THE PROBLEMS FOR THE SECOND TEST FOR 18.102 BRIEF SOLUTIONS

RICHARD MELROSE

Question.1

Show that a subset of a separable Hilbert space is compact if and only if it is closed and bounded and has the property of 'finite dimensional approximation' meaning that for any $\epsilon > 0$ there exists a linear subspace $D_N \subset H$ of finite dimension such that

$$d(K, D_N) = \sup_{u \in K} \inf_{v \in D_N} \{d(u, v)\} \le \epsilon. \tag{1}$$

Solution: A compact set K is closed and bounded and has the 'equi-small tails' property with respect to any onb. So, given $\epsilon > 0$ then exists N such that

$$\sum_{j > N} |\langle u, e_j \rangle|^2 < \epsilon^2 \ \forall \ u \in K. \tag{2}$$

Let D_N be the linear space spanned by the e_j for $j \le N$ then if the orthogonal decomposition $H = D_N \oplus D_N^\perp$ gives $u = u' + u''$ and the distance $d(u, D_N) = \|u''\|$ so the finite dimensional approximation property follows from (2).

Conversely, suppose that K is closed and bounded and has the FDAP. By the assumed boundedness, any sequence in K has a weakly convergent subsequence. Denote such a sequence u_n. For each D_N let P_N be the orthogonal projection onto D_N. The sequence $P_N u_n$ is weakly convergent in D_N (because bounded maps preserve weak convergence) but since this is finite dimensional, $P_N u_n$ is convergent, and hence Cauchy. If D_N satisfies (1) for $\epsilon/3$, and n is large enough then for $m \ge n$,

$$\|u_n - u_m\| \le \|(\mathrm{Id} - P_N)u_n\| + \|P_N u_n - P_N u_m\| + \|(\mathrm{Id} - P_N)u_m\| < \epsilon, \tag{3}$$

and it follows that u_n is Cauchy, hence convergent with limit in K. Note that $\|(\mathrm{Id} - P_N)u_n\| = d(u_n, D_N)$. Thus K is compact since every sequence in it has a convergent subsequence.

Question.2

Strong convergence of a sequence of bounded operators $A_n \in \mathcal{B}(H)$ means that for each $u \in H$, $A_n u$ converges in H. Show that $Au = \lim_n A_n u$ is necessarily a bounded linear operator on H (called the strong limit of the sequence).

Solution: By assumption, $A_n u$ converges for each u so by the uniform boundedness principle $\|A_n\|$ is bounded. Define $Au = \lim_{n\to\infty} A_n u$. This is a linear map $A : H \longrightarrow H$, since $A_n(au + bv) \to aAu + bAv$. Moreover, if $\|u\| \le 1$ and C is an upper bound for the $\|A_n\|$ then $\|Au\| \le C$, as the limit if a sequence with $\|A_n u\| \le C$. Thus A is bounded.

Queston.3

If H is a separable, infinite dimensional, Hilbert space set

$$l^2(H) = \{u : \mathbb{N} \longrightarrow H; \|u\|^2_{l^2(H)} = \sum_i \|u_i\|^2_H < \infty\}. \tag{4}$$

Show that $l^2(H)$ has a Hilbert space structure and construct an explicit isometric (norm-preserving) isomorphism (bijection) from $l^2(H)$ to H.

Solution: The inner product

$$\langle u, v\rangle = \sum_j \langle u_j, v_j\rangle_H,\ u, v \in l^2(H) \tag{5}$$

is well-defined since the series converges absolutely

$$\sum_j |\langle u_j, v_j\rangle_H| \le \frac{1}{2}\left(\sum_j \|u_j\|^2_H + \sum_j \|v_j\|^2_H\right) < \infty. \tag{6}$$

This makes $l^2(H)$ into a pre-Hilbert space. If $u_{(i)}$ is a Cauchy sequence in $l^2(H)$, then, as in any normed space it is bounded. The sequence in H formed by the jth components for any j are Cauchy since

$$\|u_{(i),j} - u_{(i'),j}\|_H \le \|u_{(i)} - u_{(i')}\|. \tag{7}$$

Since H is complete it follows that $\lim_i u_{(i),j} = v_j$ exists in H. The bound on the norm in $l^2(H)$ for the sequence implies that

$$\sum_j \|u_{(i),j}\|^2_H < C^2 \tag{8}$$

for some C independent of i. Truncating the series at some finite point and then passing to the limit in the finite sum that results shows that

$$\sum_{j\le N} \|u_{(i),j}\|^2_H \to \sum_{j\le N} \|v_j\|^2_H \le C^2 \tag{9}$$

for all N. Passing to the limit as $N \to \infty$ shows $v \in l^2(H)$. Similarly, the Cauchy condition shows that given $\epsilon > 0$ there exists M such that for $i' > i > M$

$$\sum_j \|u_{(i),j} - u_{(i'),j}\|^2_H < \epsilon^2. \tag{10}$$

Again truncating the series at N and passing to the limit as $i' \to \infty$ shows that for $i > M$,

$$\sum_{j\le N} \|u_{(i),j} - v_j\|^2_H \le \epsilon^2. \tag{11}$$

Now, letting $N \to \infty$ we conclude that $u_{(i)} \to v$ in $l^2(H)$ since for $i > M$,

$$\|u_{(i)} - v\|_{l^2(H)} \le \epsilon.$$

Thus $l^2(H)$ is complete and hence is a Hilbert space.

