = z \int f \leq \int |f|. \tag{5.189}$$

(where the second equality follows from the fact that the integral is equal to its real part).

(5) We know that the integral defines a linear map

$$I : L^1(\mathbb{R}) \ni [f] \longmapsto \int f \in \mathbb{C} \tag{5.190}$$

since $\int f = \int g$ if $f = g$ a.e. are two representatives of the same class in $L^1(\mathbb{R})$. To say this is continuous is equivalent to it being bounded, which follows from the preceeding estimate

$$|I([f])| = |\int f| \leq \int |f| = \|[f]\|_{L^1} \tag{5.191}$$

(Note that writing $[f]$ instead of $f \in L^1(\mathbb{R})$ is correct but would normally be considered pedantic – at least after you are used to it!)

(6) I should have asked – and might do on the test: What is the norm of I as an element of the dual space of $L^1(\mathbb{R})$. It is 1 – better make sure that you can prove this.

□

Problem 3.2 If $I \subset \mathbb{R}$ is an interval, including possibly $(-\infty, a)$ or (a, ∞), we *define* Lebesgue integrability of a function $f : I \longrightarrow \mathbb{C}$ to mean the Lebesgue integrability of

$$\tilde{f} : \mathbb{R} \longrightarrow \mathbb{C},\ \tilde{f}(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I. \end{cases} \tag{5.192}$$

The integral of f on I is then defined to be

$$\int_I f = \int \tilde{f}. \tag{5.193}$$

(1) Show that the space of such integrable functions on I is linear, denote it $\mathcal{L}^1(I)$.
(2) Show that is f is integrable on I then so is $|f|$.
(3) Show that if f is integrable on I and $\int_I |f| = 0$ then $f = 0$ a.e. in the sense that $f(x) = 0$ for all $x \in I \setminus E$ where $E \subset I$ is of measure zero as a subset of $\mathbb{R}$.
(4) Show that the set of null functions as in the preceeding question is a linear space, denote it $\mathcal{N}(I)$.
(5) Show that $\int_I |f|$ defines a norm on $L^1(I) = \mathcal{L}^1(I)/\mathcal{N}(I)$.
(6) Show that if $f \in \mathcal{L}^1(\mathbb{R})$ then

$$g : I \longrightarrow \mathbb{C},\ g(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I \end{cases} \tag{5.194}$$

is in $\mathcal{L}^1(\mathbb{R})$ an hence that f is integrable on I.
(7) Show that the preceeding construction gives a *surjective and continuous* linear map 'restriction to I'

$$L^1(\mathbb{R}) \longrightarrow L^1(I). \tag{5.195}$$

(Notice that these are the quotient spaces of integrable functions modulo equality a.e.)

Solution:

(1) If f and g are both integrable on I then setting $h = f + g$, $\tilde{h} = \tilde{f} + \tilde{g}$, directly from the definitions, so $f + g$ is integrable on I if f and g are by the linearity of $\mathcal{L}^1(\mathbb{R})$. Similarly if $h = cf$ then $\tilde{h} = c\tilde{f}$ is integrable for any constant c if $\tilde{f}$ is integrable. Thus $\mathcal{L}^1(I)$ is linear.
(2) Again from the definition, $|\tilde{f}| = \tilde{h}$ if $h = |f|$. Thus f integrable on I implies $\tilde{f} \in \mathcal{L}^1(\mathbb{R})$, which, as we know, implies that $|\tilde{f}| \in \mathcal{L}^1(\mathbb{R})$. So in turn $\tilde{h} \in \mathcal{L}^1(\mathbb{R})$ where $h = |f|$, so $|f| \in \mathcal{L}^1(I)$.
(3) If $f \in \mathcal{L}^1(I)$ and $\int_I |f| = 0$ then $\int_{\mathbb{R}} |\tilde{f}| = 0$ which implies that $\tilde{f} = 0$ on $\mathbb{R} \setminus E$ where $E \subset \mathbb{R}$ is of measure zero. Now, $E_I = E \cap I \subset E$ is also of measure zero (as a subset of a set of measure zero) and f vanishes outside E_I.
(4) If $f, g : I \longrightarrow \mathbb{C}$ are both of measure zero in this sense then $f + g$ vanishes on $I \setminus (E_f \cup E_g)$ where $E_f \subset I$ and $E_f \subset I$ are of measure zero. The union of two sets of measure zero (in $\mathbb{R}$) is of measure zero so this shows $f + g$ is null. The same is true of $cf + dg$ for constant c and d, so $\mathcal{N}(I)$ is a linear space.
(5) If $f \in \mathcal{L}^1(I)$ and $g \in \mathcal{N}(I)$ then $|f + g| - |f| \in \mathcal{N}(I)$, since it vanishes where g vanishes. Thus

$$\int_I |f + g| = \int_I |f|\ \forall\ f \in \mathcal{L}^1(I),\ g \in \mathcal{N}(I). \tag{5.196}$$

Thus

$$\|[f]\|_I = \int_I |f| \tag{5.197}$$

is a well-defined function on $L^1(I) = \mathcal{L}^1(\mathbb{R})/\mathcal{N}(I)$ since it is constant on equivalence classes. Now, the norm properties follow from the same properties on the whole of $\mathbb{R}$.

(6) Suppose $f \in \mathcal{L}^1(\mathbb{R})$ and g is defined in (5.194) above by restriction to I. We need to show that $g \in \mathcal{L}^1(\mathbb{R})$. If f_n is an absolutely summable series of step functions converging to f wherever, on $\mathbb{R}$, it converges absolutely consider

$$g_n(x) = \begin{cases} f_n(x) & \text{on } \tilde{I} \\ 0 & \text{on } \mathbb{R} \setminus \tilde{I} \end{cases} \tag{5.198}$$

where $\tilde{I}$ is I made half-open if it isn't already – by adding the lower end-point (if there is one) and removing the upper end-point (if there is one). Then g_n is a step function (which is why we need $\tilde{I}$). Moreover, $\int |g_n| \le \int |f_n|$ so the series g_n is absolutely summable and converges to g_n outside I and at all points inside I where the series is absolutely convergent (since it is then the same as f_n). Thus g is integrable, and since $\tilde{f}$ differs from g by its values at two points, at most, it too is integrable so f is integrable on I by definition.

(7) First we check we do have a map. Namely if $f \in \mathcal{N}(\mathbb{R})$ then g in (5.194) is certainly an element of $\mathcal{N}(I)$. We have already seen that 'restriction to I' maps $\mathcal{L}^1(\mathbb{R})$ into $\mathcal{L}^1(I)$ and since this is clearly a linear map it defines (5.195) – the image only depends on the equivalence class of f. It is clearly linear and to see that it is surjective observe that if $g \in \mathcal{L}^1(I)$ then extending it as zero outside I gives an element of $\mathcal{L}^1(\mathbb{R})$ and the class of this function maps to $[g]$ under (5.195).

Problem 3.3 Really continuing the previous one.

(1) Show that if $I = [a, b)$ and $f \in L^1(I)$ then the restriction of f to $I_x = [x, b)$ is an element of $L^1(I_x)$ for all $a \le x < b$.

(2) Show that the function

$$F(x) = \int_{I_x} f : [a, b) \longrightarrow \mathbb{C} \tag{5.199}$$

is continuous.

(3) Prove that the function $x^{-1}\cos(1/x)$ is not Lebesgue integrable on the interval $(0, 1]$. Hint: Think about it a bit and use what you have shown above.

Solution:

(1) This follows from the previous question. If $f \in L^1([a, b))$ with f' a representative then extending f' as zero outside the interval gives an element of $\mathcal{L}^1(\mathbb{R})$, by defintion. As an element of $L^1(\mathbb{R})$ this does not depend on the choice of f' and then (5.195) gives the restriction to $[x, b)$ as an element of $L^1([x, b))$. This is a linear map.

(2) Using the discussion in the preceeding question, we now that if f_n is an absolutely summable series converging to f' (a representative of f) where it converges absolutely, then for any $a \le x \le b$, we can define

$$f_n' = \chi([a, x))f_n,\ f_n'' = \chi([x, b))f_n \tag{5.200}$$

where $\chi([a,b))$ is the characteristic function of the interval. It follows that f'_n converges to $f\chi([a,x))$ and f''_n to $f\chi([x,b))$ where they converge absolutely. Thus

$$\int_{[x,b)} f = \int f\chi([x,b)) = \sum_n \int f''_n,\ \int_{[a,x)} f = \int f\chi([a,x)) = \sum_n \int f'_n. \tag{5.201}$$

Now, for step functions, we know that $\int f_n = \int f'_n + \int f''_n$ so

$$\int_{[a,b)} f = \int_{[a,x)} f + \int_{[x,b)} f \tag{5.202}$$

as we have every right to expect. Thus it suffices to show (by moving the end point from a to a general point) that

$$\lim_{x\to a} \int_{[a,x)} f = 0 \tag{5.203}$$

for any f integrable on $[a,b)$. Thus can be seen in terms of a defining absolutely summable sequence of step functions using the usual estimate that

$$|\int_{[a,x)} f| \le \int_{[a,x)} |\sum_{n\le N} f_n| + \sum_{n>N} \int_{[a,x)} |f_n|. \tag{5.204}$$

The last sum can be made small, independent of x, by choosing N large enough. On the other hand as $x \to a$ the first integral, for fixed N, tends to zero by the definition for step functions. This proves (5.204) and hence the continuity of F.

(3) If the function $x^{-1}\cos(1/x)$ were Lebesgue integrable on the interval $(0,1]$ (on which it is defined) then it would be integrable on $[0,1)$ if we define it arbitrarily, say to be 0, at 0. The same would be true of the absolute value and Riemann integration shows us easily that

$$\lim_{t\downarrow 0} \int_t^1 x|\cos(1/x)|dx = \infty. \tag{5.205}$$

This is contrary to the continuity of the integral as a function of the limits just shown.

Problem 3.4 [Harder but still doable] Suppose $f \in \mathcal{L}^1(\mathbb{R})$.

(1) Show that for each $t \in \mathbb{R}$ the translates

$$f_t(x) = f(x-t) : \mathbb{R} \longrightarrow \mathbb{C} \tag{5.206}$$

are elements of $\mathcal{L}^1(\mathbb{R})$.

(2) Show that

$$\lim_{t\to 0} \int |f_t - f| = 0. \tag{5.207}$$

This is called 'Continuity in the mean for integrable functions'. Hint: I will add one!

(3) Conclude that for each $f \in \mathcal{L}^1(\mathbb{R})$ the map (it is a 'curve')

$$\mathbb{R} \ni t \longmapsto [f_t] \in L^1(\mathbb{R}) \tag{5.208}$$

is continuous.

