PROBLEM SET 8 FOR 18.102, SPRING 2015
DUE SATURDAY 25TH APRIL BY 7AM

RICHARD MELROSE

Problem 8.1

Suppose that $E \in \mathcal{B}(H)$ is a compact self-adjoint operator on a separable Hilbert space and that E is non-negative in the sense that

$$(Eu, u) \geq 0 \ \forall \ u \in H.$$

Show that E has no negative eigenvalues and that the positive eigenvalues can be arranged in a (weakly) decreasing sequence

$$s_1 \geq s_2 \geq \cdots \to 0$$

either finite, or decreasing to zero, such that if $F \subset H$ has dimension N then

$$\min_{u \in F, \|u\|=1} (Eu, u) \leq s_N, \ \forall \ N.$$

NB. The s_j have to be repeated corresponding to the dimension of the associated eigenspace.

Problem 8.2

Extend this further to show that under the same conditions on E the eigenvalues are give by the minimax formula:-

$$s_j(E) = \max_{F \subset H; \dim F = j} \left(\min_{u \in F; \|u\|=1} (Eu, u) \right).$$

Problem 8.3

With E as above, suppose that $D \in \mathcal{B}(H)$ is a bounded self-adjoint operator. Show that

$$s_j(DED) \leq \|D\|^2 s_j(E) \ \forall \ j.$$

NB. Be a bit careful about the minimax argument.

Problem 8.4

Briefly (just so we know that you understand it) suppose $V \in \mathcal{C}([0, 2\pi])$ is real-valued and non-negative go through the argument to show that $\lambda \in \mathbb{R}$ is an eigenvalue for the Dirichlet problem

$$(-\frac{d^2}{dx^2} + V(x))u(x) = \lambda u(x) \text{ on } [0, 2\pi], \ u(0) = 0 = u(2\pi)$$

with a twice-continuously differentiable eigenfunction if and only if $s = 1/\lambda$ is an eigenvalue of the operator

$$(\mathrm{Id} + AVA)^{-\frac{1}{2}} A^2 (\mathrm{Id} + AVA)^{-\frac{1}{2}}$$

where A is positive, self-adjoint and compact with eigenvalues $\frac{2}{k}$, $k \in \mathbb{N}_0$.

Problem 8.5

Combining all these things, show that the eigenvalues of the Dirichlet problem, with V real-valued and continuous, repeated according to multiplicity (the dimension of the eigenspace) and arranged as a non-decreasing sequence,

$$\lambda_1 \le \lambda_2 \le \lambda_N \to \infty$$

satisfy

$$\lambda_k \ge \frac{k^2}{4} + \min_{[0,2\pi]} V.$$

Problem 8.6 – extra

Consider the notion of an *unbounded* self-adjoint operator (since so far an operator is bounded, you should think of this as unbounded-self-adjoint-operator, a new notion which does included bounded self-adjoint operators). Namely, if H is a separable Hilbert space and $D \subset H$ is a *dense* linear subspace then a linear map $A : D \longrightarrow H$ is an unbounded self-adjoint operator if

(1) For all $v,\ w \in D$, $\langle Av, w\rangle_H = \langle v, Aw\rangle_H$.
(2) $\{u \in H; D \ni v \longmapsto \langle Av, u\rangle \in \mathbb{C}$ extends to a continuous map on $H\} = D$.

Show that

$$\mathrm{Gr}(A) = \{(u, Au) \in H \times H; u \in D\} \tag{1}$$

is a closed subspace of $H \times H$ and that $A + i\,\mathrm{Id} : D \longrightarrow H$ is surjective with a bounded two-sided inverse $B : H \longrightarrow H$ (with range D of course).

Problem 8.7 – extra

Suppose A is a compact self-adjoint operator on a separable Hilbert space and that $\mathrm{Nul}(A) = \{0\}$. Define a dense subspace $D \subset H$ in such a way that $A^{-1} : D \longrightarrow H$ is an unbounded self-adjoint operator which is a two-sided inverse of A.

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: rbm@math.mit.edu