Functional Analysis
Lecture notes for 18.102

Richard Melrose

Department of Mathematics, MIT
E-mail address: rbm@math.mit.edu

$\boxed{\boxed{\text{Version 0.8D; Revised: 4-5-2010; Run: May 13, 2015}}}$.

Contents

Preface

These are notes for the course 'Introduction to Functional Analysis' – or in the MIT style, 18.102, from various years culminating in Spring 2015. There are many people who I should like to thank for comments on and corrections to the notes over the years, but for the moment I would simply like to thank the MIT undergraduates who have made this course a joy to teach, as a result of their interest and enthusiasm.

Introduction

This course is intended for 'well-prepared undergraduates' meaning specifically that they have a rigourous background in analysis at roughly the level of the first half of Rudin's book [**3**] – at MIT this is 18.100B. In particular the basic theory of metric spaces is used freely. Some familiarity with linear algebra is also assumed, but not at a very sophisticated level.

The main aim of the course in a mathematical sense is the presentation of the standard constructions of linear functional analysis, centred on Hilbert space and its most significant analytic realization as the Lebesgue space $L^2(\mathbb{R})$ and leading up to the spectral theory of ordinary differential operators. In a one-semester course at MIT it is only just possible to get this far. Beyond the core material I have included other topics that I believe may prove useful both in showing how to apply the 'elementary' material and more directly.

Dirichlet problem. The eigenvalue problem with potential perturbation on an interval is one of the proximate aims of this course, so let me describe it briefly here for orientation.

Let $V : [0, 1] \longrightarrow \mathbb{R}$ be a real-valued continuous function. We are interested in 'oscillating modes' on the interval; something like this arises in quantum mechanics for instance. Namely we want to know about functions $u(x)$ – twice continuously differentiable on $[0, 1]$ so that things make sense – which satisfy the differential equation

$$\begin{aligned}-\frac{d^2u}{dx^2}(x) + V(x)u(x) = \lambda u(x) \text{ and the}\\ \text{boundary conditions } u(0) = u(1) = 0.\end{aligned} \tag{1}$$

Here the eigenvalue, λ is an 'unknown' constant. More precisely we wish to know which such λ's can occur. In fact all λ's can occur with $u \equiv 0$ but this is the 'trivial solution' which will always be there for such an equation. What other solutions are there? The main result is that there is an infinite sequence of λ's for which there is a non-trivial solution of (1) $\lambda_j \in \mathbb{R}$ – they are all real, no non-real complex λ's can occur. For each of the λ_j there is at least one (and maybe more) non-trivial solution u_j to (1). We can say a lot more about everything here but one main aim of this course is to get at least to this point. From a Physical point of view, (1) represents a linearized oscillating string with fixed ends.

So the journey to a discussion of the Dirichlet problem is rather extended and apparently wayward. The relevance of Hilbert space and the Lebesgue integral is not immediately apparent, but I hope this will become clear as we proceed. In fact in this one-dimensional setting it can be avoided, although at some cost in terms of elegance. The basic idea is that we consider a space of all 'putative' solutions to the problem at hand. In this case one might take the space of all twice continuously differentiable functions on $[0, 1]$ – we will consider such spaces at least briefly below. One of the weaknesses of such an approach is that it is not closely connected with the 'energy' invariant of a solution, which is the integral

$$\int_0^1 (|\frac{du}{dx}|^2 + V(x)|u(x)|^2)dx. \tag{2}$$

It is the importance of such integrals which brings in the Lebesgue integral and leads to a Hilbert space structure.

In any case one of the significant properties of the equation (1) is that it is 'linear'. So we start with a brief discussion of linear spaces. What we are dealing with here can be thought of as the an eigenvalue problem for an 'infinite matrix'. This in fact is not a very good way of looking at things (there was such a matrix approach to quantum mechanics in the early days but it was replaced by the sort of 'operator' theory on Hilbert space that we will use here.) One of the crucial distinctions between the treatment of finite dimensional matrices and an infinite dimensional setting is that in the latter *topology* is encountered. This is enshrined in the notion of a *normed linear space* which is the first important topic treated.

After a brief treatment of normed and Banach spaces, the course proceeds to the construction of the Lebesgue integral. Usually I have done this in one dimension, on the line, leading to the definition of the space $L^1(\mathbb{R})$. To some extent I follow here the idea of Jan Mikusiński that one can simply define integrable functions as the almost everywhere limits of absolutely summable series of step functions and more significantly the basic properties can be deduced this way. While still using this basic approach I have dropped the step functions almost completely and instead emphasize the completion of the space of continuous functions to get the Lebesgue space. Even so, Mikusiński's approach still underlies the explicit identification of elements of the completion with Lebesgue 'functions'. This approach is followed in the book of Debnaith and Mikusiński.

After about a three-week stint of integration and then a little measure theory the course proceeds to the more gentle ground of Hilbert spaces. Here I have been most guided by the (old now) book of Simmons. We proceed to a short discussion of operators and the spectral theorem for compact self-adjoint operators. Then in the last third or so of the semester this theory is applied to the treatment of the Dirichlet eigenvalue problem and treatment of the harmonic oscillator with a short discussion of the Fourier transform. Finally various loose ends are brought together, or at least that is my hope.

CHAPTER 1

Normed and Banach spaces

In this chapter we introduce the basic setting of functional analysis, in the form of normed spaces and bounded linear operators. We are particularly interested in complete, i.e. Banach, spaces and the process of completion of a normed space to a Banach space. In lectures I proceed to the next chapter, on Lebesgue integration after Section 7 and then return to the later sections of this chapter at appropriate points in the course.

There are many good references for this material and it is always a good idea to get at least a couple of different views. I suggest the following on-line sources Wilde [**5**], Chen [**1**] and Ward [**4**]. The treatment here, whilst quite brief, does cover what is needed later.

1. Vector spaces

You should have some familiarity with linear, or I will usually say 'vector', spaces. Should I break out the axioms? Not here I think, but they are included in Section 14 at the end of the chapter. In short it is a space V in which we can add elements and multiply by scalars with rules quite familiar to you from the the basic examples of $\mathbb{R}^n$ or $\mathbb{C}^n$. Whilst these special cases are (very) important below, this is not what we are interested in studying here. The main examples are spaces of *functions* hence the name of the course.

Note that for us the 'scalars' are either the real numbers or the complex numbers – usually the latter. To be neutral we denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$, but of course consistently. Then our set V – the set of vectors with which we will deal, comes with two 'laws'. These are maps

$$+ : V \times V \longrightarrow V,\ \cdot : \mathbb{K} \times V \longrightarrow V. \tag{1.1}$$

which we denote not by $+(v,w)$ and $\cdot(s,v)$ but by $v+w$ and sv. Then we impose the *axioms of a vector space* – see Section (14) below! These are commutative group axioms for $+$, axioms for the action of $\mathbb{K}$ and the distributive law linking the two.

The basic examples:

- The field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$ is a vector space over itself.
- The vector spaces $\mathbb{K}^n$ consisting of ordered n-tuples of elements of $\mathbb{K}$. Addition is by components and the action of $\mathbb{K}$ is by multiplication on all components. You should be reasonably familiar with these spaces and other finite dimensional vector spaces.
- Seriously non-trivial examples such as $\mathrm{C}([0,1])$ the space of continuous functions on $[0,1]$ (say with complex values).