Solution:

(1) If f_n is an absolutely summable series of step functions converging to f where it converges absolutely then $f_n(\cdot - t)$ is such a series converging to $f(\cdot - t)$ for each $t \in \mathbb{R}$. Thus, each of the $f(x-t)$ is Lebesgue integrable, i.e. are elements of $\mathcal{L}^1(\mathbb{R})$

(2) Now, we know that if f_n is a series converging to f as above then

$$\int |f| \le \sum_n \int |f_n|. \tag{5.209}$$

We can sum the first terms and then start the series again and so it follows that for any N,

$$\int |f| \le \int |\sum_{n\le N} f_n| + \sum_{n>N} \int |f_n|. \tag{5.210}$$

Applying this to the series $f_n(\cdot - t) - f_n(\cdot)$ we find that

$$\int |f_t - f| \le \int |\sum_{n\le N} f_n(\cdot - t) - f_n(\cdot)| + \sum_{n>N} \int |f_n(\cdot - t) - f_n(\cdot)| \tag{5.211}$$

The second sum here is bounded by $2 \sum\limits_{n>N} \int |f_n|$. Given $\delta > 0$ we can choose N so large that this sum is bounded by $\delta/2$, by the absolute convergence. So the result is reduce to proving that if $|t|$ is small enough then

$$\int |\sum_{n\le N} f_n(\cdot - t) - f_n(\cdot)| \le \delta/2. \tag{5.212}$$

This however is a finite sum of step functions. So it suffices to show that

$$|\int g(\cdot - t) - g(\cdot)| \to 0 \text{ as } t \to 0 \tag{5.213}$$

for each component, i.e. a constant, c, times the characteristic function of an interval $[a,b)$ where it is bounded by $2|c||t|$.

(3) For the 'curve' f_t which is a map

$$\mathbb{R} \ni t \longmapsto f_t \in \mathcal{L}^1(\mathbb{R}) \tag{5.214}$$

it follows that $f_{t+s} = (f_t)_s$ so we can apply the argument above to show that for each s,

$$\lim_{t\to s} \int |f_t - f_s| = 0 \Longrightarrow \lim_{t\to s} \|[f_t] - [f_s]\|_{L^1} = 0 \tag{5.215}$$

which proves continuity of the map (5.214).

Problem 3.5 In the last problem set you showed that a continuous function on a compact interval, extended to be zero outside, is Lebesgue integrable. Using this, and the fact that step functions are dense in $L^1(\mathbb{R})$ show that the linear space of continuous functions on $\mathbb{R}$ each of which vanishes outside a compact set (which depends on the function) form a dense subset of $L^1(\mathbb{R})$.

Solution: Since we know that step functions (really of course the equivalence classes of step functions) are dense in $L^1(\mathbb{R})$ we only need to show that any step function is the limit of a sequence of continuous functions each vanishing outside a compact set, with respect to L^1. So, it suffices to prove this for the charactertistic function of an interval $[a,b)$ and then multiply by constants and add. The sequence

$$g_n(x) = \begin{cases} 0 & x < a - 1/n \\ n(x - a + 1/n) & a - 1/n \le x \le a \\ 0 & a < x < b \\ n(b + 1/n - x) & b \le x \le b + 1/n \\ 0 & x > b + 1/n \end{cases} \tag{5.216}$$

is clearly continuous and vanishes outside a compact set. Since

$$\int |g_n - \chi([a,b))| = \int_{a-1/n}^{1} g_n + \int_{b}^{b+1/n} g_n \le 2/n \tag{5.217}$$

it follows that $[g_n] \to [\chi([a,b))]$ in $L^1(\mathbb{R})$. This proves the density of continuous functions with compact support in $L^1(\mathbb{R})$.

Problem 3.6

(1) If $g : \mathbb{R} \longrightarrow \mathbb{C}$ is bounded and continuous and $f \in \mathcal{L}^1(\mathbb{R})$ show that $gf \in \mathcal{L}^1(\mathbb{R})$ and that

$$\int |gf| \le \sup_{\mathbb{R}} |g| \cdot \int |f|. \tag{5.218}$$

(2) Suppose now that $G \in \mathcal{C}([0,1] \times [0,1])$ is a continuous function (I use $\mathcal{C}(K)$ to denote the continuous functions on a compact metric space). Recall from the preceeding discussion that we have defined $L^1([0,1])$. Now, using the first part show that if $f \in L^1([0,1])$ then

$$F(x) = \int_{[0,1]} G(x, \cdot) f(\cdot) \in \mathbb{C} \tag{5.219}$$

(where $\cdot$ is the variable in which the integral is taken) is well-defined for each $x \in [0,1]$.

(3) Show that for each $f \in L^1([0,1])$, F is a continuous function on $[0,1]$.

(4) Show that

$$L^1([0,1]) \ni f \longmapsto F \in \mathcal{C}([0,1]) \tag{5.220}$$

is a bounded (i.e. continuous) linear map into the Banach space of continuous functions, with supremum norm, on $[0,1]$.

Solution:

(1) Let's first assume that $f = 0$ outside $[-1,1]$. Applying a result form Problem set there exists a sequence of step functions g_n such that for any R, $g_n \to g$ *uniformly* on $[0,1)$. By passing to a subsequence we can arrange that $\sup_{[-1,1]} |g_n(x) - g_{n-1}(x)| < 2^{-n}$. If f_n is an absolutly summable series of step functions converging a .e . to f we can replace it by $f_n\chi([-1,1])$ as discussed above, and still have the same conclusion. Thus, from the uniform convergence of g_n,

$$g_n(x) \sum_{k=1}^{n} f_k(x) \to g(x)f(x) \text{ a.e. on } \mathbb{R}. \tag{5.221}$$

So define $h_1 = g_1 f_1$, $h_n = g_n(x) \sum\limits_{k=1}^{n} f_k(x) - g_{n-1}(x) \sum\limits_{k=1}^{n-1} f_k(x)$. This series of step functions converges to $gf(x)$ almost everywhere and since

$$|h_n| \le A|f_n(x)| + 2^{-n} \sum_{k<n} |f_k(x)|, \ \sum_n \int |h_n| \le A \sum_n \int |f_n| + 2 \sum_n \int |f_n| < \infty \tag{5.222}$$

it is absolutely summable. Here A is a bound for $|g_n|$ independent of n. Thus $gf \in \mathcal{L}^1(\mathbb{R})$ under the assumption that $f = 0$ outside $[0,1)$ and

$$\int |gf| \le \sup |g| \int |f| \tag{5.223}$$

follows from the limiting argument. Now we can apply this argument to f_p which is the restriction of p to the interval $[p, p+1)$, for each $p \in \mathbb{Z}$. Then we get gf as the limit a .e . of the absolutely summable series gf_p where (5.223) provides the absolute summablitly since

$$\sum_p \int |gf_p| \le \sup |g| \sum_p \int_{[p,p+1)} |f| < \infty. \tag{5.224}$$

Thus, $gf \in \mathcal{L}^1(\mathbb{R})$ by a theorem in class and

$$\int |gf| \le \sup |g| \int |f|. \tag{5.225}$$

(2) If $f \in L^1[(0,1])$ has a representative f' then $G(x, \cdot) f'(\cdot) \in \mathcal{L}^1([0,1))$ so

$$F(x) = \int_{[0,1]} G(x, \cdot) f(\cdot) \in \mathbb{C} \tag{5.226}$$

is well-defined, since it is indpendent of the choice of f', changing by a null function if f' is changed by a null function.

(3) Now by the uniform continuity of continuous functions on a compact metric space such as $S = [0,1] \times [0,1]$ given $\delta > 0$ there exist $\epsilon > 0$ such that

$$\sup_{y \in [0,1]} |G(x,y) - G(x',y)| < \delta \text{ if } |x - x'| < \epsilon. \tag{5.227}$$

Then if $|x - x'| < \epsilon$,

$$|F(x) - F(x')| = |\int_{[0,1]} (G(x, \cdot) - G(x', \cdot)) f(\cdot)| \le \delta \int |f|. \tag{5.228}$$

Thus $F \in \mathcal{C}([0,1])$ is a continuous function on $[0,1]$. Moreover the map $f \longmapsto F$ is linear and

$$\sup_{[0,1]} |F| \le \sup_S |G| \int_{[0,1]} ||f| \tag{5.229}$$

which is the desired boundedness, or continuity, of the map

$$I : L^1([0,1]) \longrightarrow \mathcal{C}([0,1]),\ F(f)(x) = \int G(x, \cdot) f(\cdot), \\ \|I(f)\|_{\sup} \le \sup |G| \|f\|_{L^1}. \tag{5.230}$$

You should be thinking about using Lebesgue's dominated convergence at several points below.

Problem 5.1

Let $f:\mathbb{R}\longrightarrow\mathbb{C}$ be an element of $\mathcal{L}^1(\mathbb{R})$. Define

$$f_L(x)=\begin{cases} f(x) & x\in[-L,L] \\ 0 & \text{otherwise.}\end{cases} \tag{5.231}$$

Show that $f_L\in\mathcal{L}^1(\mathbb{R})$ and that $\int|f_L-f|\to 0$ as $L\to\infty$.

Solution. If χ_L is the characteristic function of $[-N,N]$ then $f_L=f\chi_L$. If f_n is an absolutely summable series of step functions converging a.e. to f then $f_n\chi_L$ is absolutely summable, since $\int|f_n\chi_L|\le\int|f_n|$ and converges a.e. to f_L, so $f_L\int\mathcal{L}^1(\mathbb{R})$. Certainly $|f_L(x)-f(x)|\to 0$ for each x as $L\to\infty$ and $|f_L(x)-f(x)|\le|f_l(x)|+|f(x)|\le 2|f(x)|$ so by Lebesgue's dominated convergence, $\int|f-f_L|\to 0$.

Problem 5.2 Consider a real-valued function $f:\mathbb{R}\longrightarrow\mathbb{R}$ which is locally integrable in the sense that

$$g_L(x)=\begin{cases} f(x) & x\in[-L,L] \\ 0 & x\in\mathbb{R}\setminus[-L,L]\end{cases} \tag{5.232}$$

is Lebesgue integrable of each $L\in\mathbb{N}$.

(1) Show that for each fixed L the function

$$g_L^{(N)}(x)=\begin{cases} g_L(x) & \text{if } g_L(x)\in[-N,N] \\ N & \text{if } g_L(x)>N \\ -N & \text{if } g_L(x)<-N\end{cases} \tag{5.233}$$

is Lebesgue integrable.

(2) Show that $\int|g_L^{(N)}-g_L|\to 0$ as $N\to\infty$.

