PROBLEM SET 3 FOR 18.102, SPRING 2015 SOLUTIONS

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Recall that we have defined a set $E \subset \mathbb{R}$ to be 'of measure zero' if there exists a sequence $f_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ with $\sum\limits_n \int |f_n| < \infty$ such that

$$E \subset \{\sum_n |f_n(x)| = +\infty\}. \tag{1}$$

On Thursday 19 February I expect to show in class that if $f_n \in \mathcal{L}^1(\mathbb{R})$ is a sequence with $\sum\limits_n \int |f_n| < \infty$ then $\sum\limits_n f_n(x)$ converges a.e. and the limit is an element of $\mathcal{L}^1(\mathbb{R})$. You may, and probably should, use this below.

Problem 3.1

Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is such that there is a sequence $v_n \in \mathcal{L}^1(\mathbb{R})$ with real values, such that $v_n(x)$ is increasing for each x, $\int v_n$ is bounded and

$$\lim_n v_n(x) = f(x) \tag{2}$$

whenever the limit exists. Show that $f \in \mathcal{L}^1(\mathbb{R})$. Hint: Turn this sequence into an absolutely summable series.

My solution: This is really the Monotonicity Lemmma. Set $g_1 = v_1$ and $g_j = v_j - v_{j-1}$ for $j > 1$. Then the $g_j \in \mathcal{L}^1(\mathbb{R})$ form a sequence and $g_j \ge 0$ for $j > 1$ by the assumed monotonicity of the v_n. Thus

$$\sum_{j=1}^{n} \int |g_j| = \int |v_1| + \sum_{j=2}^{n} \int v_j - v_{j-1} = \int |v_1| + \int v_n - \int v_1 \text{ for } n > 1$$

so the boundedness of $\int v_n$ implies the absolute summability of the sequence g_j in $\mathcal{L}^1(\mathbb{R})$. It follows from the result recalled above that

$$\sum_j g_j(x) = \lim v_n \text{ exists } a.e.$$

and that $f \in \mathcal{L}^1(\mathbb{R})$.

Problem 3.2

(1) Suppose that $O \subset \mathbb{R}$ is a *bounded* open subset, so $O \subset (-R, R)$ for some R. Show that the characteristic function of O

$$\chi_O(x) = \begin{cases} 1 & x \in O \\ 0 & x \notin O \end{cases} \tag{3}$$

is an element of $\mathcal{L}^1(\mathbb{R})$.

(2) If O is bounded and open define the length (or Lebesgue measure) of O to be $l(O) = \int \chi_O$. Show that if $U = \bigcup_j O_j$ is a (n at most) countable union of bounded open sets such that $\sum\limits_j l(O_j) < \infty$ then $\chi_U \in \mathcal{L}^1(\mathbb{R})$; again we set $l(U) = \int \chi_U$.

(3) Conversely show that if U is open and $\chi_U \in \mathcal{L}^1(\mathbb{R})$ then $U = \bigcup_j O_j$ *is* the union of a countable collection of bounded open sets with $\sum\limits_j l(O_j) < \infty$.

(4) Show that if $K \subset \mathbb{R}$ is compact then its characteristic function is an element of $\mathcal{L}^1(\mathbb{R})$.

My solution:

(1) A bounded open set is a union of an at most countable collection of disjoint intervals, $O = \bigcup_j (a_j, b_j)$, necessarily with $|a_j|, |b_j| < R$. We know that each $\chi_{(a_j,b_j)} \in \mathcal{L}^1(\mathbb{R})$ has integral $b_j - a_j$. If the sum is finite the integrability of χ_O follows and otherwise the χ_{O_N}, $O_N = \bigcup_j^N (a_j, b_j)$ form an increasing sequence with integral bounded by $\sum\limits_{j=1}^N (b_j - a_j) \le 2R$. Since χ_O is the pointwise limit of the χ_{O_N} the result above shows that $\chi_O \in \mathcal{L}^1(\mathbb{R})$ with integral $\sum\limits_j (b_j - a_j) \le 2R$.