In these and many other examples we will encounter below the 'component addition' corresponds to the addition of functions.

LEMMA 1. *If X is a set then the spaces of all functions*

$$\mathcal{F}(X;\mathbb{R}) = \{u : X \longrightarrow \mathbb{R}\},\ \mathcal{F}(X;\mathbb{C}) = \{u : X \longrightarrow \mathbb{C}\} \tag{1.2}$$

are vector spaces over $\mathbb{R}$ and $\mathbb{C}$ respectively.

NON-PROOF. Since I have not written out the axioms of a vector space it is hard to check this – and I leave it to you as the first of many important exercises. In fact, better do it more generally as in Problem 5.1 – then you can sound sophisticated by saying 'if V is a linear space then $\mathcal{F}(X;V)$ inherits a linear structure'. The main point to make sure you understand is that, because we *do* know how to add and multiply in either $\mathbb{R}$ and $\mathbb{C}$, we can add functions and multiply them by constants (we can multiply functions by each other but that is not part of the definition of a vector space so we ignore it for the moment since many of the spaces of functions we consider below are *not* multiplicative in this sense):-

$$(c_1f_1 + c_2f_2)(x) = c_1f_1(x) + c_2f_2(x) \tag{1.3}$$

defines the function $c_1f_1 + c_2f_2$ if $c_1,\ c_2 \in \mathbb{K}$ and $f_1,\ f_2 \in \mathcal{F}(X;\mathbb{K})$. □

You should also be familiar with the notions of linear subspace and quotient space. These are discussed a little below and most of the linear spaces we will meet are either subspaces of these function-type spaces, or quotients of such subspaces – see Problems 5.2 and 5.3.

Although you are probably most comfortable with finite-dimensional vector spaces it is the infinite-dimensional case that is most important here. The notion of dimension is based on the concept of the linear independence of a subset of a vector space. Thus a subset $E \subset V$ is said to be *linearly independent* if for any finite collection of elements $v_i \in E$, $i = 1, \dots, N$, and any collection of 'constants' $a_i \in \mathbb{K}$, $i = 1, \dots, N$ we have the following implication

$$\sum_{i=1}^{N} a_iv_i = 0 \Longrightarrow a_i = 0\ \forall\ i. \tag{1.4}$$

That is, it is a set in which there are 'no non-trivial finite linear dependence relations between the elements'. A vector space is finite-dimensional if every linearly independent subset is finite. It follows in this case that there is a finite and maximal linearly independent subset – a basis – where maximal means that if any new element is added to the set E then it is no longer linearly independent. A basic result is that any two such 'bases' in a finite dimensional vector space have the same number of elements – an outline of the finite-dimensional theory can be found in Problem 1.

Still it is time to leave this secure domain behind, since we are most interested in the other case, namely infinite-dimensional vector spaces. As usual with such mysterious-sounding terms as 'infinite-dimensional' it is defined by negation.

DEFINITION 1. A vector space is infinite-dimensional if it is not finite dimensional, i.e. for any $N \in \mathbb{N}$ there exist N elements with no, non-trivial, linear dependence relation between them.

As is quite typical the idea of an infinite-dimensional space, which you may be quite keen to understand, appears just as the non-existence of something. That is, it is the 'residual' case, where there is no finite basis. This means that it is 'big'.

So, finite-dimensional vector spaces have finite bases, infinite-dimensional vector spaces do not. The notion of a basis in an infinite-dimensional vector spaces needs to be modified to be useful analytically. Convince yourself that the vector space in Lemma 1 is infinite dimensional if and only if X is infinite.

2. Normed spaces

In order to deal with infinite-dimensional vector spaces we need the control given by a metric (or more generally a non-metric topology, but we will not quite get that far). A norm on a vector space leads to a metric which is 'compatible' with the linear structure.

Definition 2. A norm on a vector space is a function, traditionally denoted

$$\|\cdot\| : V \longrightarrow [0,\infty), \tag{1.5}$$

with the following properties

(*Definiteness*)

$$v \in V,\ \|v\| = 0 \Longrightarrow v = 0. \tag{1.6}$$

(*Absolute homogeneity*) For any $\lambda \in \mathbb{K}$ and $v \in V$,

$$\|\lambda v\| = |\lambda|\|v\|. \tag{1.7}$$

(*Triangle Inequality*) The triangle inequality holds, in the sense that for any two elements v, $w \in V$

$$\|v + w\| \le \|v\| + \|w\|. \tag{1.8}$$

Note that (1.7) implies that $\|0\| = 0$. Thus (1.6) means that $\|v\| = 0$ is equivalent to $v = 0$. This definition is based on the same properties holding for the standard norm(s), $|z|$, on $\mathbb{R}$ and $\mathbb{C}$. You should make sure you understand that

$$\begin{gathered}\mathbb{R} \ni x \longrightarrow |x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x \le 0 \end{cases} \in [0,\infty) \text{ is a norm as is} \\ \mathbb{C} \ni z = x + iy \longrightarrow |z| = (x^2 + y^2)^{\frac{1}{2}}.\end{gathered} \tag{1.9}$$

Situations do arise in which we do not have (1.6):-

Definition 3. A function (1.5) which satisfes (1.7) and (1.8) but possibly not (1.6) is called a seminorm.

A metric, or distance function, on a set is a map

$$d : X \times X \longrightarrow [0,\infty) \tag{1.10}$$

satisfying three standard conditions

$$d(x,y) = 0 \Longleftrightarrow x = y, \tag{1.11}$$

$$d(x,y) = d(y,x)\ \forall\ x,y \in X \text{ and} \tag{1.12}$$

$$d(x,y) \le d(x,z) + d(z,y)\ \forall\ x,y,z \in X. \tag{1.13}$$

As you are no doubt aware, a set equipped with such a metric function is called a metric space.

If you do not know about metric spaces, then you are in trouble. I suggest that you take the appropriate course now and come back next year. You could read the first few chapters of Rudin's book [**3**] before trying to proceed much further but it will be a struggle to say the least. The point of course is

PROPOSITION 1. *If* $\|\cdot\|$ *is a norm on* V *then*

$$d(v,w) = \|v - w\| \tag{1.14}$$

is a distance on V *turning it into a metric space.*

PROOF. Clearly (1.11) corresponds to (1.6), (1.12) arises from the special case $\lambda = -1$ of (1.7) and (1.13) arises from (1.8). □

We will not use any special notation for the metric, nor usually mention it explicitly – we just subsume all of metric space theory from now on. So $\|v - w\|$ is the distance between two points in a normed space.

Now, we need to talk about a few examples; there are more in Section 7. The most basic ones are the usual finite-dimensional spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ with their Euclidean norms

$$|x| = \left(\sum_i |x_i|^2\right)^{\frac{1}{2}} \tag{1.15}$$

where it is at first confusing that we just use single bars for the norm, just as for $\mathbb{R}$ and $\mathbb{C}$, but you just need to get used to that.