(3) Show that there is a sequence, h_n, of step functions such that

$$h_n(x)\to f(x) \text{ a.e. in } \mathbb{R}. \tag{5.234}$$

(4) Defining

$$h_{n,L}^{(N)}=\begin{cases} 0 & x\notin[-L,L] \\ h_n(x) & \text{if } h_n(x)\in[-N,N],\ x\in[-L,L] \\ N & \text{if } h_n(x)>N,\ x\in[-L,L] \\ -N & \text{if } h_n(x)<-N,\ x\in[-L,L]\end{cases}. \tag{5.235}$$

Show that $\int|h_{n,L}^{(N)}-g_L^{(N)}|\to 0$ as $n\to\infty$.

Solution:

(1) By definition $g_L^{(N)}=\max(-N\chi_L,\min(N\chi_L,g_L))$ where χ_L is the characteristic funciton of $-[L,L]$, thus it is in $\mathcal{L}^1(\mathbb{R})$.
(2) Clearly $g_L^{(N)}(x)\to g_L(x)$ for every x and $|g_L^{(N)}(x)|\le|g_L(x)|$ so by Dominated Convergence, $g_L^{(N)}\to g_L$ in L^1, i.e. $\int|g_L^{(N)}-g_L|\to 0$ as $N\to\infty$ since the sequence converges to 0 pointwise and is bounded by $2|g(x)|$.
(3) Let $S_{L,n}$ be a sequence of step functions converging a.e. to g_L – for example the sequence of partial sums of an absolutely summable series of step functions converging to g_L which exists by the assumed integrability.

Then replacing $S_{L,n}$ by $S_{L,n}\chi_L$ we can assume that the elements all vanish outside $[-N,N]$ but still have convergence a.e. to g_L. Now take the sequence

$$h_n(x) = \begin{cases} S_{k,n-k} & \text{on } [k,-k]\setminus[(k-1),-(k-1)],\ 1\le k\le n,\\ 0 & \text{on } \mathbb{R}\setminus[-n,n]. \end{cases} \tag{5.236}$$

This is certainly a sequence of step functions – since it is a finite sum of step functions for each n – and on $[-L,L]\setminus[-(L-1),(L-1)]$ for large integral L is just $S_{L,n-L}\to g_L$. Thus $h_n(x)\to f(x)$ outside a countable union of sets of measure zero, so also almost everywhere.

(4) This is repetition of the first problem, $h^{(N)}_{n,L}(x)\to g^{(N)}_L$ almost everywhere and $|h^{(N)}_{n,L}|\le N\chi_L$ so $g^{(N)}_L\in\mathcal{L}^1(\mathbb{R})$ and $\int|h^{(N)}_{n,L}-g^{(N)}_L|\to 0$ as $n\to\infty$.

Problem 5.3 Show that $\mathcal{L}^2(\mathbb{R})$ is a Hilbert space – since it is rather central to the course I wanted you to go through the details carefully!

First working with real functions, define $\mathcal{L}^2(\mathbb{R})$ as the set of functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ which are locally integrable and such that $|f|^2$ is integrable.

(1) For such f choose h_n and define g_L, $g^{(N)}_L$ and $h^{(N)}_n$ by (5.232), (5.233) and (5.235).
(2) Show using the sequence $h^{(N)}_{n,L}$ for fixed N and L that $g^{(N)}_L$ and $(g^{(N)}_L)^2$ are in $\mathcal{L}^1(\mathbb{R})$ and that $\int|(h^{(N)}_{n,L})^2-(g^{(N)}_L)^2|\to 0$ as $n\to\infty$.
(3) Show that $(g_L)^2\in\mathcal{L}^1(\mathbb{R})$ and that $\int|(g^{(N)}_L)^2-(g_L)^2|\to 0$ as $N\to\infty$.
(4) Show that $\int|(g_L)^2-f^2|\to 0$ as $L\to\infty$.
(5) Show that f, $g\in\mathcal{L}^2(\mathbb{R})$ then $fg\in\mathcal{L}^1(\mathbb{R})$ and that

$$|\int fg|\le\int|fg|\le\|f\|_{L^2}\|g\|_{L^2},\ \|f\|^2_{L^2}=\int|f|^2. \tag{5.237}$$

(6) Use these constructions to show that $\mathcal{L}^2(\mathbb{R})$ is a linear space.
(7) Conclude that the quotient space $L^2(\mathbb{R})=\mathcal{L}^2(\mathbb{R})/\mathcal{N}$, where $\mathcal{N}$ is the space of null functions, is a real Hilbert space.
(8) Extend the arguments to the case of complex-valued functions.

Solution:

(1) Done. I think it should have been $h^{(N)}_{n,L}$.
(2) We already checked that $g^{(N)}_L\in\mathcal{L}^1(\mathbb{R})$ and the same argument applies to $(g^{(N)}_L)$, namely $(h^{(N)}_{n,L})^2\to g^{(N)}_L$ almost everywhere and both are bounded by $N^2\chi_L$ so by dominated convergence

$$\begin{gathered}(h^{(N)}_{n,L})^2\to g^{(N)}_L)^2\le N^2\chi_L \text{ a.e. } \Longrightarrow g^{(N)}_L)^2\in\mathcal{L}^1(\mathbb{R}) \text{ and}\\ |h^{(N)}_{n,L})^2-g^{(N)}_L)^2|\to 0 \text{ a.e. },\\ |h^{(N)}_{n,L})^2-g^{(N)}_L)^2|\le 2N^2\chi_L\Longrightarrow\int|h^{(N)}_{n,L})^2-g^{(N)}_L)^2|\to 0.\end{gathered} \tag{5.238}$$

(3) Now, as $N\to\infty$, $(g^{(N)}_L)^2\to(g_L)^2$ a .e . and $(g^{(N)}_L)^2\to(g_L)^2\le f^2$ so by dominated convergence, $(g_L)^2\in\mathcal{L}^1$ and $\int|(g^{(N)}_L)^2-(g_L)^2|\to 0$ as $N\to\infty$.
(4) The same argument of dominated convergence shows now that $g^2_L\to f^2$ and $\int|g^2_L-f^2|\to 0$ using the bound by $f^2\in\mathcal{L}^1(\mathbb{R})$.

(5) What this is all for is to show that $fg \in \mathcal{L}^1(\mathbb{R})$ if f, $F = g \in \mathcal{L}^2(\mathbb{R})$ (for easier notation). Approximate each of them by sequences of step functions as above, $h_{n,L}^{(N)}$ for f and $H_{n,L}^{(N)}$ for g. Then the product sequence is in $\mathcal{L}^1$ – being a sequence of step functions – and

$$h_{n,L}^{(N)}(x)H_{n,L}^{(N)}(x) \to g_L^{(N)}(x)G_L^{(N)}(x) \tag{5.239}$$

almost everywhere and with absolute value bounded by $N^2\chi_L$. Thus by dominated convergence $g_L^{(N)}G_L^{(N)} \in \mathcal{L}^1(\mathbb{R})$. Now, let $N \to \infty$; this sequence converges almost everywhere to $g_L(x)G_L(x)$ and we have the bound

$$|g_L^{(N)}(x)G_L^{(N)}(x)| \le |f(x)F(x)|\frac{1}{2}(f^2 + F^2) \tag{5.240}$$

so as always by dominated convergence, the limit $g_LG_L \in \mathcal{L}^1$. Finally, letting $L \to \infty$ the same argument shows that $fF \in \mathcal{L}^1(\mathbb{R})$. Moreover, $|fF| \in \mathcal{L}^1(\mathbb{R})$ and

$$|\int fF| \le \int |fF| \le \|f\|_{L^2}\|F\|_{L^2} \tag{5.241}$$

where the last inequality follows from Cauchy's inequality – if you wish, first for the approximating sequences and then taking limits.

(6) So if f, $g \in \mathcal{L}^2(\mathbb{R})$ are real-value, $f + g$ is certainly locally integrable and

$$(f + g)^2 = f^2 + 2fg + g^2 \in \mathcal{L}^1(\mathbb{R}) \tag{5.242}$$

by the discussion above. For constants $f \in \mathcal{L}^2(\mathbb{R})$ implies $cf \in \mathcal{L}^2(\mathbb{R})$ is directly true.

(7) The argument is the same as for $\mathcal{L}^1$ versus L^1. Namely $\int f^2 = 0$ implies that $f^2 = 0$ almost everywhere which is equivalent to $f = 0$ a@ė. Then the norm is the same for all $f + h$ where h is a null function since fh and h^2 are null so $(f + h)^2 = f^2 + 2fh + h^2$. The same is true for the inner product so it follows that the quotient by null functions

$$L^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R})/\mathcal{N} \tag{5.243}$$

is a preHilbert space.

However, it remains to show completeness. Suppose $\{[f_n]\}$ is an absolutely summable series in $L^2(\mathbb{R})$ which means that $\sum\limits_n \|f_n\|_{L^2} < \infty$. It follows that the cut-off series $f_n\chi_L$ is absolutely summable in the L^1 sense since

$$\int |f_n\chi_L| \le L^{\frac{1}{2}}(\int f_n^2)^{\frac{1}{2}} \tag{5.244}$$

by Cauchy's inequality. Thus if we set $F_n = \sum\limits_{k-1}^{n} f_k$ then $F_n(x)\chi_L$ converges almost everywhere for each L so in fact

$$F_n(x) \to f(x) \text{ converges almost everywhere.} \tag{5.245}$$

We want to show that $f \in \mathcal{L}^2(\mathbb{R})$ where it follows already that f is locally integrable by the completeness of L^1. Now consider the series

$$g_1 = F_1^2,\ g_n = F_n^2 - F_{n-1}^2. \tag{5.246}$$

The elements are in $\mathcal{L}^1(\mathbb{R})$ and by Cauchy's inequality for $n > 1$,

$$\begin{aligned}\int |g_n| = \int |F_n^2 - F_{n-1}|^2 \le \|F_n - F_{n-1}\|_{L^2}\|F_n + F_{n-1}\|_{L^2} \\ \le \|f_n\|_{L^2} 2\sum_k \|f_k\|_{L^2}\end{aligned} \tag{5.247}$$

where the triangle inequality has been used. Thus in fact the series g_n is absolutely summable in $\mathcal{L}^1$

$$\sum_n \int |g_n| \le 2(\sum_n \|f_n\|_{L^2})^2. \tag{5.248}$$

So indeed the sequence of partial sums, the F_n^2 converge to $f^2 \in \mathcal{L}^1(\mathbb{R})$. Thus $f \in \mathcal{L}^2(\mathbb{R})$ and moroever

$$\int (F_n - f)^2 = \int F_n^2 + \int f^2 - 2\int F_n f \to 0 \text{ as } n \to \infty. \tag{5.249}$$

Indeed the first term converges to $\int f^2$ and, by Cauchys inequality, the series of products $f_n f$ is absulutely summable in L^1 with limit f^2 so the third term converges to $-2\int f^2$. Thus in fact $[F_n] \to [f]$ in $L^2(\mathbb{R})$ and we have proved completeness.

