PROBLEM SET 4 FOR 18.102, 'SPRING' 2015
DUE SAURDAY 28 FEBRUARY, BY 7AM

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Problem 4.1

Combining the original definition with Lebesgue's dominated convergence, show that $f : \mathbb{R} \longrightarrow \mathbb{C}$ is in $\mathcal{L}^1(\mathbb{R})$ if and only if there exists a sequence $u_n \in \mathcal{C}(\mathbb{R})$ and $F \in \mathcal{L}^1(\mathbb{R})$ such that $|u_n(x)| \le F(x)$ a.e. and $u_n(x) \to f(x)$ a.e.

Problem 4.2

Define $\mathcal{L}^\infty(\mathbb{R})$ as the set of functions $g : \mathbb{R} \longrightarrow \mathbb{C}$ such that there exists $C > 0$ and $v_n \in \mathcal{C}(\mathbb{R})$ with $|v_n(x)| \le C$ and $v_n(x) \to g(x)$ a.e. Show that $\mathcal{L}^\infty$ is a linear space, that

$$\|g\|_\infty = \inf\{\sup_{\mathbb{R}\setminus E} |g(x)|; E \text{ has measure zero and } \sup_{\mathbb{R}\setminus E} |g(x)| < \infty\}$$

is a seminorm on $\mathcal{L}^\infty(\mathbb{R})$ and that this makes $L^\infty(\mathbb{R}) = \mathcal{L}^\infty(\mathbb{R})/\mathcal{N}$ into a Banach space, where $\mathcal{N}$ is the space of null functions.

Problem 4.3

Show that if $g \in \mathcal{L}^\infty(\mathbb{R})$ and $f \in \mathcal{L}^1(\mathbb{R})$ then $gf \in \mathcal{L}^1(\mathbb{R})$ and that this defines a map

$$L^\infty \times L^1(\mathbb{R}) \longrightarrow L^1(\mathbb{R})$$

which satisfies $\|gf\|_{L^1} \le \|g\|_{L^\infty} \|f\|_{L^1}$.

Problem 4.4

Define a set $U \subset \mathbb{R}$ to be (Lebesgue) measureable if its characteristic function

$$\chi_U(x) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases}$$

is in $\mathcal{L}^\infty(\mathbb{R})$. Letting $\mathcal{M}$ be the collection of measureable sets, show

(1) $\mathbb{R} \in \mathcal{M}$
(2) $U \in \mathcal{M} \Longrightarrow \mathbb{R} \setminus U \in \mathcal{M}$
(3) $U_j \in \mathcal{M}$ for $j \in \mathbb{N}$ then $\bigcup_{j=1}^\infty U_j \in \mathcal{M}$
(4) If $U \subset \mathbb{R}$ is open then $U \in \mathcal{M}$

Problem 4.5

If $U \subset \mathbb{R}$ is measureable and $f \in \mathcal{L}^1(\mathbb{R})$ show that

$$\int_U f = \int \chi_U f \in \mathbb{C}$$

is well-defined. Prove that if $f \in \mathcal{L}^1(\mathbb{R})$ then

$$I_f(x) = \begin{cases} \int_{(0,x)} f & x \ge 0 \\ -\int_{(0,-x)} f & x < 0 \end{cases}$$

is a bounded continuous function on $\mathbb{R}$.

Problem 4.6 – Extra

Recall (from Rudin's book for instance) that if $F : [a,b] \longrightarrow [A,B]$ is an increasing continuously differentiable map, in the strong sense that $F'(x) > 0$, between finite intervals then for any continuous function $f : [A,B] \longrightarrow \mathbb{C}$, (Rudin shows it for Riemann integrable functions)

$$\int_A^B f(y)dy = \int_a^b f(F(x))F'(x)dx. \tag{1}$$

Prove the correspondng identity for every $f \in \mathcal{L}^1((A,B))$, which in particular requires the right side to make sense.

Problem 4.7 – Extra

Show that if $f \in \mathcal{L}^1(\mathbb{R})$ and I_f in Problem 4.5 vanishes identically then $f \in \mathcal{N}$.

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