(2) The same argument applies. If $U = \bigcup_j O_j$ then χ_U is the pointwise limit of the increasing sequence χ_{U_N}, $U_N = \bigcup_{j=1}^N O_j$. Since $\sum\limits_{j=1}^N \chi_{O_j} \ge \chi_{U_N}$ it follows that the integral of the χ_{U_N} is bounded by $\sum\limits_j l(O_j) < \infty$ so indeed $\chi_U \in \mathcal{L}^1(\mathbb{R})$.

(3) Conversely if $U = \bigcup_j (a_j, b_j)$ and $\chi_U \in \mathcal{L}^1(\mathbb{R})$ then for each j and N $\chi_{(a_j,b_j)\cap(-N,N)} \in \mathcal{L}^1(\mathbb{R})$ with integral bounded by $\int \chi_U$. These functions increase monotonically with N so by the result above, $\chi_{(a_j,b_j)} \in \mathcal{L}^1(\mathbb{R})$. Similarly the $\sum\limits_{j=1}^n \chi_{(a_j,b_j)} \le \chi_U$ are in $\mathcal{L}^1(\mathbb{R})$ are increase montonically to χ_U so $\sum\limits_j (b_j, a_j) = l(U)$ is finite as claimed.

(4) If $K \subset \mathbb{R}$ is compact then it is bounded and hence the $U_j = \{y \in \mathbb{R}; B(y, 1/n) \cap K \ne \emptyset\}$ form a decreasing sequence of bounded open sets. Thus the $-\chi_{U_j}$ form an increasing sequence in $\mathcal{L}^1(\mathbb{R})$ which have integrals bounded above and which converge pointwise to $-\chi_K$ since $\chi_{U_j}(y) \to 0$ unless $\chi_{U_j}(y) = 1$ for all j which implies the existence of a sequence $x_j \in B(y, 1/j) \cap K$ and $x_j \to y$ implies $y \in K$. Thus by the Montonicity result above, $\chi_K \in \mathcal{L}^1(\mathbb{R})$.

Problem 3.3

Suppose $F \subset \mathbb{R}$ has the following (well-known) property:-

$$\forall\ \epsilon > 0\ \exists \text{ a countable collection of open sets } O_i \text{ s.t.}$$

(4)
$$\sum_i l(O_i) < \epsilon,\ F \subset \bigcup_i O_i.$$

Show that F is a set of measure zero in the sense above (the same sense as in lectures).

My solution: Taking $\epsilon = 1/n$, let U_n' be the union of the corresponding open sets containing F with $l(U_n) \le \sum_i l(O_i) < 1/n$. We may replace U_n' by $U_n = \bigcap_{j=1}^n U_n'$ and the conclusion that U_n is open, $F \subset U_n$ and $l(U_n) < 1/n$ remains correct. Thus the χ_{U_n} form a decreasing set of non-negative functions with $\chi_{U_n} \ge \chi_F$ for all n. It follows from the monotonicity lemma again that $\chi_G \in \mathcal{L}^1(\mathbb{R})$ where $G = \cap_n U_n$, which is the pointwise limit, is a null function. So $\chi_G = 0$ a.e. and hence $F \subset G$ is a set of measure zero.

Problem 3.4

Suppose $f_n \in \mathcal{C}_c(\mathbb{R})$ form an absolutely summable series with respect to the L^1 norm and set

$$E = \{x \in \mathbb{R}; \sum_n |f_n(x)| = \infty\}. \tag{5}$$

(1) Show that if $a > 0$ then the set

$$\{x \in \mathbb{R}; \sum_n |f_n(x)| \le a\} \tag{6}$$

is closed.

(2) Deduce that if $\epsilon > 0$ is given then there is an open set $O_\epsilon \supset E$ with $\sum_n |f_n(x)| > 1/\epsilon$ for each $x \in O_\epsilon$.