(8) For the complex case we need to check linearity, assuming f is locally integrable and $|f|^2 \in \mathcal{L}^1(\mathbb{R})$. The real part of f is locally integrable and the approximation $F_L^{(N)}$ discussed above is square integrable with $(F_L^{(N)})^2 \le |f|^2$ so by dominated convergence, letting first $N \to \infty$ and then $L \to \infty$ the real part is in $\mathcal{L}^2(\mathbb{R})$. Now linearity and completeness follow from the real case.

Problem 5.4
Consider the sequence space

$$h^{2,1} = \left\{ c : \mathbb{N} \ni j \longmapsto c_j \in \mathbb{C}; \sum_j (1+j^2)|c_j|^2 < \infty \right\}. \tag{5.250}$$

(1) Show that

$$h^{2,1} \times h^{2,1} \ni (c,d) \longmapsto \langle c, d\rangle = \sum_j (1+j^2) c_j \overline{d_j} \tag{5.251}$$

is an Hermitian inner form which turns $h^{2,1}$ into a Hilbert space.

(2) Denoting the norm on this space by $\|\cdot\|_{2,1}$ and the norm on l^2 by $\|\cdot\|_2$, show that

$$h^{2,1} \subset l^2,\ \|c\|_2 \le \|c\|_{2,1}\ \forall\ c \in h^{2,1}. \tag{5.252}$$

Solution:

(1) The inner product is well defined since the series defining it converges absolutely by Cauchy's inequality:

$$\begin{gathered}\langle c, d\rangle = \sum_j (1+j^2)^{\frac12} c_j \overline{(1+j^2)^{\frac12} d_j}, \\ \sum_j |(1+j^2)^{\frac12} c_j \overline{(1+j^2)^{\frac12} d_j}| \le (\sum_j (1+j^2)|c_j|^2)^{\frac12} (\sum_j (1+j^2)|d_j|^2)^{\frac12}.\end{gathered} \tag{5.253}$$

It is sesquilinear and positive definite since

$$\|c\|_{2,1} = (\sum_j (1+j^2)|c_j|^2)^{\frac{1}{2}} \tag{5.254}$$

only vanishes if all c_j vanish. Completeness follows as for l^2 – if $c^{(n)}$ is a Cauchy sequence then each component $c_j^{(n)}$ converges, since $(1+j)^{\frac{1}{2}} c_j^{(n)}$ is Cauchy. The limits c_j define an element of $h^{2,1}$ since the sequence is bounded and

$$\sum_{j=1}^{N} (1+j^2)^{\frac{1}{2}} |c_j|^2 = \lim_{n\to\infty} \sum_{j=1}^{N} (1+j^2)|c_j^{(n)}|^2 \le A \tag{5.255}$$

where A is a bound on the norms. Then from the Cauchy condition $c^{(n)} \to c$ in $h^{2,1}$ by passing to the limit as $m \to \infty$ in $\|c^{(n)} - c^{(m)}\|_{2,1} \le \epsilon$.

(2) Clearly $h^{2,2} \subset l^2$ since for any finite N

$$\sum_{j=1}^{N} |c_j|^2 \sum_{j=1}^{N} (1+j)^2 |c_j|^2 \le \|c\|_{2,1}^2 \tag{5.256}$$

and we may pass to the limit as $N \to \infty$ to see that

$$\|c\|_{l^2} \le \|c\|_{2,1}. \tag{5.257}$$

Problem 5.5 In the separable case, prove Riesz Representation Theorem directly.

Choose an orthonormal basis $\{e_i\}$ of the separable Hilbert space H. Suppose $T : H \longrightarrow \mathbb{C}$ is a bounded linear functional. Define a sequence

$$w_i = \overline{T(e_i)},\ i \in \mathbb{N}. \tag{5.258}$$

(1) Now, recall that $|Tu| \le C\|u\|_H$ for some constant C. Show that for every finite N,

$$\sum_{j=1}^{N} |w_i|^2 \le C^2. \tag{5.259}$$

(2) Conclude that $\{w_i\} \in l^2$ and that

$$w = \sum_i w_i e_i \in H. \tag{5.260}$$

(3) Show that

$$T(u) = \langle u, w\rangle_H \ \forall\ u \in H \text{ and } \|T\| = \|w\|_H. \tag{5.261}$$

Solution:

(1) The finite sum $w_N = \sum\limits_{i=1}^{N} w_i e_i$ is an element of the Hilbert space with norm $\|w_N\|_N^2 = \sum\limits_{i=1}^{N} |w_i|^2$ by Bessel's identity. Expanding out

$$T(w_N) = T(\sum_{i=1}^{N} w_i e_i) = \sum_{i=1}^{n} w_i T(e_i) = \sum_{i=1}^{N} |w_i|^2 \tag{5.262}$$

and from the continuity of T,

$$|T(w_N)| \le C\|w_N\|_H \Longrightarrow \|w_N\|_H^2 \le C\|w_N\|_H \Longrightarrow \|w_N\|^2 \le C^2 \tag{5.263}$$

which is the desired inequality.

(2) Letting $N \to \infty$ it follows that the infinite sum converges and

$$\sum_i |w_i|^2 \le C^2 \Longrightarrow w = \sum_i w_i e_i \in H \tag{5.264}$$

since $\|w_N - w\| \le \sum\limits_{j>N} |w_i|^2$ tends to zero with N.

(3) For any $u \in H$ $u_N = \sum\limits_{i=1}^{N} \langle u, e_i\rangle e_i$ by the completness of the $\{e_i\}$ so from the continuity of T

$$\begin{aligned} T(u) = \lim_{N\to\infty} T(u_N) = \lim_{N\to\infty} \sum_{i=1}^{N} \langle u, e_i\rangle T(e_i) \\ = \lim_{N\to\infty} \sum_{i=1}^{N} \langle u, w_i e_i\rangle = \lim_{N\to\infty} \langle u, w_N\rangle = \langle u, w\rangle \end{aligned} \tag{5.265}$$

where the continuity of the inner product has been used. From this and Cauchy's inequality it follows that $\|T\| = \sup_{\|u\|_H=1} |T(u)| \le \|w\|$. The converse follows from the fact that $T(w) = \|w\|_H^2$.

Solution 21. *If $f \in L^1(\mathbb{R}^k \times \mathbb{R}^p)$ show that there exists a set of measure zero $E \subset \mathbb{R}^k$ such that*

$$x \in \mathbb{R}^k \setminus E \Longrightarrow g_x(y) = f(x, y) \text{ defines } g_x \in L^1(\mathbb{R}^p), \tag{5.266}$$

that $F(x) = \int g_x$ defines an element $F \in L^1(\mathbb{R}^k)$ and that

$$\int_{\mathbb{R}^k} F = \int_{\mathbb{R}^k \times \mathbb{R}^p} f. \tag{5.267}$$

Note: These identities are usually written out as an equality of an iterated integral and a 'regular' integral:

$$\int_{\mathbb{R}^k} \int_{\mathbb{R}^p} f(x, y) = \int f. \tag{5.268}$$

It is often used to 'exchange the order of integration' since the hypotheses are the same if we exchange the variables.

Solution. This is not hard but is a little tricky (I believe Fubini never understood what the fuss was about).

Certainly this result holds for step functions, since ultimately it reduces to the case of the characterisitic function for a 'rectrangle'.

In the general case we can take an absolutely summable sequence f_j of step functions summing to f

$$f(x, y) = \sum_j f_j(x, y) \text{ whenever } \sum_j |f_j(x, y)| < \infty. \tag{5.269}$$

This, after all, is our definition of integrability.

Now, consider the functions

$$h_j(x) = \int_{\mathbb{R}^p} |f_j(x, \cdot)| \tag{5.270}$$

which are step functions. Moreover this series is absolutely summable since

$$\sum_j \int_{\mathbb{R}^k} |h_j| = \sum_j \int_{\mathbb{R}^k \times \mathbb{R}^p} |f_j|. \tag{5.271}$$

Thus the series $\sum_j h_j(x)$ converges (absolutely) on the complement of a set $E \subset \mathbb{R}^k$ of measure zero. It follows that the series of step functions

$$F_j(x) = \int_{\mathbb{R}^p} f_j(x, \cdot) \tag{5.272}$$

converges absolutely on $\mathbb{R}^k \setminus E$ since $|f_j(x)| \le h_j(x)$. Thus,

$$F(x) = \sum_j F_j(x) \text{ converges absolutely on } \mathbb{R}^k \setminus E \tag{5.273}$$

defines $F \in L^1(\mathbb{R}^k)$ with

$$\int_{\mathbb{R}^k} F = \sum_j \int_{\mathbb{R}^k} F_j = \sum_j \int_{\mathbb{R}^k \times \mathbb{R}^p} f_j = \int_{\mathbb{R}^k \times \mathbb{R}^p} f. \tag{5.274}$$

The absolute convergence of $\sum_j h_j(x)$ for a given x is precisely the absolutely summability of $f_k(x, y)$ as a series of functions of y,

$$\sum_j \int_{\mathbb{R}^p} |f_j(x, \cdot)| = \sum_j h_j(x). \tag{5.275}$$

Thus for each $x \notin E$ the series $\sum_j f_k(x, y)$ must converge absolutely for $y \in (\mathbb{R}^p \setminus E_x)$ where E_x is a set of measure zero. But (5.269) shows that the sum is $g_x(y) = f(x, y)$ at all such points, so for $x \notin E$, $f(x, \cdot) \in L^1(\mathbb{R}^p)$ (as the limit of an absolutely summable series) and

$$F(x) = \int_{\mathbb{R}^p} g_x. \tag{5.276}$$

With (5.274) this is what we wanted to show. □

Problem 4.1

Let H be a normed space in which the norm satisfies the parallelogram law:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2) \ \forall \ u, v \in H. \tag{5.277}$$

Show that the norm comes from a positive definite sesquilinear (i.e. Hermitian) inner product. Big Hint:- Try

$$(u, v) = \frac{1}{4} \left(\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2 \right)! \tag{5.278}$$

Solution: Setting $u = v$, even without the parallelogram law,

$$(u, u) = \frac{1}{4} \left\|2u\|^2 + i\|(1 + i)u\|^2 - i\|(1 - i)u\|^2 \right) = \|u\|^2. \tag{5.279}$$