To find an isomorphism of $l^2(H)$ to H choose an orthonormal basis e_i of H. Then the sequence

$$E_{j,i,k} = \begin{cases} 0 & \text{if } k \ne j \\ e_i & \text{if } k = j \end{cases}$$

(the sequence which is zero except for e_i in the jth position) form an orthonormal basis of $l^2(H)$. They are orthonormal by inspection and complete since if $u \in l^2(H)$ then $\langle u, E_{j,i}\rangle = \langle u_j, e_i\rangle_H$ and if this vanishes for all j and i then $u = 0$. Choosing an ordering of the $E_{j,i}$ to a sequence f_l, the isometric isomorphism follows by linear extension of the map on bases

$$U(f_i) = e_i,\ U : l^2(H) \longrightarrow H$$

which is norm-preserving, linear and a bijection.

Question.4

Show that, in a separable Hilbert space, a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if

$$\|v\|_H = \lim_{n\to\infty} \|v_n\|_H \tag{12}$$

where v is the weak limit.

Solution: If a weakly convergent sequence is strongly convergent it has the same limit and (12) follows from the continuity of the norm.

Conversely, if u_j conveges weakly to u then

$$\|u_j - u\|^2 = \langle u_j - u, u_j - u\rangle = \|u_j\|^2 - \langle u_j, u\rangle - \langle u, u_j\rangle + \|u\|^2.$$

By weak convergent the middle two terms on the right each converge to $-\|u\|^2$ so if $\|u_j\| \to \|u\|$ it follows that $\|u_j - u\| \to 0$ and $u_j \to u$.

Question.5

Let e_k, $k \in \mathbb{N}$, be an orthonormal basis in a separable Hilbert space, H. Show that there is a uniquely defined bounded linear operator $T : H \longrightarrow H$, satisfying

$$Te_j = e_{j-1}\ \forall\ j \geq 2,\ Te_1 = 0, \tag{13}$$

and that $T + B$ has one-dimensional null space if B is bounded and $\|B\| < 1$.

Solution: Extending T by linearity gives a bounded operator since

$$T(\sum_{i\geq 1} c_i e_i) = \sum_{i\geq 2} c_i e_{i-1} \Longrightarrow \|Tu\| \leq \|u\|. \tag{14}$$

Define $B' = B(\mathrm{Id} - P_1)$ where P_1 is the orthogonal projection onto e_1. Now, both T and B' can be restricted to bounded operators between the Hilbert spaces $e_1^\perp$ and H. Moreover, T is then invertible with inverse fixed by

$$Se_j = e_{j+1},\ j \geq 1 \tag{15}$$

also of norm one. Since $\|B'\| \leq \|B\| < 1$, it follows that $T + B'$ is inverible with inverse $S' : H \longrightarrow e_1^\perp$. Now, an element of the null space of $T + B$ is a vector $u = u_1 + u'$, $u_1 = P_1 u$, which satisfies

$$(T + B)u = (T + B)(\mathrm{Id} - P_1)u + BP_1u = 0 \Longleftrightarrow u' = -S'BP_1u_1. \tag{16}$$

This is one-dimensional since if $u_1 = e_1$ then $u' \in e_1^\perp$ is uniquely determined by (16) and conversely any element of the null space is a multiple of this vector.

Question.6

Show that a continuous function $K : [0,1] \longrightarrow L^2(0,2\pi)$ has the property that the Fourier series of $K(x) \in L^2(0,2\pi)$, for $x \in [0,1]$, converges uniformly in the sense that if $K_n(x)$ is the sum of the Fourier series over $|k| \le n$ then $K_n : [0,1] \longrightarrow L^2(0,2\pi)$ is also continuous and

$$\sup_{x\in[0,1]} \|K(x) - K_n(x)\|_{L^2(0,2\pi)} \to 0. \tag{17}$$

Solution: The image under K of $[0,1]$ is compact and hence the Fourier series converges uniformly on the range, which is precisely the content of (17). The $K_n(x)$ are given as $P_nK(x)$ where P_n is the self-adjoint projection onto the span of the part of the Fourier basis with $|k| \le n$; since P_n is continuous, $K_n : [0,1] \longrightarrow L^2(0,2\pi)$ is continuous as the composite of continuous maps.

