PROBLEM SET 7 FOR 18.102
DUE 7AM SATURDAY 4 APRIL, 2015.

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Problem 7.1

Give an example of a closed subset in an infinite-dimensional Hilbert space which is not weakly closed.

Problem 7.2

Let $A \in \mathcal{B}(H)$, H a separable Hilbert space, be such that for some orthonormal basis $\{e_i\}$

$$\sum_i \|Ae_i\|_H^2 < \infty. \tag{5.1}$$

Show that the same inequality is true for any other orthonormal basis and that A^* satisfies the same inequality.

Hint: For another basis, expand each norm $\|Ae_i\|^2$ using Bessel's identity and then use the adjoint identity and undo the double sum the opposite way.

Problem 7.3

The elements of $A \in \mathcal{B}(H)$ as in Problem 7.2 are called 'Hilbert-Schmidt operators'. Show that these form a 2-sided $*$-closed ideal $\mathrm{HS}(H)$, inside the compact operators and that

$$\langle A, B \rangle = \sum_i \langle Ae_i, Be_i \rangle, \tag{5.2}$$

for any choice of orthonormal basis, makes this into a Hilbert space.

Problem 7.4

Consider the space of functions $u \in L^2(0, 2\pi)$ which have Fourier coefficients

$$c_k = \frac{1}{2\pi} \int_{(0,2\pi)} u(x) e^{-ikx}$$

satisfying

$$\sum_k (1 + k^2)|c_k|^2 < \infty. \tag{5.3}$$

Show that these functions form a Hilbert space (denoted $H^1_{\mathrm{per}}([0, 2\pi])$ below) and each such function is continuous (has a continuous representative) on $[0, 2\pi]$ with $u(0) = u(2\pi)$.

Hint: Use (5.3) to show that the Fourier series for u converges uniformly and absolutely.

Problem 7.5

Show that the Hilbert space in Problem 7.4, has an orthonormal basis of the form $d_k e^{ikx}$ for some positive constants d_k and that differentiation gives a bounded linear operator

$$\frac{d}{dx} : H^1_{\text{per}}([0, 2\pi]) \longrightarrow L^2(0, 2\pi).$$

Hint. Work out d/dx on the basis and extend by linearity!

Problem 7.6-extra

An operator T on a separable Hilbert space is said to be 'of trace class' (where this is just old-fashioned language) if it can be written as a finite sum

$$T = \sum_{i=1}^{N} A_i B_i$$

where all the A_i, B_i are Hilbert-Schmidt. Show that these trace class operators form a 2-sided ideal in the bounded operators, closed under passage to adjoints and that

$$\sup_{\{e_i\},\{f_i\}} \sum_i |\langle Te_i, f_i\rangle| < \infty$$

where the sup is over all pairs of orthonormal bases. Show that the trace functonal

$$\operatorname{Tr}(T) = \sum_i \langle Te_i e_i\rangle$$

is well-defined on trace class operators, independent of the orthonormal basis $\{e_i\}$ used to compute it.

Problem 7.7-extra

Suppose $A : L^2(0, 2\pi) \longleftarrow H^1_{\text{per}}([0, 2\pi])$ is a bounded linear operator between these Hilbert spaces. Show that as a bounded linear operator on $L^2(0, 2\pi)$ (since we may regard $H^1_{\text{per}}([0, 2\pi]) \subset L^2(0, 2\pi)$ as a subspace) such an A is Hilbert-Schmidt.

Hint: Show that by evaluation at any point $x \in [0, 2\pi]$, $(Au)(x)$ defines a continuous linear functional on $L^2(0, 2\pi)$ which varies continuously with x and use the resulting continuous map coming from Riesz' Theorem $[0, 2\pi] \longrightarrow L^2(0, 2\pi)$ to estimate the Hilbert Schmidt norm.

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