There are other norms on $\mathbb{C}^n$ (I will mostly talk about the complex case, but the real case is essentially the same). The two most obvious ones are

$$\begin{gathered} |x|_\infty = \max |x_i|, \ x = (x_1, \dots, x_n) \in \mathbb{C}^n, \\ |x|_1 = \sum_i |x_i| \end{gathered} \tag{1.16}$$

but as you will see (if you do the problems) there are also the norms

$$|x|_p = (\sum_i |x_i|^p)^{\frac{1}{p}}, \ 1 \le p < \infty. \tag{1.17}$$

In fact, for $p = 1$, (1.17) reduces to the second norm in (1.16) and in a certain sense the case $p = \infty$ is consistent with the first norm there.

In lectures I usually do not discuss the notion of equivalence of norms straight away. However, two norms on the one vector space – which we can denote $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(2)}$ are *equivalent* if there exist constants C_1 and C_2 such that

$$\|v\|_{(1)} \le C_1 \|v\|_{(2)}, \ \|v\|_{(2)} \le C_2 \|v\|_{(1)} \ \forall \ v \in V. \tag{1.18}$$

The equivalence of the norms implies that the metrics define the same open sets – the topologies induced are the same. You might like to check that the reverse is also true, if two norms induced the same topologies (just meaning the same collection of open sets) through their associated metrics, then they are equivalent in the sense of (1.18) (there are more efficient ways of doing this if you wait a little).

Look at Problem 5.6 to see why we are not so interested in norms in the finite-dimensional case – namely any two norms on a finite-dimensional vector space are equivalent and so in that case a choice of norm does not tell us much, although it certainly has its uses.

One important class of normed spaces consists of the spaces of bounded continuous functions on a metric space X :

$$\mathcal{C}_\infty(X) = \mathcal{C}_\infty(X;\mathbb{C}) = \{u : X \longrightarrow \mathbb{C}, \text{ continuous and bounded}\}. \tag{1.19}$$

That this is a linear space follows from the (obvious) result that a linear combination of bounded functions is bounded and the (less obvious) result that a linear combination of continuous functions is continuous; this we know. The norm is the best bound

$$\|u\|_\infty = \sup_{x\in X} |u(x)|. \tag{1.20}$$

That this *is* a norm is straightforward to check. Absolute homogeneity is clear, $\|\lambda u\|_\infty = |\lambda|\|u\|_\infty$ and $\|u\|_\infty = 0$ means that $u(x) = 0$ for all $x \in X$ which is exactly what it means for a function to vanish. The triangle inequality 'is inherited from $\mathbb{C}$' since for any two functions and any point,

$$|(u+v)(x)| \le |u(x)| + |v(x)| \le \|u\|_\infty + \|v\|_\infty \tag{1.21}$$

by the definition of the norms, and taking the supremum of the left gives

$$\|u+v\|_\infty \le \|u\|_\infty + \|v\|_\infty.$$

Of course the norm (1.20) is defined even for bounded, not necessarily continuous functions on X. Note that convergence of a sequence $u_n \in \mathcal{C}_\infty(X)$ (remember this means with respect to the distance induced by the norm) is precisely *uniform convergence*

$$\|u_n - v\|_\infty \to 0 \Longleftrightarrow u_n(x) \to v(x) \text{ uniformly on } X. \tag{1.22}$$

Other examples of infinite-dimensional normed spaces are the spaces l^p, $1 \le p \le \infty$ discussed in the problems below. Of these l^2 is the most important for us. It is in fact one form of Hilbert space, with which we are primarily concerned:-

$$l^2 = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_j |a(j)|^2 < \infty\}. \tag{1.23}$$

It is not immediately obvious that this is a linear space, nor that

$$\|a\|_2 = \left(\sum_j |a(j)|^2\right)^{\frac{1}{2}} \tag{1.24}$$

is a norm. It is. From now on we will generally use sequential notation and think of a map from $\mathbb{N}$ to $\mathbb{C}$ as a sequence, so setting $a(j) = a_j$. Thus the 'Hilbert space' l^2 consists of the square summable sequences.

3. Banach spaces

You are supposed to remember from metric space theory that there are three crucial properties, completeness, compactness and connectedness. It turns out that normed spaces are always connected, so that is not very interesting, and they are never compact (unless you consider the trivial case $V = \{0\}$) so that is not very interesting either – although we will ultimately be very interested in compact subsets – so that leaves completeness. That is so important that we give it a special name in honour of Stefan Banach.

Definition 4. A normed space which is complete with respect to the induced metric is a *Banach* space.

Lemma 2. *The space $\mathcal{C}_\infty(X)$, defined in* (1.19) *for any metric space X, is a Banach space.*

PROOF. This is a standard result from metric space theory – basically that the uniform limit of a sequence of (bounded) continuous functions on a metric space is continuous. However, it is worth recalling how one proves completeness at least in outline. Suppose u_n is a Cauchy sequence in $\mathcal{C}_\infty(X)$. This means that given $\delta > 0$ there exists N such that

$$n, m > N \Longrightarrow \|u_n - u_m\|_\infty = \sup_X |u_n(x) - u_m(x)| < \delta. \tag{1.25}$$

Fixing $x \in X$ this implies that the sequence $u_n(x)$ is Cauchy in $\mathbb{C}$. We know that this space is complete, so each sequence $u_n(x)$ must converge (we say the sequence of functions converges pointwise). Since the limit of $u_n(x)$ can only depend on x, we define $u(x) = \lim_n u_n(x)$ in $\mathbb{C}$ for each $x \in X$ and so define a function $u : X \longrightarrow \mathbb{C}$. Now, we need to show that this is bounded and continuous and is the limit of u_n with respect to the norm. Any Cauchy sequence is bounded in norm – take $\delta = 1$ in (1.25) and it follows from the triangle inequality that

$$\|u_m\|_\infty \le \|u_{N+1}\|_\infty + 1, \ m > N \tag{1.26}$$

and the finite set $\|u_n\|_\infty$ for $n \le N$ is certainly bounded. Thus $\|u_n\|_\infty \le C$, but this means $|u_n(x)| \le C$ for all $x \in X$ and hence $|u(x)| \le C$ by properties of convergence in $\mathbb{C}$ and thus $\|u\|_\infty \le C$.

The uniform convergence of u_n to u now follows from (1.25) since we may pass to the limit in the inequality to find

$$\begin{aligned} n > N \Longrightarrow |u_n(x) - u(x)| &= \lim_{m\to\infty} |u_n(x) - u_m(x)| \le \delta \\ &\Longrightarrow \|u_n - u\|_{|infty} \le \delta. \end{aligned} \tag{1.27}$$

The continuity of u at $x \in X$ follows from the triangle inequality in the form

$$\begin{aligned} |u(y) - u(x)| \le |u(y) - u_n(y)| + |u_n(y) - u_n(x)| + |u_n(x) - u_n(x)| \\ \le 2\|u - u_n\|_\infty + |u_n(x) - u_n(y)|. \end{aligned}$$

Give $\delta > 0$ the first term on the far right can be make less than $\delta/2$ by choosing n large using (1.27) and then the second term can be made less than $\delta/2$ by choosing $d(x, y)$ small enough. □

I have written out this proof (succinctly) because this general structure arises often below – first find a candidate for the limit and then show it has the properties that are required.

