

PROBLEM SET 9 FOR 18.102, SPRING 2015
DUE 7AM SATURDAY 9 MAY.

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This of course is the last problem set for the semester; I believe it is reasonably short.

Problem 9.1

Work out the precise effect of the creation and annihilation operators on the Hermite basis – the orthonormal basis of eigenfunctions of the harmonic oscillator.

Problem 9.2

Define a subspace $H_{\text{iso}}^1 \subset L^2(\mathbb{R})$ by the condition $u \in H_{\text{iso}}^1(\mathbb{R})$ if the coefficients with respect to the Hermite basis satisfy

$$\sum_{i=0}^{\infty} (2i+1) |\langle u, e_i \rangle|^2 < \infty.$$

Show that this has a norm with respect to which it is a Hilbert space and in which $\mathcal{S}(\mathbb{R})$ is dense.

Problem 9.3

Show that the creation and annihilation operators extend from $\mathcal{S}(\mathbb{R})$ to define bounded operators $H_{\text{iso}}^1(\mathbb{R}) \rightarrow L^2(\mathbb{R})$.

Problem 9.4

Show that if $u \in H_{\text{iso}}^1(\mathbb{R})$ then $xu \in L^2(\mathbb{R})$ and $\xi\hat{u} \in L^2(\mathbb{R})$ where \hat{u} is the Fourier transform of u .

Problem 9.5

Show conversely that if $u \in L^2(\mathbb{R})$, $xu \in L^2(\mathbb{R})$ and $\xi\hat{u} \in L^2(\mathbb{R})$ then $u \in H_{\text{iso}}^1(\mathbb{R})$.

Here is a hint which I have given to a few people who have asked: Since the basis elements are in Schwartz space you can manipulate the inner product, say $\langle u, C_n e_j \rangle$ to express it in terms of $\langle xu, e_j \rangle$ and $\langle \xi\hat{u}, \hat{e}_j \rangle$. From this and a similar argument for the annihilation operator and the fact that the Fourier-Bessel series for xu and g , where $\hat{g} = \xi\hat{u}$ converge, you can deduce the desired bound on the coefficients of the Fourier-Bessel series for u . Said another way, the Fourier-Bessel coefficients for xu and g , where $\hat{g} = \xi\hat{u}$, make sense (since the e_j are Schwartz) and can be expressed in terms of the Fourier-Bessel coefficients for u .

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