Question.7

Prove that for appropriate constants d_k, the functions $d_k \sin(kx/2)$, $k \in \mathbb{N}$, form an orthonormal basis for $L^2(0,2\pi)$.

Solution: We can embed $L^2(0,2\pi)$ as a closed subspace of $L^2(-2\pi,2\pi)$ by extending each element to be odd:

$$A : L^2(0,2\pi) \ni f \longmapsto 2^{-\frac{1}{2}} g \in L^2(-2\pi,2\pi),\ g(x) = \begin{cases} f(x) & \text{if } x > 0 \\ -f(x) & \text{if } x < 0. \end{cases}$$

Consider the Fourier basis $e^{ikx/2}/2\sqrt{\pi}$ for $L^2(-2\pi,2\pi)$. The Fourier coefficients of an inner product with an odd function reduced to multiples of the pairing with $\sin(kx/2)$ which therefore are complete as a subspace of the odd part and hence restrict to be complete on $L^2(0,2\pi)$ where they are still orthogonal. It follows that they form a basis for the correct choice of constants d_k – which I did not ask you to compute although who knows why!

Question.8

Show that a separable Hilbert space in which every bounded operator is compact is finite dimensional.

Solution: If every bounded operator on H is compact then the identity operator is compact and hence the unit ball in H is equal to its image under a compact operator so it is contained in a compact set and hence is itself compact, being closed. It follows that H is finite-dimensional.

Question.9

Show that if B is a compact operator on a separable Hilbert space H and A is an invertible operator then

$$\{u \in H; Bu = Au\} \tag{18}$$

is finite dimensional.

Solution: The linear space $N = \{u \in H; Bu = Au\}$ is the null space of the bounded operator $A - B$, so is closed and hence a Hilbert subspace of H. Since A is invertible, any element $u \in N$ is equal to $A^{-1}Bu$. Thus the unit ball of N is equal to its image under $A^{-1}B$, which is compact. It follows that N is contained in a compact set and since it is closed it is also compact. The unit ball of a Hilbert space is non-compact if the space is infinite dimensional, so N is finite-dimensional.

Question.10

Show that there is a complete orthonormal basis of $L^2([0, 2\pi])$ consisting of polynomials.

Solution: The Stone-Weierstrass Theorem asserts that the polynomial on any compact interval are dense in the continuous functions in the supremum norm. It follows that the countable collection of polynomials with coefficients which have rational real and imaginary parts are dense in $L^2([0, 2\pi])$. Applying the Gram-Schmidt procedure to this set gives an orthonormal basis of polynomials.

Question.11

Let H be a separable Hilbert space and let $\mathcal{C}(\mathbb{R}; H)$ be the linear space of continuous maps from $\mathbb{R}$ to H which vanish outside some interval $[-R, R]$ depending on the function. Show that

$$\|u\|^2 = \int_{\mathbb{R}} \|u(x)\|_H^2 \tag{19}$$

defines a norm which comes from a preHilbert structure on $\mathcal{C}(\mathbb{R}; H)$. Show that if u_n is a Cauchy sequence in this preHilbert space and $h \in H$ then $\langle u_n(x), h\rangle_H$ converges in $L^2(\mathbb{R})$.

Solution: By the continuity of the norm on a Hilbert space, if $u \in \mathcal{C}(\mathbb{R}; H)$ then $\|u(x)\|_H \in \mathcal{C}(\mathbb{R})$ and hence (19) is well-defined. It is non-negative and vanishes precisely when $\|u(x)\|_H = 0$ and hence when $u = 0$ as an element of $\mathcal{C}(\mathbb{R}; H)$. The underlying inner product

$$\langle u, v\rangle = \int_{\mathbb{R}} \langle u(x), v(x)\rangle_H$$

is well-defined for the same reason, that the integrand is a continuous function vanishing outside some bounded set. Moreover, it is Hermitian symmetric and as noted above positive definite. Thus (19) is a pre-Hilbert norm.

A sequence u_n in $\mathcal{C}(\mathbb{R}; H)$ is Cauchy if given $\epsilon > 0$ there exists N such that

$$\int_{\mathbb{R}} \|u_n(x) - u_m\|_H^2 < \epsilon^2 \ \forall\ n, m > N,$$

If $h \in H$ is fixed then the sequence

$$|\langle u_n(x), h\rangle_H - \langle u_m(x), h\rangle|^2 \le \|u_n(x) - u_m(x)\|_H^2 \|h\|_H^2$$

so integrating gives

$$\|\langle u_n(x), h\rangle_H - \langle u_m(x), h\rangle\|_{L^2} \le \epsilon \|h\|_H$$

from which it follows that the sequence $\langle u_n(x), h\rangle_H$ is Cauchy, and hence converges, in $L^2(\mathbb{R})$.

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: rbm@math.mit.edu