PROBLEM SET 6 FOR 18.102
DUE 7AM SATURDAY MARCH 28, 2015.

Problem 6.1

Recall the space $h^{2,1}$ (discussed in an earlier problem set) consisting of the complex valued sequences c_i such that

$$\|c\|^2 = \sum_i (1 + |i|^2)|c_i|^2 < \infty.$$

Show that the unit ball in this space, considered as a subset of l^2, has compact closure.

Hint: You may use the criterion for compactness of a set in a separable Hilbert space which was proved in lectures and is in the notes.

Problem 6.2

Define the space $\mathcal{L}^2(0,1)$ as consisting of those elements of $\mathcal{L}^2(\mathbb{R})$ which vanish outside $(0,1]$ and show that the quotient $L^2(0,1) = \mathcal{L}^2(0,1)/\mathcal{N}(0,1)$ by the null functions in $\mathcal{L}^2(0,1)$ is a Hilbert space.

Remark: This is indeed easy, but make sure you do it properly (for instance identify $L^2(0,1)$ with a closed subspace of $L^2(\mathbb{R})$ by treating the null functions properly).

Problem 6.3

Identify $\mathcal{C}[0,1]$, the space of continuous functions on the closed interval, as a subspace of $L^2(0,1)$. For each $n \in \mathbb{N}$ let $F_n \subset L^2(0,1)$ be the subspace of functions which are constant on each interval $((m-1)/n, m/n]$ for $m = 1, \dots, n$ Show that if $f \in \mathcal{C}[0,1]$ there exists $g_n \in F_n$ such that

$$\delta_n = \sup_{|t-s| \le 1/n} |f(t) - f(s)| \Longrightarrow \|f - g_n\|_{L^2} \le \delta_n.$$

Problem 6.4

Show that a bounded and equicontinuous subset of $\mathcal{C}[0,1]$ has compact closure in $L^2(0,1)$. Note that equicontinuity means 'uniform equicontinuity' so for each $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all elements f of the set.

Hint: Use the approximation by the finite dimensional spaces F_n.

Problem 6.5

If $K \in \mathcal{C}([0,1] \times [0,1])$ is a continuous function of two variables, show that

$$Af(x) = \int K(x,y)f(y) \tag{6.1}$$

defines a compact linear operator on $L^2(0,1)$.

Hint: Show that A defines a bounded linear map from $L^2(0,1)$ to $\mathcal{C}[0,1]$ and that the image of the unit ball is *equicontinuous* using the uniform continuity of K.

Problem 6.6 – extra

Show that a closed and bounded subset of $L^2(\mathbb{R})$ is compact if and only if it is 'uniformly equicontinuous in the mean' and 'uniformly small at infinity' so that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_{-`/\delta}^{1/\delta} |f|^2 < \epsilon^2 \text{ and } |t| < \delta \Longrightarrow \int |f(x) - f(x-t)|^2 < \epsilon^2$$

for all elements of the set.

Problem 6.7 – extra

Consider the space of continuous functions on $\mathbb{R}$ vanishing outside $(0,1)$ which are of the form

$$u(x) = \int_0^x v,\ \ v \in L^2(0,1).$$

Show that these form a Hilbert space and that the unity ball of this space has compact closure in $L^2(0,1)$.