There is a space of sequences which is really an example of this Lemma. Consider the space c_0 consisting of all the sequences $\{a_j\}$ (valued in $\mathbb{C}$) such that $\lim_{j\to\infty} a_j = 0$. As remarked above, sequences are just functions $\mathbb{N} \longrightarrow \mathbb{C}$. If we make $\{a_j\}$ into a function $\alpha : D = \{1, 1/2, 1/3, \dots\} \longrightarrow \mathbb{C}$ by setting $\alpha(1/j) = a_j$ then we get a function on the metric space D. Add 0 to D to get $\overline{D} = D \cup \{0\} \subset [0, 1] \subset \mathbb{R}$; clearly 0 is a limit point of D and $\overline{D}$ is, as the notation dangerously indicates, the closure of D in $\mathbb{R}$. Now, you will easily check (it is really the definition) that $\alpha : D \longrightarrow \mathbb{C}$ corresponding to a sequence, extends to a *continuous* function on $\overline{D}$ vanishing at 0 if and only if $\lim_{j\to\infty} a_j = 0$, which is to say, $\{a_j\} \in c_0$. Thus it follows, with a little thought which you should give it, that c_0 is a Banach space with the norm

$$\|a\|_\infty = \sup_j \|a_j\|. \tag{1.28}$$

What is an example of a non-complete normed space, a normed space which is *not* a Banach space? These are legion of course. The simplest way to get one is to 'put the wrong norm' on a space, one which does not correspond to the definition. Consider for instance the linear space $\mathcal{T}$ of sequences $\mathbb{N} \longrightarrow \mathbb{C}$ which 'terminate', i.e. each element $\{a_j\} \in \mathcal{T}$ has $a_j = 0$ for $j > J$, where of course the J may depend on the particular sequence. Then $\mathcal{T} \subset c_0$, the norm on c_0 defines a norm on $\mathcal{T}$ but it cannot be complete, since the closure of $\mathcal{T}$ is easily seen to be all of c_0 – so there are Cauchy sequences in $\mathcal{T}$ without limit in $\mathcal{T}$. Make sure you are not lost here – you need to get used to the fact that we often need to discuss the 'convergence of sequences of convergent sequences' as here.

One result we will exploit below, and I give it now just as preparation, concerns *absolutely summable series.* Recall that a series is just a sequence where we 'think' about adding the terms. Thus if v_n is a sequence in some vector space V then there is the corresponding sequence of partial sums $w_N = \sum\limits_{i=1}^{N} v_i$. I will say that $\{v_n\}$ is a series if I am thinking about summing it.

So a sequence $\{v_n\}$ with partial sums $\{w_N\}$ is said to be *absolutely summable* if

$$\sum_n \|v_n\|_V < \infty, \text{ i.e. } \sum_{N>1} \|w_N - w_{N-1}\|_V < \infty. \tag{1.29}$$

PROPOSITION 2. *The sequence of partial sums of any absolutely summable series in a normed space is Cauchy and a normed space is complete if and only if every absolutely summable series in it converges, meaning that the sequence of partial sums converges.*

PROOF. The sequence of partial sums is

$$w_n = \sum_{j=1}^{n} v_j. \tag{1.30}$$

Thus, if $m \ge n$ then

$$w_m - w_n = \sum_{j=n+1}^{m} v_j. \tag{1.31}$$

It follows from the triangle inequality that

$$\|w_n - w_m\|_V \le \sum_{j=n+1}^{m} \|v_j\|_V. \tag{1.32}$$

So if the series is absolutely summable then

$$\sum_{j=1}^{\infty} \|v_j\|_V < \infty \text{ and } \lim_{n\to\infty} \sum_{j=n+1}^{\infty} \|v_j\|_V = 0.$$

Thus $\{w_n\}$ is Cauchy if $\{v_j\}$ is absolutely summable. Hence if V is complete then every absolutely summable series is summable, i.e. the sequence of partial sums converges.

Conversely, suppose that every absolutely summable series converges in this sense. Then we need to show that every Cauchy sequence in V converges. Let u_n be a Cauchy sequence. It suffices to show that this has a subsequence which converges, since a Cauchy sequence with a convergent subsequence is convergent.

To do so we just proceed inductively. Using the Cauchy condition we can for every k find an integer N_k such that

$$n, m > N_k \Longrightarrow \|u_n - u_m\| < 2^{-k}. \tag{1.33}$$

Now choose an increasing sequence n_k where $n_k > N_k$ and $n_k > n_{k-1}$ to make it increasing. It follows that

$$\|u_{n_k} - u_{n_{k-1}}\| \le 2^{-k+1}. \tag{1.34}$$

Denoting this subsequence as $u'_k = u_{n_k}$ it follows from (1.34) and the triangle inequality that

$$\sum_{n=1}^{\infty} \|u'_n - u'_{n-1}\| \le 4 \tag{1.35}$$

so the sequence $v_1 = u'_1$, $v_k = u'_k - u'_{k-1}$, $k > 1$, is absolutely summable. Its sequence of partial sums is $w_j = u'_j$ so the assumption is that this converges, hence the original Cauchy sequence converges and V is complete. □

Notice the idea here, of 'speeding up the convergence' of the Cauchy sequence by dropping a lot of terms. We will use this idea of absolutely summable series heavily in the discussion of Lebesgue integration.

4. Operators and functionals

As above, I suggest that you read this somewhere else (as well) for instance Wilde, [**5**], Chapter 2 to 2.7, Chen, [**1**], the first part of Chapter 6 and of Chapter 7 and/or Ward, [**4**], Chapter 3, first 2 sections.

The vector spaces we are most interested in are, as already remarked, spaces of functions (or something a little more general). The elements of these are the objects of primary interest but they are related by linear maps. A map between two vector spaces (over the same field, for us either $\mathbb{R}$ or $\mathbb{C}$) is linear if it takes linear combinations to linear combinations:-

$$T : V \longrightarrow W,\ T(a_1v_1+a_2v_2) = a_1T(v_1)+a_2T(v_2),\ \forall\, v_1,\ v_2 \in V,\ a_1, a_2 \in \mathbb{K}. \tag{1.36}$$

The sort of examples we have in mind are differential, or more especially, integral operators. For instance if $u \in \mathcal{C}([0,1])$ then its indefinite Riemann integral

$$(Tu)(x) = \int_0^x u(s)ds \tag{1.37}$$

is continuous in $x \in [0,1]$ and so this defines a map

$$T : \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1]). \tag{1.38}$$

This is a linear map, with linearity being one of the standard properties of the Riemann integral.

In the finite-dimensional case linearity is enough to allow maps to be studied. However in the case of infinite-dimensional normed spaces we need to impose continuity. Of course it makes perfectly good sense to say, demand or conclude, that a map as in (1.36) is continuous if V and W are normed spaces since they are then metric spaces. Recall that for metric spaces there are several different equivalent conditions that ensure a map, $T : V \longrightarrow W$, is continuous:

$$v_n \to v \text{ in } V \Longrightarrow Tv_n \to Tv \text{ in } W \tag{1.39}$$

$$O \subset W \text{ open } \Longrightarrow T^{-1}(O) \subset V \text{ open} \tag{1.40}$$

$$C \subset W \text{ closed } \Longrightarrow T^{-1}(C) \subset V \text{ closed.} \tag{1.41}$$

For a linear map between normed spaces there is a direct characterization of continuity in terms of the norm.