So the point is that the parallelogram law shows that (u, v) is indeed an Hermitian inner product. Taking complex conjugates and using properties of the norm, $\|u + iv\| = \|v - iu\|$ etc

$$\overline{(u, v)} = \frac{1}{4} \left(\|v + u\|^2 - \|v - u\|^2 - i\|v - iu\|^2 + i\|v + iu\|^2 \right) = (v, u). \tag{5.280}$$

Thus we only need check the linearity in the first variable. This *is* a little tricky! First compute away. Directly from the identity $(u, -v) = -(u, v)$ so $(-u, v) = -(u, v)$ using (5.280). Now,

(5.281)

$$\begin{aligned}(2u, v) = &\frac{1}{4}\big(\|u + (u + v)\|^2 - \|u + (u - v)\|^2\\
&+ i\|u + (u + iv)\|^2 - i\|u + (u - iv)\|^2\big)\\
=&\frac{1}{2}\big(\|u + v\|^2 + \|u\|^2 - \|u - v\|^2 - \|u\|^2\\
&+ i\|(u + iv)\|^2 + i\|u\|^2 - i\|u - iv\|^2 - i\|u\|^2\big)\\
&- \frac{1}{4}\left(\|u - (u + v)\|^2 - \|u - (u - v)\|^2 + i\|u - (u + iv)\|^2 - i\|u - (u - iv)\|^2\right)\\
=&2(u, v).\end{aligned}$$

Using this and (5.280), for any u, u' and v,

$$\begin{aligned}(u + u', v) = &\frac{1}{2}(u + u', 2v)\\
=&\frac{1}{2}\frac{1}{4}\big(\|(u + v) + (u' + v)\|^2 - \|(u - v) + (u' - v)\|^2\\
&+ i\|(u + iv) + (u - iv)\|^2 - i\|(u - iv) + (u' - iv)\|^2\big)\\
=&\frac{1}{4}\big(\|u + v\| + \|u' + v\|^2 - \|u - v\| - \|u' - v\|^2\\
&+ i\|(u + iv)\|^2 + i\|u - iv\|^2 - i\|u - iv\| - i\|u' - iv\|^2\big)\\
&- \frac{1}{2}\frac{1}{4}\big(\|(u + v) - (u' + v)\|^2 - \|(u - v) - (u' - v)\|^2\\
&+ i\|(u + iv) - (u - iv)\|^2 - i\|(u - iv) = (u' - iv)\|^2\big)\\
&= (u, v) + (u', v).\end{aligned}\tag{5.282}$$

Using the second identity to iterate the first it follows that $(ku, v) = k(u, v)$ for any u and v and any positive integer k. Then setting $nu' = u$ for any other positive integer and $r = k/n$, it follows that

$$(ru, v) = (ku', v) = k(u', v) = rn(u', v) = r(u, v)\tag{5.283}$$

where the identity is reversed. Thus it follows that $(ru, v) = r(u, v)$ for any rational r. Now, from the definition both sides are continuous in the first element, with respect to the norm, so we can pass to the limit as $r \to x$ in $\mathbb{R}$. Also directly from the definition,

$$(iu, v) = \frac{1}{4}\left(\|iu + v\|^2 - \|iu - v\|^2 + i\|iu + iv\|^2 - i\|iu - iv\|^2\right) = i(u, v)\tag{5.284}$$

so now full linearity in the first variable follows and that is all we need.

Problem 4.2

Let H be a finite dimensional (pre)Hilbert space. So, by definition H has a basis $\{v_i\}_{i=1}^n$, meaning that any element of H can be written

$$v = \sum_i c_i v_i\tag{5.285}$$

and there is no dependence relation between the v_i's – the presentation of $v = 0$ in the form (5.285) is unique. Show that H has an orthonormal basis, $\{e_i\}_{i=1}^n$ satisfying $(e_i, e_j) = \delta_{ij}$ $(= 1$ if $i = j$ and 0 otherwise). Check that for the orthonormal basis the coefficients in (5.285) are $c_i = (v, e_i)$ and that the map

$$T : H \ni v \longmapsto ((v, e_i)) \in \mathbb{C}^n \tag{5.286}$$

is a linear isomorphism with the properties

$$(u, v) = \sum_i (Tu)_i \overline{(Tv)_i},\ \ \|u\|_H = \|Tu\|_{\mathbb{C}^n}\ \forall\ u, v \in H. \tag{5.287}$$

Why is a finite dimensional preHilbert space a Hilbert space?

Solution: Since H is assumed to be finite dimensional, it has a basis v_i, $i = 1, \ldots, n$. This basis can be replaced by an orthonormal basis in n steps. First replace v_1 by $e_1 = v_1/\|v_1\|$ where $\|v_1\| \neq 0$ by the linear indepedence of the basis. Then replace v_2 by

$$e_2 = w_2/\|w_2\|,\ w_2 = v_2 - (v_2, e_1)e_1. \tag{5.288}$$

Here $w_2 \perp e_1$ as follows by taking inner products; w_2 cannot vanish since v_2 and e_1 must be linearly independent. Proceeding by finite induction we may assume that we have replaced v_1, v_2, $\ldots$, v_k, $k < n$, by e_1, e_2, $\ldots$, e_k which are orthonormal and span the same subspace as the v_i's $i = 1, \ldots, k$. Then replace v_{k+1} by

$$e_{k+1} = w_{k+1}/\|w_{k+1}\|,\ w_{k+1} = v_{k+1} - \sum_{i=1}^k (v_{k+1}, e_i)e_i. \tag{5.289}$$

By taking inner products, $w_{k+1} \perp e_i$, $i = 1, \ldots, k$ and $w_{k+1} \neq 0$ by the linear independence of the v_i's. Thus the orthonormal set has been increased by one element preserving the same properties and hence the basis can be orthonormalized.

Now, for each $u \in H$ set

$$c_i = (u, e_i). \tag{5.290}$$

It follows that $U = u - \sum\limits_{i=1}^n c_i e_i$ is orthogonal to all the e_i since

$$(u, e_j) = (u, e_j) - \sum_i c_i (e_i, e_j) = (u.e_j) - c_j = 0. \tag{5.291}$$

This implies that $U = 0$ since writing $U = \sum\limits_i d_i e_i$ it follows that $d_i = (U, e_i) = 0$.

Now, consider the map (5.286). We have just shown that this map is injective, since $Tu = 0$ implies $c_i = 0$ for all i and hence $u = 0$. It is linear since the c_i depend linearly on u by the linearity of the inner product in the first variable. Moreover it is surjective, since for any $c_i \in \mathbb{C}$, $u = \sum\limits_i c_i e_i$ reproduces the c_i through (5.290). Thus T is a linear isomorphism and the first identity in (5.287) follows by direct computation:-

$$\begin{aligned} \sum_{i=1}^n (Tu)_i \overline{(Tv)_i} &= \sum_i (u, e_i) \\ &= (u, \sum_i (v, e_i) e_i) \\ &= (u, v). \end{aligned} \tag{5.292}$$

Setting $u = v$ shows that $\|Tu\|_{\mathbb{C}^n} = \|u\|_H$.

Now, we know that $\mathbb{C}^n$ is complete with its standard norm. Since T is an isomorphism, it carries Cauchy sequences in H to Cauchy sequences in $\mathbb{C}^n$ and T^{-1} carries convergent sequences in $\mathbb{C}^n$ to convergent sequences in H, so every Cauchy sequence in H is convergent. Thus H is complete.

Hint: Don't pay too much attention to my hints, sometimes they are a little off-the-cuff and may not be very helpfult. An example being the old hint for Problem 6.2!

Problem 6.1 Let H be a separable Hilbert space. Show that $K \subset H$ is compact if and only if it is closed, bounded and has the property that any sequence in K which is weakly convergent sequence in H is (strongly) convergent.

Hint:- In one direction use the result from class that any bounded sequence has a weakly convergent subsequence.

Problem 6.2 Show that, in a separable Hilbert space, a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if the weak limit, v satisfies

$$\|v\|_H = \lim_{n\to\infty} \|v_n\|_H. \tag{5.293}$$

Hint:- To show that this condition is sufficient, expand

$$(v_n - v, v_n - v) = \|v_n\|^2 - 2\operatorname{Re}(v_n, v) + \|v\|^2. \tag{5.294}$$

Problem 6.3 Show that a subset of a separable Hilbert space is compact if and only if it is closed and bounded and has the property of 'finite dimensional approximation' meaning that for any $\epsilon > 0$ there exists a linear subspace $D_N \subset H$ of finite dimension such that

$$d(K, D_N) = \sup_{u\in K} \inf_{v\in D_N} \{d(u, v)\} \le \epsilon. \tag{5.295}$$

Hint:- To prove necessity of this condition use the 'equi-small tails' property of compact sets with respect to an orthonormal basis. To use the finite dimensional approximation condition to show that any weakly convergent sequence in K is strongly convergent, use the convexity result from class to define the sequence $\{v'_n\}$ in D_N where v'_n is the closest point in D_N to v_n. Show that v'_n is weakly, hence strongly, convergent and hence deduce that $\{v_n\}$ is Cauchy.

Problem 6.4 Suppose that $A : H \longrightarrow H$ is a bounded linear operator with the property that $A(H) \subset H$ is finite dimensional. Show that if v_n is weakly convergent in H then Av_n is strongly convergent in H.

Problem 6.5 Suppose that H_1 and H_2 are two different Hilbert spaces and $A : H_1 \longrightarrow H_2$ is a bounded linear operator. Show that there is a unique bounded linear operator (the adjoint) $A^* : H_2 \longrightarrow H_1$ with the property

$$(Au_1, u_2)_{H_2} = (u_1, A^*u_2)_{H_1} \ \forall\ u_1 \in H_1,\ u_2 \in H_2. \tag{5.296}$$

Problem 8.1 Show that a continuous function $K : [0, 1] \longrightarrow L^2(0, 2\pi)$ has the property that the Fourier series of $K(x) \in L^2(0, 2\pi)$, for $x \in [0, 1]$, converges uniformly in the sense that if $K_n(x)$ is the sum of the Fourier series over $|k| \le n$ then $K_n : [0, 1] \longrightarrow L^2(0, 2\pi)$ is also continuous and

$$\sup_{x\in[0,1]} \|K(x) - K_n(x)\|_{L^2(0,2\pi)} \to 0. \tag{5.297}$$

Hint. Use one of the properties of compactness in a Hilbert space that you proved earlier.