(3) Deduce that the characteristic function of O_ϵ is in $\mathcal{L}^1(\mathbb{R})$ and that $l(O_\epsilon) \le \epsilon C$, $C = \sum_n \int |f_n(x)|$.

(4) Conclude that E satisfies the condition (4).

My solution:

(1) By the continuity of the f_n, $A_N = \{x \in \mathbb{R}; \sum_{n=1}^N |f_n(x)| > a\}$ is closed for each N. Since $A = \bigcap_N A_N$ is the set above, it too is closed.

(2) As the complement of a closed set, $O_\epsilon = \{x \in \mathbb{R}; \sum_n |f_n(x)| > 1/\epsilon\}$ is open and contains E.

(3) Each O_ϵ is an at most countable union of intervals (a_j, b_j). On each of these $\sum_{n \le N} |f_n(x)|$ is an increasing sequence in $\mathcal{L}^1(\mathbb{R})$ with integral bounded by $\sum_j \int |f_j|$ so the pointwise limit of $g_j = \chi_{(a_j,b_j)} \sum_n |f_n|$ exists a.e. and is a element of $\mathcal{L}^1(\mathbb{R})$. Since $g_j \ge \epsilon^{-1}\chi_{(a_j,b_j)}$ and $\chi_{(a_j,b_j)}$ bounds from above, and is the pointwise limit of, $\chi_{(a_j,b_j)\cap(-N,N)}$ it follows that $\chi_{(a_j,b_j)} \in \mathcal{L}^1(\mathbb{R})$. Thus the $\sum_{j=1}^N \chi_{(a_j,b_j)}$ form an increasing sequence with integral bounded above by ϵC, $C = \sum_n \int |f_n|$ so it follows that $\chi_{O_\epsilon} \in \mathcal{L}^1(\mathbb{R})$ and that $l(O_\epsilon) \le \epsilon C$ as claimed.

(4) The O_ϵ provide the cover as required in the standard definition of sets of measure zero.

Problem 3.5

Show that the function with$F(0)=0$ and

$$F(x)=\begin{cases}0 & x>1\\ \exp(i/x) & 0<|x|<1\\ 0 & x<-1,\end{cases}$$

is an element of $\mathcal{L}^1(\mathbb{R})$.

My solution: On the closure of each interval $(-1,-1/n)$ and $(1/n,1)$, $F(x)$ extends to a continuous function with absolute value bounded by 1. Thus $F_n = (\chi_{(-1,-1/n)}+\chi_{(1/n,1)})F \in \mathcal{L}^1(\mathbb{R})$ for $n\ge 2$ and for $n>2$ $\int|F_n-F_{n-1}|\le 2(1/(n-1)-1/n)=2/n(n-1)$. Thus the F_n are the partial sums of an absolutely convergent sequence and since $F_n\to F(x)$ everywhere, it follows that $F\in\mathcal{L}^1(\mathbb{R})$.

Problem 3.6 – Extra

(1) Recall the definition of a Riemann integrable function $g:[a,b]\longrightarrow\mathbb{R}$ – that there exist a sequence of successively finer partitions for which the upper Riemann sum approaches the lower Riemann sum. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be the extension of this function to be zero outside the interval.
(2) Translate this condition into a statement about two sequences of piecewise-constant functions (with respect to the partition), u_n, l_n with $l_n(x)\le f(x)\le u_n(x)$ and conclude that $\int(u_n-l_n)\to 0$.
(3) Deduce that $f\in\mathcal{L}^1(\mathbb{R})$ and the Lebesgue integral of f on $\mathbb{R}$ is equal to the Riemann integral of g on $[a,b]$.
(4) Show that there is a Lebesgue integrable function $f:\mathbb{R}\longrightarrow\mathbb{R}$ which vanishes outside $[a,b]$ but that no function equal to it a.e. can be Riemann integrable.

Problem 3.7 – Extra

Prove that the Cantor set has measure zero.

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