Proposition 3. *A linear map* (1.36) *between normed spaces is continuous if and only if it is* bounded *in the sense that there exists a constant C such that*

$$\|Tv\|_W \le C\|v\|_V \ \forall \ v \in V. \tag{1.42}$$

Of course bounded for a function on a metric space already has a meaning and this is not it! The usual sense would be $\|Tv\| \le C$ but this would imply $\|T(av)\| = |a|\|Tv\| \le C$ so $Tv = 0$. Hence it is not so dangerous to use the term 'bounded' for (1.42) – it is really 'relatively bounded', i.e. takes bounded sets into bounded sets. From now on, bounded for a linear map means (1.42).

Proof. If (1.42) holds then if $v_n \to v$ in V it follows that $\|Tv - Tv_n\| = \|T(v - v_n)\| \le C\|v - v_n\| \to 0$ as $n \to \infty$ so $Tv_n \to Tv$ and continuity follows.

For the reverse implication we use the second characterization of continuity above. Denote the ball around $v \in V$ of radius $\epsilon > 0$ by

$$B_V(v, \epsilon) = \{w \in V; \|v - w\| < \epsilon\}.$$

Thus if T is continuous then the inverse image of the the unit ball around the origin, $T^{-1}(B_W(0, 1)) = \{v \in V; \|Tv\|_W < 1\}$, contains the origin in V and so, being open, must contain some $B_V(0, \epsilon)$. This means that

$$T(B_V(0, \epsilon)) \subset B_W(0, 1) \text{ so } \|v\|_V < \epsilon \Longrightarrow \|Tv\|_W \le 1. \tag{1.43}$$

Now proceed by scaling. If $0 \ne v \in V$ then $\|v'\| < \epsilon$ where $v' = \epsilon v/2\|v\|$. So (1.43) shows that $\|Tv'\| \le 1$ but this implies (1.42) with $C = 2/\epsilon$ – it is trivially true if $v = 0$. □

As a general rule we drop the distinguishing subscript for norms, since which norm we are using can be determined by what it is being applied to.

So, if $T : V \longrightarrow W$ is continous and linear between normed spaces, or from now on 'bounded', then

$$\|T\| = \sup_{\|v\|=1} \|Tv\| < \infty. \tag{1.44}$$

Lemma 3. *The bounded linear maps between normed spaces V and W form a linear space $\mathcal{B}(V, W)$ on which $\|T\|$ defined by* (1.44) *or equivalently*

$$\|T\| = \inf\{C;\ (1.42)\ \textit{holds}\} \tag{1.45}$$

is a norm.

Proof. First check that (1.44) is equivalent to (1.45). Define $\|T\|$ by (1.44). Then for any $v \in V$, $v \ne 0$,

$$\|T\| \ge \|T(\frac{v}{\|v\|})\| = \frac{\|Tv\|}{\|v\|} \Longrightarrow \|Tv\| \le \|T\|\|v\| \tag{1.46}$$

since as always this is trivially true for $v = 0$. Thus $C = \|T\|$ is a constant for which (1.42) holds.

Conversely, from the definition of $\|T\|$, if $\epsilon > 0$ then there exists $v \in V$ with $\|v\| = 1$ such that $\|T\| - \epsilon < \|Tv\| \le C$ for any C for which (1.42) holds. Since $\epsilon > 0$ is arbitrary, $\|T\| \le C$ and hence $\|T\|$ is given by (1.45).

From the definition of $\|T\|$, $\|T\| = 0$ implies $Tv = 0$ for all $v \in V$ and for $\lambda \neq 0$,

$$\|\lambda T\| = \sup_{\|v\|=1} \|\lambda Tv\| = |\lambda|\|T\| \tag{1.47}$$

and this is also obvious for $\lambda = 0$. This only leaves the triangle inequality to check and for any T, $S \in \mathcal{B}(V, W)$, and $v \in V$ with $\|v\| = 1$

$$\|(T + S)v\|_W = \|Tv + Sv\|_W \le \|Tv\|_W + \|Sv\|_W \le \|T\| + \|S\| \tag{1.48}$$

so taking the supremum, $\|T + S\| \le \|T\| + \|S\|$. □

Thus we see the very satisfying fact that the space of bounded linear maps between two normed spaces is itself a normed space, with the norm being the best constant in the estimate (1.42). Make sure you absorb this! Such bounded linear maps between normed spaces are often called 'operators' because we are thinking of the normed spaces as being like function spaces.

You might like to check boundedness for the example of a linear operator in (1.38), namely that in terms of the supremum norm on $\mathcal{C}([0, 1])$, $\|T\| \le 1$.

One particularly important case is when $W = \mathbb{K}$ is the field, for us usually $\mathbb{C}$. Then a simpler notation is handy and one sets $V' = \mathcal{B}(V, \mathbb{C})$ – this is called the *dual space* of V (also sometimes denoted V^*.)

Proposition 4. *If W is a Banach space then $\mathcal{B}(V, W)$, with the norm* (1.44)*, is a Banach space.*

Proof. We simply need to show that if W is a Banach space then every Cauchy sequence in $\mathcal{B}(V, W)$ is convergent. The first thing to do is to find the limit. To say that $T_n \in \mathcal{B}(V, W)$ is Cauchy, is just to say that given $\epsilon > 0$ there exists N such that n, $m > N$ implies $\|T_n - T_m\| < \epsilon$. By the definition of the norm, if $v \in V$ then $\|T_n v - T_m v\|_W \le \|T_n - T_m\|\|v\|_V$ so $T_n v$ is Cauchy in W for each $v \in V$. By assumption, W is complete, so

$$T_n v \longrightarrow w \text{ in } W. \tag{1.49}$$

However, the limit can only depend on v so we can define a map $T : V \longrightarrow W$ by $Tv = w = \lim_{n\to\infty} T_n v$ as in (1.49).

This map defined from the limits is linear, since $T_n(\lambda v) = \lambda T_n v \longrightarrow \lambda Tv$ and $T_n(v_1 + v_2) = T_n v_1 + T_n v_2 \longrightarrow Tv_2 + Tv_2 = T(v_1 + v_2)$. Moreover, $|\|T_n\| - \|T_m\|| \le \|T_n - T_m\|$ so $\|T_n\|$ is Cauchy in $[0, \infty)$ and hence converges, with limit S, and

$$\|Tv\| = \lim_{n\to\infty} \|T_n v\| \le S\|v\| \tag{1.50}$$

so $\|T\| \le S$ shows that T is bounded.

Returning to the Cauchy condition above and passing to the limit in $\|T_n v - T_m v\| \le \epsilon\|v\|$ as $m \to \infty$ shows that $\|T_n - T\| \le \epsilon$ if $n > M$ and hence $T_n \to T$ in $\mathcal{B}(V, W)$ which is therefore complete. □

Note that this proof is structurally the same as that of Lemma 2.

One simple consequence of this is:-

Corollary 1. *The dual space of a normed space is always a Banach space.*

However you should be a little suspicious here since we have not shown that the dual space V' is non-trivial, meaning we have not eliminated the possibility that $V' = \{0\}$ even when $V \neq \{0\}$. The Hahn-Banach Theorem, discussed below, takes care of this.