Problem 8.2

Consider an integral operator acting on $L^2(0,1)$ with a kernel which is continuous – $K \in \mathcal{C}([0,1]^2)$. Thus, the operator is

$$Tu(x) = \int_{(0,1)} K(x,y)u(y). \tag{5.298}$$

Show that T is bounded on L^2 (I think we did this before) and that it is in the norm closure of the finite rank operators.

Hint. Use the previous problem! Show that a continuous function such as K in this Problem defines a continuous map $[0,1] \ni x \longmapsto K(x,\cdot) \in \mathcal{C}([0,1])$ and hence a continuous function $K : [0,1] \longrightarrow L^2(0,1)$ then apply the previous problem with the interval rescaled.

Here is an even more expanded version of the hint: You can think of $K(x,y)$ as a continuous function of x with values in $L^2(0,1)$. Let $K_n(x,y)$ be the continuous function of x and y given by the previous problem, by truncating the Fourier series (in y) at some point n. Check that this defines a finite rank operator on $L^2(0,1)$ – yes it maps into continuous functions but that is fine, they are Lebesgue square integrable. Now, the idea is the difference $K - K_n$ defines a bounded operator with small norm as n becomes large. It might actually be clearer to do this the other way round, exchanging the roles of x and y.

Problem 8.3 Although we have concentrated on the Lebesgue integral in one variable, you proved at some point the covering lemma in dimension 2 and that is pretty much all that was needed to extend the discussion to 2 dimensions. Let's just assume you have assiduously checked everything and so you know that $L^2((0,2\pi)^2)$ is a Hilbert space. Sketch a proof – noting anything that you are not sure of – that the functions $\exp(ikx + ily)/2\pi$, $k, l \in \mathbb{Z}$, form a complete orthonormal basis.

P9.1: Periodic functions

Let $\mathbb{S}$ be the circle of radius 1 in the complex plane, centered at the origin, $\mathbb{S} = \{z; |z| = 1\}$.

(1) Show that there is a 1-1 correspondence

$$\begin{aligned}\mathcal{C}^0(\mathbb{S}) = \{u : \mathbb{S} \longrightarrow \mathbb{C},\ \text{continuous}\} \longrightarrow \\ \{u : \mathbb{R} \longrightarrow \mathbb{C};\ \text{continuous and satisfying } u(x+2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}.\end{aligned} \tag{5.299}$$

Solution: The map $E : \mathbb{R} \ni \theta \longmapsto e^{2\pi i\theta} \in \mathbb{S}$ is continuous, surjective and 2π-periodic and the inverse image of any point of the circle is precisly of the form $\theta + 2\pi\mathbb{Z}$ for some $\theta \in \mathbb{R}$. Thus composition defines a map

$$E^* : \mathcal{C}^0(\mathbb{S}) \longrightarrow \mathcal{C}^0(\mathbb{R}),\ E^*f = f \circ E. \tag{5.300}$$

This map is a linear bijection.

(2) Show that there is a 1-1 correspondence

$$\begin{aligned}L^2(0,2\pi) \longleftrightarrow \{u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}); u\big|_{(0,2\pi)} \in \mathcal{L}^2(0,2\pi) \\ \text{and } u(x+2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}/\mathcal{N}_P\end{aligned} \tag{5.301}$$

where $\mathcal{N}_P$ is the space of null functions on $\mathbb{R}$ satisfying $u(x+2\pi) = u(x)$ for all $x \in \mathbb{R}$.

Solution: Our original definition of $L^2(0,2\pi)$ is as functions on $\mathbb{R}$ which are square-integrable and vanish outside $(0,2\pi)$. Given such a function u we can define an element of the right side of (5.301) by assigning a value at 0 and then extending by periodicity

$$\tilde{u}(x) = u(x - 2n\pi),\ n \in \mathbb{Z} \tag{5.302}$$

where for each $x \in \mathbb{R}$ there is a unique integer n so that $x - 2n\pi \in [0, 2\pi)$. Null functions are mapped to null functions his way and changing the choice of value at 0 changes $\tilde{u}$ by a null function. This gives a map as in (5.301) and restriction to $(0, 2\pi)$ is a 2-sided invese.

(3) If we denote by $L^2(\mathbb{S})$ the space on the left in (5.301) show that there is a dense inclusion

$$\mathcal{C}^0(\mathbb{S}) \longrightarrow L^2(\mathbb{S}). \tag{5.303}$$

Solution: Combining the first map and the inverse of the second gives an inclusion. We know that continuous functions vanishing near the endpoints of $(0, 2\pi)$ are dense in $L^2(0, 2\pi)$ so density follows.

So, the idea is that we can freely think of functions on $\mathbb{S}$ as 2π-periodic functions on $\mathbb{R}$ and conversely.

P9.2: Schrödinger's operator

Since that is what it is, or at least it is an example thereof:

$$-\frac{d^2u(x)}{dx^2} + V(x)u(x) = f(x),\ x \in \mathbb{R}, \tag{5.304}$$

(1) First we will consider the special case $V = 1$. Why not $V = 0$? – Don't try to answer this until the end!

Solution: The reason we take $V = 1$, or at least some other positive constant is that there is 1-d space of periodic solutions to $d^2u/dx^2 = 0$, namely the constants.

(2) Recall how to solve the differential equation

$$-\frac{d^2u(x)}{dx^2} + u(x) = f(x),\ x \in \mathbb{R}, \tag{5.305}$$

where $f(x) \in \mathcal{C}^0(\mathbb{S})$ is a continuous, 2π-periodic function on the line. Show that there is a unique 2π-periodic and twice continuously differentiable function, u, on $\mathbb{R}$ satisfying (5.305) and that this solution can be written in the form

$$u(x) = (Sf)(x) = \int_{0,2\pi} A(x, y)f(y) \tag{5.306}$$

where $A(x, y) \in \mathcal{C}^0(\mathbb{R}^2)$ satisfies $A(x+2\pi, y+2\pi) = A(x, y)$ for all $(x, y) \in \mathbb{R}$.

Extended hint: In case you managed to avoid a course on differential equations! First try to find a solution, igonoring the periodicity issue. To do so one can (for example, there are other ways) factorize the differential operator involved, checking that

$$-\frac{d^2u(x)}{dx^2} + u(x) = -(\frac{dv}{dx} + v) \text{ if } v = \frac{du}{dx} - u \tag{5.307}$$

since the cross terms cancel. Then recall the idea of integrating factors to see that

$$\begin{aligned} \frac{du}{dx} - u &= e^x \frac{d\phi}{dx},\ \phi = e^{-x}u, \\ \frac{dv}{dx} + v &= e^{-x}\frac{d\psi}{dx},\ \psi = e^x v. \end{aligned} \tag{5.308}$$

Now, solve the problem by integrating twice from the origin (say) and hence get a solution to the differential equation (5.305). Write this out explicitly as a double integral, and then change the order of integration to write the solution as

$$u'(x) = \int_{0,2\pi} A'(x,y)f(y)dy \tag{5.309}$$

where A' is continuous on $\mathbb{R}\times[0,2\pi]$. Compute the difference $u'(2\pi)-u'(0)$ and $\frac{du'}{dx}(2\pi) - \frac{du'}{dx}(0)$ as integrals involving f. Now, add to u' as solution to the *homogeneous* equation, for $f=0$, namely $c_1e^x + c_2e^{-x}$, so that the new solution to (5.305) satisfies $u(2\pi) = u(0)$ and $\frac{du}{dx}(2\pi) = \frac{du}{dx}(0)$. Now, check that u is given by an integral of the form (5.306) with A as stated.

Solution: Integrating once we find that if $v = \frac{du}{dx} - u$ then

$$v(x) = -e^{-x}\int_0^x e^s f(s)ds,\ u'(x) = e^x\int_0^x e^{-t}v(t)dt \tag{5.310}$$

gives a solution of the equation $-\frac{d^2u'}{dx^2} + u'(x) = f(x)$ so combinging these two and changing the order of integration

$$\begin{aligned} u'(x) &= \int_0^x \tilde{A}(x,y)f(y)dy,\ \tilde{A}(x,y) = \frac{1}{2}\left(e^{y-x} - e^{x-y}\right) \\ u'(x) &= \int_{(0,2\pi)} A'(x,y)f(y)dy,\ A'(x,y) = \begin{cases} \frac{1}{2}\left(e^{y-x} - e^{x-y}\right) & x \ge y \\ 0 & x \le y. \end{cases} \end{aligned} \tag{5.311}$$

Here A' is continuous since $\tilde{A}$ vanishes at $x=y$ where there might otherwise be a discontinuity. This is the only solution which vanishes with its derivative at 0. If it is to extend to be periodic we need to add a solution of the homogeneous equation and arrange that

$$u = u' + u'',\ u'' = ce^x + de^{-x},\ u(0) = u(2\pi),\ \frac{du}{dx}(0) = \frac{du}{dx}(2\pi). \tag{5.312}$$

So, computing away we see that

$$u'(2\pi) = \int_0^{2\pi}\frac{1}{2}\left(e^{y-2\pi} - e^{2\pi-y}\right)f(y),\ \frac{du'}{dx}(2\pi) = -\int_0^{2\pi}\frac{1}{2}\left(e^{y-2\pi} + e^{2\pi-y}\right)f(y). \tag{5.313}$$

Thus there is a unique solution to (5.312) which must satify

$$\begin{aligned} c(e^{2\pi}-1) + d(e^{-2\pi}-1) &= -u'(2\pi),\ c(e^{2\pi}-1) - d(e^{-2\pi}-1) = -\frac{du'}{dx}(2\pi) \\ (e^{2\pi}-1)c &= \frac{1}{2}\int_0^{2\pi}\left(e^{2\pi-y}\right)f(y),\ (e^{-2\pi}-1)d = -\frac{1}{2}\int_0^{2\pi}\left(e^{y-2\pi}\right)f(y). \end{aligned} \tag{5.314}$$

Putting this together we get the solution in the desired form:

$$u(x)=\int_{(0.2\pi)}A(x,y)f(y),\ A(x,y)=A'(x,y)+\frac{1}{2}\frac{e^{2\pi-y+x}}{e^{2\pi}-1}-\frac{1}{2}\frac{e^{-2\pi+y-x}}{e^{-2\pi}-1}\Longrightarrow$$
$$A(x,y)=\frac{\cosh(|x-y|-\pi)}{e^{\pi}-e^{-\pi}}. \tag{5.315}$$

(3) Check, either directly or indirectly, that $A(y,x)=A(x,y)$ and that A is real.

Solution: Clear from (5.315).

(4) Conclude that the operator S extends by continuity to a bounded operator on $L^2(\mathbb{S})$.

Solution. We know that $\|S\|\le\sqrt{2\pi}\sup|A|$.

