

**TEST 1 FOR 18.102, MARCH 5, 2015**

Will consist (without the hints) of some of the questions below.

You may use standard results for the Riemann integral (anything from 18.100B).  
You may use the results we have shown up to this point including the definition

$$(1) \quad \mathcal{L}^2(\mathbb{R}) = \{u : \mathbb{R} \rightarrow \mathbb{C}; F \in \mathcal{L}^1(\mathbb{R}), \\ \exists u_n \in \mathcal{C}_c(\mathbb{R}), u_n(x) \rightarrow u(x) \text{ a.e. and } |u_n(x)|^2 \leq F \text{ a.e.}\}.$$

T1-pre.1

Show that the function

$$(2) \quad u(x) = \begin{cases} 0 & x \leq 0 \\ \min(x^{-\frac{1}{2}}, x^{-2}) & x > 0 \end{cases}$$

is in  $\mathcal{L}^1(\mathbb{R})$ .

Hint: Use continuity on  $[\delta, 1/\delta]$ ,  $\delta > 0$ , to deduce integrability for  $u$  multiplied by the characteristic function for this region. Estimate the integral using Riemann integrals and show that it is bounded and then the result follows by monotonicity.

T1-pre.2

Prove from the definition above that the product of two functions in  $\mathcal{L}^2(\mathbb{R})$  is in  $\mathcal{L}^1(\mathbb{R})$ .

Hint: Take approximating sequences  $U_n$  for  $f$  and  $V_n$  for  $g \in \mathcal{L}^2(\mathbb{R})$ . The product of these converges a.e. to the product and if  $|U_n|^2 \leq |F|$  and  $|V_n|^2 \leq |G|$  are bounds by  $\mathcal{L}^1$  functions then  $2|U_n V_n| \leq |F| + |G|$  gives the desired result.

T1-pre.3

Show that the product of a bounded continuous function on  $\mathbb{R}$  and an element of  $\mathcal{L}^2(\mathbb{R})$  is in  $\mathcal{L}^2(\mathbb{R})$ .

T1-pre.4

Give an example of a function which is in  $\mathcal{L}^1(\mathbb{R})$  but not in  $\mathcal{L}^2(\mathbb{R})$  and another example of a function which is in  $\mathcal{L}^2(\mathbb{R})$  but not in  $\mathcal{L}^1(\mathbb{R})$ ; justify both.

T1-pre.5

Show that if  $t \in \mathbb{R}$  and  $f \in \mathcal{L}^1(\mathbb{R})$  then

$$(3) \quad f_t(x) = f(x - t)$$

is an element of  $\mathcal{L}^1(\mathbb{R})$ . Prove that  $f \in \mathcal{L}^1(\mathbb{R})$  is *continuous-in-the-mean* in the sense that given  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$(4) \quad |t| < \delta \implies \int |f_t - f| < \epsilon.$$

Hint: If  $U_n$  is an approximating sequence in  $\mathcal{C}_c$  for  $f$  the the translates,  $U_n(\cdot - t)$  are an approximating sequence for  $f_t$ . Given  $\epsilon > 0$  and using the convergence of the norms, we can choose  $n$  so large that  $|\int (f - U_n)| < \epsilon/3$ . The same estimate applies to the sequence for  $f_t$  (simultaneously for all  $t$ ). Having fixed  $n$  observe that the estimate with  $U_n$  in place of  $f$  follows from compactness of support and (uniform) continuity. Combining these three estimates gives what we want by the usual  $\epsilon/3$  trick.

## T1-pre.6

We know that the characteristic function of a finite interval,  $\chi_{[a,b]}$ ,  $a < b$  real, is integrable. A (real-valued) step function is a finite sum of real multiples of such characteristic functions. If a function  $f : [a, b] \rightarrow \mathbb{R}$ , on a finite interval, is *Riemann integrable* then it is easy to see from the definition (you do not need to check this) that given  $\epsilon > 0$  there are two step functions  $U$  and  $L$  such that

$$(5) \quad L(x) \leq f(x) \leq U(x) \text{ on } [a, b] \text{ and } \int U - \int L < \epsilon$$

where we are using the Lebesgue integral. Deduce that a Riemann integrable function on an interval  $[a, b]$ , extended as zero outside its domain of definition, is Lebesgue integrable.

Hint: Everything here is to be extended as zero outside  $[a, b]$ . Choose a sequence  $U_n$  and  $L_n$  by taking  $\epsilon = 1/n$ . Replace  $U_n$  by the minimum of the  $U_k$  for  $k \leq n$  and similarly for the  $L_n$ 's. The sequence  $U_n$  is then decreasing with integrals bounded below and hence by monotonicity converges pointwise to  $f_+ \in \mathcal{L}^1$  and similarly the  $L_n$ 's converge up to  $f_-$  where  $f_- \leq f \leq f_+$  and  $\int(f_+ - f_-) = 0$ . From this it follows that  $f_+ = f = f_-$  a.e.

## T1-pre.6

Suppose  $g \in \mathcal{L}^1(\mathbb{R})$  is non-negative and vanishes outside some bounded interval. Show that if  $t \in (0, 1)$  then

$$(6) \quad g_t(x) = \begin{cases} 0 & \text{if } g(x) = 0 \\ g(x)^t & \text{if } g(x) > 0 \end{cases}$$

is an element of  $\mathcal{L}^1(\mathbb{R})$  and that  $\lim_{t \downarrow 0} g_t(x) = \chi_V(x)$  is the characteristic function of some set and that the limit is in  $\mathcal{L}^1(\mathbb{R})$ .

## T1-pre.7

Suppose that  $B : L^1(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is a bounded linear operator. Show that if  $B(\phi) = 0$  for all  $\phi \in \mathcal{C}_c(\mathbb{R})$  then  $B = 0$  as an operator.

## T1-pre.8

Find the real numbers  $s$  such that  $(1+|x|)^{s/2} \in L^2(\mathbb{R})$  and justify your conclusion.

## T1-pre.9

Suppose that  $[f] \in L^2(\mathbb{R})$  and  $\phi \in \mathcal{C}_c(\mathbb{R})$ , explain why  $\int f\phi$  exists and show that if  $\int f\phi = 0$  for all  $\phi \in \mathcal{C}_c(\mathbb{R})$  then  $[f] = 0$ .