One game you can play is 'what is the dual of that space'. Of course the dual is the dual, but you may well be able to identify the dual space of V with some other Banach space by finding a linear bijection between V' and the other space, W, which identifies the norms as well. We will play this game a bit later.

5. Subspaces and quotients

The notion of a linear subspace of a vector space is natural enough, and you are likely quite familiar with it. Namely $W \subset V$ where V is a vector space is a (linear) subspace if any linear combinations $\lambda_1 w_1 + \lambda_2 w_2 \in W$ if $\lambda_1, \lambda_2 \in \mathbb{K}$ and $w_1, w_2 \in W$. Thus W 'inherits' its linear structure from V. Since we also have a topology from the metric we will be especially interested in closed subspaces. Check that you understand the (elementary) proof of

Lemma 4. *A subspace of a Banach space is a Banach space in terms of the restriction of the norm if and only if it is closed.*

There is a second very important way to construct new linear spaces from old. Namely we want to make a linear space out of 'the rest' of V, given that W is a linear subspace. In finite dimensions one way to do this is to give V an inner product and then take the subspace orthogonal to W. One problem with this is that the result depends, although not in an essential way, on the inner product. Instead we adopt the usual 'myopia' approach and take an equivalence relation on V which identifies points which differ by an element of W. The equivalence classes are then 'planes parallel to W'. I am going through this construction quickly here under the assumption that it is familiar to most of you, if not you should think about it carefully since we need to do it several times later.

So, if $W \subset V$ is a linear subspace of V we define a relation on V – remember this is just a subset of $V \times V$ with certain properties – by

$$v \sim_W v' \Longleftrightarrow v - v' \in W \Longleftrightarrow \exists\, w \in W \text{ s.t. } v = v' + w. \tag{1.51}$$

This satisfies the three conditions for an equivalence relation:

(1) $v \sim_W v$
(2) $v \sim_W v' \Longleftrightarrow v' \sim_W v$
(3) $v \sim_W v',\ v' \sim_W w'' \Longrightarrow v \sim_W v''$

which means that we can regard it as a 'coarser notion of equality.'

Then V/W is the set of equivalence classes with respect to $\sim_W$. You can think of the elements of V/W as being of the form $v + W$ – a particular element of V plus an arbitrary element of W. Then of course $v' \in v + W$ if and only if $v' - v \in W$ meaning $v \sim_W v'$.

The crucial point here is that

$$V/W \text{ is a vector space.} \tag{1.52}$$

You should check the details – see Problem 1. Note that the 'is' in (1.52) should really be expanded to 'is in a natural way' since as usual the linear structure is inherited from V :

$$\lambda(v+W)=\lambda v+W,\ (v_1+W)+(v_2+W)=(v_1+v_2)+W. \tag{1.53}$$

The subspace W appears as the origin in $V/W.$

Now, two cases of this are of special interest to us.

Proposition 5. *If $\|\cdot\|$ is a seminorm on V then*

$$E=\{v\in V;\|v\|=0\}\subset V \tag{1.54}$$

is a linear subspace and

$$\|v+E\|_{V/E}=\|v\| \tag{1.55}$$

defines a norm on V/E.

Proof. That E is linear follows from the properties of a seminorm, since $\|\lambda v\|=|\lambda|\|v\|$ shows that $\lambda v\in E$ if $v\in E$ and $\lambda\in\mathbb{K}$. Similarly the triangle inequality shows that $v_1+v_2\in E$ if $v_1,\ v_2\in E$.

To check that (1.55) defines a norm, first we need to check that it makes sense as a function $\|\cdot\|_{V/E}\longrightarrow[0,\infty)$. This amounts to the statement that $\|v'\|$ is the same for all elements $v'=v+e\in v+E$ for a fixed v. This however follows from the triangle inequality applied twice:

$$\|v'\|\le\|v\|+\|e\|=\|v\|\le\|v'\|+\|-e\|=\|v'\|. \tag{1.56}$$

Now, I leave you the exercise of checking that $\|\cdot\|_{V/E}$ is a norm, see Problem 1. □

The second application is more serious, but in fact we will not use it for some time so I usually do not do this in lectures at this stage.

Proposition 6. *If $W\subset V$ is a closed subspace of a normed space then*

$$\|v+W\|_{V/W}=\inf_{w\in W}\|v+w\|_V \tag{1.57}$$

defines a norm on V/W; if V is a Banach space then so is V/W.

For the proof see Problems 1 and 1.

6. Completion

A normed space not being complete, not being a Banach space, is considered to be a defect which we might, indeed will, wish to rectify.

Let V be a normed space with norm $\|\cdot\|_V$. A *completion* of V is a Banach space B with the following properties:-

(1) There is an injective (i.e. 1-1) linear map $I:V\longrightarrow B$
(2) The norms satisfy

$$\|I(v)\|_B=\|v\|_V\ \forall\ v\in V. \tag{1.58}$$

(3) The range $I(V)\subset B$ is dense in B.

Notice that if V is itself a Banach space then we can take $B=V$ with I the identity map.

So, the main result is:

Theorem 1. *Each normed space has a completion.*

There are several ways to prove this, we will come across a more sophisticated one (using the Hahn-Banach Theorem) later. In the meantime I will give two proofs. In the first the fact that any metric space has a completion in a similar sense is recalled and then it is shown that the linear structure extends to the completion. A second, 'hands-on', proof is also outlined with the idea of motivating the construction of the Lebesgue integral – which is in our near future.

PROOF 1. One of the neater proofs that any metric space has a completion is to use Lemma 2. Pick a point in the metric space of interest, $p \in M$, and then define a map

$$M \ni q \longmapsto f_q \in \mathcal{C}_\infty(M),\ f_q(x) = d(x,q) - d(x,p)\ \forall\ x \in M. \tag{1.59}$$

That $f_q \in \mathcal{C}_\infty(M)$ is straightforward to check. It is bounded (because of the second term) by the reverse triangle inequality

$$|f_q(x)| = |d(x,q) - d(x,p)| \le d(p,q)$$

and is continuous, as the difference of two continuous functions. Moreover the distance between two functions in the image is

$$\sup_{x\in M} |f_q(x) - f_{q'}(x)| = \sup_{x\in M} |d(x,q) - d(x,q')| = d(q,q') \tag{1.60}$$

using the reverse triangle inequality (and evaluating at $x = q$). Thus the map (1.59) is well-defined, injective and even distance-preserving. Since $\mathcal{C}^0_\infty(M)$ is complete, the closure of the image of (1.59) is a complete metric space, X, in which M can be identified as a dense subset.