(5) Check, probably indirectly rather than directly, that

$$S(e^{ikx})=(k^2+1)^{-1}e^{ikx},\ k\in\mathbb{Z}. \tag{5.316}$$

Solution. We know that Sf is the unique solution with periodic boundary conditions and e^{ikx} satisfies the boundary conditions and the equation with $f=(k^2+1)e^{ikx}$.

(6) Conclude, either from the previous result or otherwise that S is a compact self-adjoint operator on $L^2(\mathbb{S})$.

Soluion: Self-adjointness and compactness follows from (5.316) since we know that the $e^{ikx}/\sqrt{2\pi}$ form an orthonormal basis, so the eigenvalues of S tend to 0. (Myabe better to say it is approximable by finite rank operators by truncating the sum).

(7) Show that if $g\in\mathcal{C}^0(\mathbb{S}))$ then Sg is twice continuously differentiable. Hint: Proceed directly by differentiating the integral.

Solution: Clearly Sf is continuous. Going back to the formula in terms of $u'+u''$ we see that both terms are twice continuously differentiable.

(8) From (5.316) conclude that $S=F^2$ where F is also a compact self-adjoint operator on $L^2(\mathbb{S})$ with eigenvalues $(k^2+1)^{-\frac{1}{2}}$.

Solution: Define $F(e^{ikx})=(k^2+1)^{-\frac{1}{2}}e^{ikx}$. Same argument as above applies to show this is compact and self-adjoint.

(9) Show that $F:L^2(\mathbb{S})\longrightarrow\mathcal{C}^0(\mathbb{S})$.

Solution. The series for Sf

$$Sf(x)=\frac{1}{2\pi}\sum_k(2k^2+1)^{-\frac{1}{2}}(f,e^{ikx})e^{ikx} \tag{5.317}$$

converges absolutely and uniformly, using Cauchy's inequality – for instance it is Cauchy in the supremum norm:

$$\|\sum_{|k|>p}(2k^2+1)^{-\frac{1}{2}}(f,e^{ikx})e^{ikx}|\le\epsilon\|f\|_{L^2} \tag{5.318}$$

for p large since the sum of the squares of the eigenvalues is finite.

(10) Now, going back to the real equation (5.304), we assume that V is continuous, real-valued and 2π-periodic. Show that if u is a twice-differentiable 2π-periodic function satisfying (5.304) for a given $f\in\mathcal{C}^0(\mathbb{S})$ then

$$u+S((V-1)u)=Sf\text{ and hence }u=-F^2((V-1)u)+F^2f \tag{5.319}$$

and hence conclude that

$$u = Fv \text{ where } v \in L^2(\mathbb{S}) \text{ satisfies } v + (F(V-1)F)v = Ff \tag{5.320}$$

where $V-1$ is the operator defined by multiplication by $V-1$.

Solution: If u satisfies (5.304) then

$$-\frac{d^2u(x)}{dx^2} + u(x) = -(V(x)-1)u(x) + f(x) \tag{5.321}$$

so by the uniqueness of the solution with periodic boundary conditions, $u = -S(V-1)u + Sf$ so $u = F(-F(V-1)u + Ff)$. Thus indeed $u = Fv$ with $v = -F(V-1)u + Ff$ which means that v satisfies

$$v + F(V-1)Fv = Ff. \tag{5.322}$$

(11) Show the converse, that if $v \in L^2(\mathbb{S})$ satisfies

$$v + (F(V-1)F)v = Ff,\ f \in \mathcal{C}^0(\mathbb{S}) \tag{5.323}$$

then $u = Fv$ is 2π-periodic and twice-differentiable on $\mathbb{R}$ and satisfies (5.304).

Solution. If $v \in L^2(0, 2\pi)$ satisfies (5.323) then $u = Fv \in \mathcal{C}^0(\mathbb{S})$ satisfies $u + F^2(V-1)u = F^2 f$ and since $F^2 = S$ maps $\mathcal{C}^0(\mathbb{S})$ into twice continuously differentiable functions it follows that u satisfies (5.304).

(12) Apply the Spectral theorem to $F(V-1)F$ (including why it applies) and show that there is a sequence λ_j in $\mathbb{R} \setminus \{0\}$ with $|\lambda_j| \to 0$ such that for all $\lambda \in \mathbb{C} \setminus \{0\}$, the equation

$$\lambda v + (F(V-1)F)v = g,\ g \in L^2(\mathbb{S}) \tag{5.324}$$

has a unique solution for every $g \in L^2(\mathbb{S})$ if and only if $\lambda \neq \lambda_j$ for any j.

Solution: We know that $F(V-1)F$ is self-adjoint and compact so $L^2(0.2\pi)$ has an orthonormal basis of eigenfunctions of $-F(V-1)F$ with eigenvalues λ_j. This sequence tends to zero and (5.324), for given $\lambda \in \mathbb{C} \setminus \{0\}$, if and only if has a solution if and only if it is an isomorphism, meaning $\lambda \neq \lambda_j$ is not an eigenvalue of $-F(V-1)F$.

(13) Show that for the λ_j the solutions of

$$\lambda_j v + (F(V-1)F)v = 0, v \in L^2(\mathbb{S}), \tag{5.325}$$

are all continuous 2π-periodic functions on $\mathbb{R}$.

Solution: If v satisfies (5.325) with $\lambda_j \neq 0$ then $v = -F(V-1)F/\lambda_j \in \mathcal{C}^0(\mathbb{S})$.

(14) Show that the corresponding functions $u = Fv$ where v satisfies (5.325) are all twice continuously differentiable, 2π-periodic functions on $\mathbb{R}$ satisfying

$$-\frac{d^2u}{dx^2} + (1 - s_j + s_j V(x))u(x) = 0,\ s_j = 1/\lambda_j. \tag{5.326}$$

Solution: Then $u = Fv$ satisfies $u = -S(V-1)u/\lambda_j$ so is twice continuously differentiable and satisfies (5.326).

(15) Conversely, show that if u is a twice continuously differentiable and 2π-periodic function satisfying

$$-\frac{d^2u}{dx^2} + (1 - s + sV(x))u(x) = 0,\ s \in \mathbb{C}, \tag{5.327}$$

and u is not identically 0 then $s = s_j$ for some j.

Solution: From the uniquess of periodic solutions $u = -S(V-1)u/\lambda_j$ as before.

(16) Finally, conclude that Fredholm's alternative holds for the equation in (5.304)

THEOREM 5.2. *For a given real-valued, continuous 2π-periodic function V on $\mathbb{R}$, either* (5.304) *has a unique twice continuously differentiable, 2π-periodic, solution for each f which is continuous and 2π-periodic or else there exists a finite, but positive, dimensional space of twice continuously differentiable 2π-periodic solutions to the homogeneous equation*

$$-\frac{d^2w(x)}{dx^2} + V(x)w(x) = 0, \ x \in \mathbb{R}, \tag{5.328}$$

and (5.304) *has a solution if and only if $\int_{(0,2\pi)} fw = 0$ for every 2π-periodic solution, w, to* (5.328).

Solution: This corresponds to the special case $\lambda_j = 1$ above. If λ_j is not an eigenvalue of $-F(V-1)F$ then

$$v + F(V-1)Fv = Ff \tag{5.329}$$

has a unque solution for all f, otherwise the necessary and sufficient condition is that $(v, Ff) = 0$ for all v' satisfying $v' + F(V-1)Fv' = 0$. Correspondingly either (5.304) has a unique solution for all f or the necessary and sufficient condition is that $(Fv', f) = 0$ for all $w = Fv'$ (remember that F is injetive) satisfying (5.328).

Problem P10.1 Let H be a separable, infinite dimensional Hilbert space. Show that the direct sum of two copies of H is a Hilbert space with the norm

$$H \oplus H \ni (u_1, u_2) \longmapsto (\|u_1\|_H^2 + \|u_2\|_H^2)^{\frac{1}{2}} \tag{5.330}$$

either by constructing an isometric isomorphism

$$T : H \longrightarrow H \oplus H, \text{ 1-1 and onto, } \|u\|_H = \|Tu\|_{H\oplus H} \tag{5.331}$$

or otherwise. In any case, construct a map as in (5.331).

Solution: Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of H, which exists by virtue of the fact that it is an infinite-dimensional but separable Hilbert space. Define the map

$$T : H \ni u \longrightarrow (\sum_{i=1}^{\infty}(u, e_{2i-1})e_i, \sum_{i=1}^{\infty}(u, e_{2i})e_i) \in H \oplus H \tag{5.332}$$

The convergence of the Fourier Bessel series shows that this map is well-defined and linear. Injectivity similarly follows from the fact that $Tu = 0$ in the image implies that $(u, e_i) = 0$ for all i and hence $u = 0$. Surjectivity is also clear from the fact that

$$S : H \oplus H \ni (u_1, u_2) \longmapsto \sum_{i=1}^{\infty}((u_1, e_i)e_{2i-1} + (u_2, e_i)e_{2i}) \in H \tag{5.333}$$

is a 2-sided inverse and Bessel's identity implies isometry since $\|S(u_1, u_2)\|^2 = \|u_1\|^2 + \|u_2\|^2$

Problem P10.2 One can repeat the preceding construction any finite number of times. Show that it can be done 'countably often' in the sense that if H is a separable, infinite dimensional, Hilbert space then

$$l_2(H) = \{u : \mathbb{N} \longrightarrow H; \|u\|^2_{l_2(H)} = \sum_i \|u_i\|^2_H < \infty\} \tag{5.334}$$

has a Hilbert space structure and construct an explicit isometric isomorphism from $l_2(H)$ to H.

Solution: A similar argument as in the previous problem works. Take an orthormal basis e_i for H. Then the elements $E_{i,j} \in l_2(H)$, which for each i, i consist of the sequences with 0 entries except the jth, which is e_i, given an orthonromal basis for $l_2(H)$. Orthormality is clear, since with the inner product is

$$(u, v)_{l_2(H)} = \sum_j (u_j, v_j)_H. \tag{5.335}$$

Completeness follows from completeness of the orthonormal basis of H since if $v = \{v_j\}$ $(v, E_{j,i}) = 0$ for all j implies $v_j = 0$ in H. Now, to construct an isometric isomorphism just choose an isomorphism $m : \mathbb{N}^2 \longrightarrow \mathbb{N}$ then

$$Tu = v,\ v_j = \sum_i (u, e_{m(i,j)})e_i \in H. \tag{5.336}$$

I would expect you to go through the argument to check injectivity, surjectivity and that the map is isometric.

Problem P10.3 Recall, or perhaps learn about, the winding number of a closed curve with values in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We take as given the following fact:[3] If $Q = [0, 1]^N$ and $f : Q \longrightarrow \mathbb{C}^*$ is continuous then for each choice of $b \in \mathbb{C}$ satisfying $\exp(2\pi ib) = f(0)$, there exists a unique continuous function $F : Q \longrightarrow \mathbb{C}$ satisfying

$$\exp(2\pi iF(q)) = f(q),\ \forall\ q \in Q \text{ and } F(0) = b. \tag{5.337}$$

Of course, you are free to change b to $b + n$ for any $n \in \mathbb{Z}$ but then F changes to $F + n$, just shifting by the same integer.

(1) Now, suppose $c : [0, 1] \longrightarrow \mathbb{C}^*$ is a closed curve – meaning it is continuous and $c(1) = c(0)$. Let $C : [0, 1] \longrightarrow \mathbb{C}$ be a choice of F for $N = 1$ and $f = c$. Show that the winding number of the closed curve c may be defined *unambiguously* as

$$\mathrm{wn}(c) = C(1) - C(0) \in \mathbb{Z}. \tag{5.338}$$

Solution: Let C', be another choice of F in this case. Now, $g(t) = C'(t) - C(t)$ is continuous and satisfies $\exp(2\pi g(t)) = 1$ for all $t \in [0, 1]$ so by the uniqueness must be constant, thus $C'(1) - C'(0) = C(1) - C(0)$ and the winding number is well-defined.

(2) Show that $\mathrm{wn}(c)$ is *constant under homotopy*. That is if $c_i : [0, 1] \longrightarrow \mathbb{C}^*$, $i = 1, 2$, are two closed curves so $c_i(1) = c_i(0)$, $i = 1, 2$, which are *homotopic* through closed curves in the sense that there exists $f : [0, 1]^2 \longrightarrow \mathbb{C}^*$ continuous and such that $f(0, x) = c_1(x)$, $f(1, x) = c_2(x)$ for all $x \in [0, 1]$ and $f(y, 0) = f(y, 1)$ for all $y \in [0, 1]$, then $\mathrm{wn}(c_1) = \mathrm{wn}(c_2)$.

Solution: Choose F using the 'fact' corresponding to this homotopy f. Since f is periodic in the second variable – the two curves $f(y, 0)$, and $f(y, 1)$ are the same – so by the uniquess $F(y, 0) - F(y, 1)$ must be constant, hence $\mathrm{wn}(c_2) = F(1, 1) - F(1, 0) = F(0, 1) - F(0, 0) = \mathrm{wn}(c_1)$.

[3] Of course, you are free to give a proof – it is not hard.

(3) Consider the closed curve $L_n : [0,1] \ni x \longmapsto e^{2\pi ix}\,\mathrm{Id}_{n\times n}$ of $n\times n$ matrices. Using the standard properties of the determinant, show that this curve is not homotopic to the identity through closed curves in the sense that there does not exist a continuous map $G : [0,1]^2 \longrightarrow \mathrm{GL}(n)$, with values in the invertible $n\times n$ matrices, such that $G(0,x) = L_n(x)$, $G(1,x) \equiv \mathrm{Id}_{n\times n}$ for all $x \in [0,1]$, $G(y,0) = G(y,1)$ for all $y \in [0,1]$.

Solution: The determinant is a continuous (actually it is analytic) map which vanishes precisely on non-invertible matrices. Moreover, it is given by the product of the eigenvalues so

$$\det(L_n) = \exp(2\pi ixn). \tag{5.339}$$

This is a periodic curve with winding number n since it has the 'lift' xn. Now, if there were to exist such an homotopy of periodic curves of matrices, always invertible, then by the previous result the winding number of the determinant would have to remain constant. Since the winding number for the constant curve with value the identity is 0 such an homotopy cannot exist.

Problem P10.4 Consider the closed curve corresponding toL_n above in the case of a separable but now infinite dimensional Hilbert space:

$$L : [0,1] \ni x \longmapsto e^{2\pi ix}\,\mathrm{Id}_H \in \mathrm{GL}(H) \subset \mathcal{B}(H) \tag{5.340}$$

taking values in the invertible operators on H. Show that after identifying H with $H \oplus H$ as above, there is a continuous map

$$M : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H) \tag{5.341}$$

with values in the invertible operators and satisfying

$$M(0,x) = L(x),\ M(1,x)(u_1,u_2) = (e^{4\pi ix}u_1, u_2),\ M(y,0) = M(y,1),\ \forall\, x,y \in [0,1]. \tag{5.342}$$

Hint: So, think of $H \oplus H$ as being 2-vectors (u_1,u_2) with entries in H. This allows one to think of 'rotation' between the two factors. Indeed, show that

$$U(y)(u_1,u_2) = (\cos(\pi y/2)u_1 + \sin(\pi y/2)u_2, -\sin(\pi y/2)u_1 + \cos(\pi y/2)u_2) \tag{5.343}$$

defines a continuous map $[0,1] \ni y \longmapsto U(y) \in \mathrm{GL}(H \oplus H)$ such that $U(0) = \mathrm{Id}$, $U(1)(u_1,u_2) = (u_2, -u_1)$. Now, consider the 2-parameter family of maps

$$U^{-1}(y)V_2(x)U(y)V_1(x) \tag{5.344}$$

where $V_1(x)$ and $V_2(x)$ are defined on $H \oplus H$ as multiplication by $\exp(2\pi ix)$ on the first and the second component respectively, leaving the other fixed.

Solution: Certainly $U(y)$ is invertible since its inverse is $U(-y)$ as follows in the two dimensional case. Thus the map $W(x,y)$ on $[0,1]^2$ in (5.344) consists of invertible and bounded operators on $H \oplus H$, meaning a continuous map $W : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H)$. When $x = 0$ or $x = 1$, both $V_1(x)$ and $v_2(x)$ reduce to the identiy, and hence $W(0,y) = W(1,y)$ for all y, so W is periodic in x. Moreove at $y = 0$ $W(x,0) = V_2(x)V_1(x)$ is exactly $L(x)$, a multiple of the identity. On the other hand, at $x = 1$ we can track composite as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} e^{2\pi ix}u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} u_2 \\ -e^{2\pi x}u_1 \end{pmatrix} \longmapsto \begin{pmatrix} u_2 \\ -e^{4\pi x}u_1 \end{pmatrix} \longmapsto \begin{pmatrix} e^{4\pi x}u_1 \\ u_2 \end{pmatrix}. \tag{5.345}$$

This is what is required of M in (5.342).

Problem P10.5 Using a rotation similar to the one in the preceeding problem (or otherwise) show that there is a continuous map

$$G : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H) \tag{5.346}$$

such that

$$\begin{aligned} G(0,x)(u_1,u_2) &= (e^{2\pi i x}u_1, e^{-2\pi i x}u_2), \\ G(1,x)(u_1,u_2) &= (u_1,u_2),\ G(y,0) = G(y,1)\ \forall\ x,y \in [0,1]. \end{aligned} \tag{5.347}$$

Solution: We can take

$$G(y,x) = U(-y)\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & e^{-2\pi i x} \end{pmatrix} U(y) \begin{pmatrix} e^{2\pi i x} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{5.348}$$

By the same reasoning as above, this is an homotopy of closed curves of invertible operators on $H \oplus H$ which satisfies (5.347).

Problem P10.6 Now, think about combining the various constructions above in the following way. Show that on $l_2(H)$ there is an homotopy like (5.346), $\tilde G : [0,1]^2 \longrightarrow \mathrm{GL}(l_2(H))$, (very like in fact) such that

$$\begin{aligned} \tilde G(0,x)\left\{u_k\right\}_{k=1}^{\infty} &= \left\{\exp((-1)^k 2\pi i x)u_k\right\}_{k=1}^{\infty}, \\ \tilde G(1,x) &= \mathrm{Id},\ \tilde G(y,0) = \tilde G(y,1)\ \forall\ x,y \in [0,1]. \end{aligned} \tag{5.349}$$

Solution: We can divide $l_2(H)$ into its odd an even parts

$$D : l_2(H) \ni v \longmapsto (\{v_{2i-1}\}, \{v_{2i}\}) \in l_2(H) \oplus l_2(H) \longleftrightarrow H \oplus H. \tag{5.350}$$

and then each copy of $l_2(H)$ on the right with H (using the same isometric isomorphism). Then the homotopy in the previous problem is such that

$$\tilde G(x,y) = D^{-1} G(y,x) D \tag{5.351}$$

accomplishes what we want.

Problem P10.7: Eilenberg's swindle For any separable, infinite-dimensional, Hilbert space, construct an homotopy – meaning a continuous map $G : [0,1]^2 \longrightarrow \mathrm{GL}(H)$ – with $G(0,x) = L(x)$ in (5.340) and $G(1,x) = \mathrm{Id}$ and of course $G(y,0) = G(y,1)$ for all $x,y \in [0,1]$.

Hint: Just put things together – of course you can rescale the interval at the end to make it all happen over $[0,1]$. First 'divide H into 2 copies of itself' and deform from L to $M(1,x)$ in (5.342). Now, 'divide the second H up into $l_2(H)$' and apply an argument just like the preceding problem to turn the identity on this factor into alternating terms multiplying by $\exp(\pm 4\pi i x)$ – starting with $-$. Now, you are on $H \oplus l_2(H)$, 'renumbering' allows you to regard this as $l_2(H)$ again and when you do so your curve has become alternate multiplication by $\exp(\pm 4\pi i x)$ (with $+$ first). Finally then, apply the preceding problem again, to deform to the identity (always of course through closed curves). Presto, Eilenberg's swindle!

Solution: By rescaling the variables above, we now have three homotopies, always through periodic families. On $H \oplus H$ between $L(x) = e^{2\pi i x}\,\mathrm{Id}$ and the matrix

$$\begin{pmatrix} e^{4\pi i x}\,\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{5.352}$$

Then on $H \oplus l_2(H)$ we can deform from

$$\begin{pmatrix} e^{4\pi ix}\,\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \text{ to } \begin{pmatrix} e^{4\pi ix}\,\mathrm{Id} & 0 \\ 0 & \tilde{G}(0,x) \end{pmatrix} \tag{5.353}$$

with $\tilde{G}(0,x)$ in (5.349). However we can then identify

$$H \oplus l_2(H) = l_2(H),\ (u,v) \longmapsto w = \{w_j\},\ w_1 = u,\ w_{j+1} = v_j,\ j \geq 1. \tag{5.354}$$

This turns the matrix of operators in (5.353) into $\tilde{G}(0,x)^{-1}$. Now, we can apply the same construction to deform this curve to the identity. Notice that this really does ultimately give an homotopy, which we can renormalize to be on $[0,1]$ if you insist, of curves of operators on H – at each stage we transfer the homotopy back to H.

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