Now, in case that $M = V$ is a normed space this all goes through. The disconcerting thing is that the map $q \longrightarrow f_q$ is *not* linear. Nevertheless, we can give X a linear structure so that it becomes a Banach space in which V is a dense linear subspace. Namely for any two elements $f_i \in X$, $i = 1, 2$, define

$$\lambda_1 f_1 + \lambda_2 f_2 = \lim_{n\to\infty} f_{\lambda_1 p_n + \lambda_2 q_n} \tag{1.61}$$

where p_n and q_n are sequences in V such that $f_{p_n} \to f_1$ and $f_{q_n} \to f_2$. Such sequences exist by the construction of X and the result does not depend on the choice of sequence – since if p'_n is another choice in place of p_n then $f_{p'_n} - f_{p_n} \to 0$ in X (and similarly for q_n). So the element of the left in (1.61) is well-defined. All of the properties of a linear space and normed space now follow by continuity from $V \subset X$ and it also follows that X is a Banach space (since a closed subset of a complete space is complete). Unfortunately there are quite a few annoying details to check! $\square$

'PROOF 2' (THE LAST BIT IS LEFT TO YOU). Let V be a normed space. First we introduce the rather large space

$$\widetilde{V} = \left\{ \{u_k\}_{k=1}^\infty; u_k \in V \text{ and } \sum_{k=1}^\infty \|u_k\| < \infty \right\} \tag{1.62}$$

the elements of which, if you recall, are said to be absolutely summable. Notice that the elements of $\widetilde{V}$ are *sequences*, valued in V so two sequences are equal, are the same, only when each entry in one is equal to the corresponding entry in the other – no shifting around or anything is permitted as far as equality is concerned. We think of these as series (remember this means nothing except changing the name, a series is a sequence and a sequence is a series), the only difference is that we 'think' of taking the limit of a sequence but we 'think' of summing the elements of a series, whether we can do so or not being a different matter.

Now, each element of $\widetilde{V}$ is a Cauchy sequence – meaning the corresponding sequence of partial sums $v_N = \sum\limits_{k=1}^{N} u_k$ is Cauchy if $\{u_k\}$ is absolutely summable. As noted earlier, this is simply because if $M \geq N$ then

$$\|v_M - v_N\| = \|\sum_{j=N+1}^{M} u_j\| \leq \sum_{j=N+1}^{M} \|u_j\| \leq \sum_{j\geq N+1} \|u_j\| \tag{1.63}$$

gets small with N by the assumption that $\sum\limits_j \|u_j\| < \infty$.

Moreover, $\widetilde{V}$ is a linear space, where we add sequences, and multiply by constants, by doing the operations on each component:-

$$t_1\{u_k\} + t_2\{u'_k\} = \{t_1 u_k + t_2 u'_k\}. \tag{1.64}$$

This always gives an absolutely summable series by the triangle inequality:

$$\sum_k \|t_1 u_k + t_2 u'_k\| \leq |t_1| \sum_k \|u_k\| + |t_2| \sum_k \|u'_k\|. \tag{1.65}$$

Within $\widetilde{V}$ consider the linear subspace

$$S = \left\{ \{u_k\}; \sum_k \|u_k\| < \infty,\ \sum_k u_k = 0 \right\} \tag{1.66}$$

of those which sum to 0. As discussed in Section 5 above, we can form the quotient

$$B = \widetilde{V}/S \tag{1.67}$$

the elements of which are the 'cosets' of the form $\{u_k\} + S \subset \widetilde{V}$ where $\{u_k\} \in \widetilde{V}$. This is our completion, we proceed to check the following properties of this B.

(1) A norm on B (via a seminorm on $\tilde{V}$) is defined by

$$\|b\|_B = \lim_{n\to\infty} \|\sum_{k=1}^{n} u_k\|,\ b = \{u_k\} + S \in B. \tag{1.68}$$

(2) The original space V is imbedded in B by

$$V \ni v \longmapsto I(v) = \{u_k\} + S,\ u_1 = v,\ u_k = 0\ \forall\ k > 1 \tag{1.69}$$

and the norm satisfies (1.58).

(3) $I(V) \subset B$ is dense.

(4) B is a Banach space with the norm (1.68).

So, first that (1.68) is a norm. The limit on the right does exist since the limit of the norm of a Cauchy sequence always exists – namely the sequence of norms is itself Cauchy but now in $\mathbb{R}$. Moreover, adding an element of S to $\{u_k\}$ does not change the norm of the sequence of partial sums, since the additional term tends to zero in norm. Thus $\|b\|_B$ is well-defined for each element $b \in B$ and $\|b\|_B = 0$ means exactly that the sequence $\{u_k\}$ used to define it tends to 0 in norm, hence is in S hence $b = 0$ in B. The other two properties of norm are reasonably clear, since if b, $b' \in B$ are represented by $\{u_k\}$, $\{u'_k\}$ in $\widetilde{V}$ then tb and $b+b'$ are represented by $\{tu_k\}$ and $\{u_k+u'_k\}$ and

(1.70)
$$\lim_{n\to\infty}\|\sum_{k=1}^{n} tu_k\| = |t|\lim_{n\to\infty}\|\sum_{k=1}^{n} u_k\|, \Longrightarrow \|tb\| = |t|\|b\|$$
$$\lim_{n\to\infty}\|\sum_{k=1}^{n}(u_k+u'_k)\| = A \Longrightarrow$$
$$\text{for } \epsilon>0\ \exists\ N \text{ s.t. } \ \forall\ n\ge N,\ A-\epsilon \le \|\sum_{k=1}^{n}(u_k+u'_k)\| \Longrightarrow$$
$$A-\epsilon \le \|\sum_{k=1}^{n} u_k\| + \|\sum_{k=1}^{n} u'_k)\|\ \forall\ n\ge N \Longrightarrow A-\epsilon\le \|b\|_B+\|b'\|_B\ \forall\ \epsilon>0 \Longrightarrow$$
$$\|b+b'\|_B \le \|b\|_B+\|b'\|_B.$$

Now the norm of the element $I(v) = v, 0, 0, \cdots$, is the limit of the norms of the sequence of partial sums and hence is $\|v\|_V$ so $\|I(v)\|_B = \|v\|_V$ and $I(v) = 0$ therefore implies $v = 0$ and hence I is also injective.

We need to check that B is complete, and also that $I(V)$ is dense. Here is an extended discussion of the difficulty – of course maybe you can see it directly yourself (or have a better scheme). Note that I suggest that you to write out your own version of it carefully in Problem 1.

Okay, what does it mean for B to be a Banach space, as discussed above it means that every absolutely summable series in B is convergent. Such a series $\{b_n\}$ is given by $b_n = \{u_k^{(n)}\} + S$ where $\{u_k^{(n)}\} \in \widetilde{V}$ and the summability condition is that

$$\infty > \sum_n \|b_n\|_B = \sum_n \lim_{N\to\infty}\|\sum_{k=1}^{N} u_k^{(n)}\|_V. \tag{1.71}$$

So, we want to show that $\sum\limits_n b_n = b$ converges, and to do so we need to find the limit b. It is supposed to be given by an absolutely summable series. The 'problem' is that this series should look like $\sum\limits_n\sum\limits_k u_k^{(n)}$ in some sense – because it is supposed to represent the sum of the b_n's. Now, it would be very nice if we had the estimate

$$\sum_n\sum_k \|u_k^{(n)}\|_V < \infty \tag{1.72}$$

since this should allow us to break up the double sum in some nice way so as to get an absolutely summable series out of the whole thing. The trouble is that (1.72) need not hold. We know that *each* of the sums over k – for given n – converges, but not the sum of the sums. All we know here is that the sum of the 'limits of the norms' in (1.71) converges.

So, that is the problem! One way to see the solution is to note that we do not have to choose the original $\{u_k^{(n)}\}$ to 'represent' b_n – we can add to it any element of S. One idea is to rearrange the $u_k^{(n)}$ – I am thinking here of fixed n – so that it 'converges even faster.' I will not go through this in full detail but rather do it later when we need the argument for the completeness of the space of Lebesgue integrable functions. Given $\epsilon > 0$ we can choose p_1 so that for all $p \ge p_1$,

$$|\|\sum_{k\le p} u_k^{(n)}\|_V - \|b_n\|_B| \le \epsilon, \ \sum_{k\ge p}\|u_k^{(n)}\|_V \le \epsilon. \tag{1.73}$$

Then in fact we can choose successive $p_j > p_{j-1}$ (remember that little n is fixed here) so that

$$|\|\sum_{k\le p_j} u_k^{(n)}\|_V - \|b_n\|_B| \le 2^{-j}\epsilon, \ \sum_{k\ge p_j}\|u_k^{(n)}\|_V \le 2^{-j}\epsilon \ \forall \ j. \tag{1.74}$$

Now, 'resum the series' defining instead $v_1^{(n)} = \sum\limits_{k=1}^{p_1} u_k^{(n)}$, $v_j^{(n)} = \sum\limits_{k=p_{j-1}+1}^{p_j} u_k^{(n)}$ and do this setting $\epsilon = 2^{-n}$ for the nth series. Check that now

$$\sum_n \sum_k \|v_k^{(n)}\|_V < \infty. \tag{1.75}$$

Of course, you should also check that $b_n = \{v_k^{(n)}\} + S$ so that these new summable series work just as well as the old ones.

After this fiddling you can now try to find a limit for the sequence as

$$b = \{w_k\} + S, \ w_k = \sum_{l+p=k+1} v_l^{(p)} \in V. \tag{1.76}$$

So, you need to check that this $\{w_k\}$ is absolutely summable in V and that $b_n \to b$ as $n \to \infty$.

Finally then there is the question of showing that $I(V)$ is dense in B. You can do this using the same idea as above – in fact it might be better to do it first. Given an element $b \in B$ we need to find elements in V, v_k such that $\|I(v_k) - b\|_B \to 0$ as $k \to \infty$. Take an absolutely summable series u_k representing b and take $v_j = \sum\limits_{k=1}^{N_j} u_k$ where the p_j's are constructed as above and check that $I(v_j) \to b$ by computing

$$\|I(v_j) - b\|_B = \lim_{\to\infty} \|\sum_{k>p_j} u_k\|_V \le \sum_{k>p_j} \|u_k\|_V. \tag{1.77}$$

$\square$

7. More examples

Let me collect some examples of normed and Banach spaces. Those mentioned above and in the problems include:

- c_0 the space of convergent sequences in $\mathbb{C}$ with supremum norm, a Banach space.
- l^p one space for each real number $1 \le p < \infty$; the space of p-summable series with corresponding norm; all Banach spaces. The most important of these for us is the case $p = 2$, which is (a) Hilbert space.
- l^∞ the space of bounded sequences with supremum norm, a Banach space with $c_0 \subset l^\infty$ as a closed subspace with the same norm.
- $\mathcal{C}([a, b])$ or more generally $\mathcal{C}(M)$ for any compact metric space M – the Banach space of continuous functions with supremum norm.
- $\mathcal{C}_\infty(\mathbb{R})$, or more generally $\mathcal{C}_\infty(M)$ for any metric space M – the Banach space of bounded continuous functions with supremum norm.

- $\mathcal{C}_0(\mathbb{R})$, or more generally $\mathcal{C}_0(M)$ for any metric space M – the Banach space of continuous functions which 'vanish at infinity' (see Problem 1) with supremum norm. A closed subspace, with the same norm, in $\mathcal{C}^0_\infty(M)$.
- $\mathcal{C}^k([a,b])$ the space of k times continuously differentiable (so $k \in \mathbb{N}$) functions on $[a,b]$ with norm the sum of the supremum norms on the function and its derivatives. Each is a Banach space – see Problem 1.
- The space $\mathcal{C}([0,1])$ with norm

$$\|u\|_{L^1} = \int_0^1 |u| dx \tag{1.78}$$

given by the Riemann integral of the absolute value. A normed space, but not a Banach space. We will construct the concrete completion, $L^1([0,1])$ of Lebesgue integrable 'functions'.
- The space $\mathcal{R}([a,b])$ of Riemann integrable functions on $[a,b]$ with $\|u\|$ defined by (1.78). This is only a seminorm, since there are Riemann integrable functions (note that u Riemann integrable does imply that $|u|$ is Riemann integrable) with $|u|$ having vanishing Riemann integral but which are not identically zero. This cannot happen for continuous functions. So the quotient is a normed space, but it is not complete.
- The same spaces – either of continuous or of Riemann integrable functions but with the (semi- in the second case) norm

$$\|u\|_{L^p} = \left(\int_a^b |u|^p\right)^{\frac{1}{p}}. \tag{1.79}$$

Not complete in either case even after passing to the quotient to get a norm for Riemann integrable functions. We can, and indeed will, define $L^p(a,b)$ as the completion of $\mathcal{C}([a,b])$ with respect to the L^p norm. However we will get a concrete realization of it soon.
- Suppose $0 < \alpha < 1$ and consider the subspace of $\mathcal{C}([a,b])$ consisting of the 'Hölder continuous functions' with exponent α, that is those $u : [a,b] \longrightarrow \mathbb{C}$ which satisfy

$$|u(x) - u(y)| \le C|x-y|^\alpha \text{ for some } C \ge 0. \tag{1.80}$$

Note that this already implies the continuity of u. As norm one can take the sum of the supremum norm and the 'best constant' which is the same as

$$\|u\|_{\mathcal{C}^\alpha} = \sup_{x\in[a,b]|} |u(x)| + \sup_{x\neq y\in[a,b]} \frac{|u(x)-u(y)|}{|x-y|^\alpha}; \tag{1.81}$$

it is a Banach space usually denoted $\mathcal{C}^\alpha([a,b])$.
- Note the previous example works for $\alpha = 1$ as well, then it is not denoted $\mathcal{C}^1([a,b])$, since that is the space of once continuously differentiable functions; this is the space of Lipschitz functions – again it is a Banach space.
- We will also talk about Sobolev spaces later. These are functions with 'Lebesgue integrable derivatives'. It is perhaps not easy to see how to

define these, but if one takes the norm on $\mathcal{C}^1([a,b])$

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \|\frac{du}{dx}\|_{L^2}^2\right)^{\frac{1}{2}} \tag{1.82}$$

and completes it, one gets the Sobolev space $H^1([a,b])$ – it is a Banach space (and a Hilbert space). In fact it is a subspace of $\mathcal{C}([a,b]) = \mathcal{C}^0([a,b])$.

Here is an example to see that the space of continuous functions on $[0,1]$ with norm (1.78) is not complete; things are even worse than this example indicates! It is a bit harder to show that the quotient of the Riemann integrable functions is not complete, feel free to give it a try.

Take a simple non-negative continuous function on $\mathbb{R}$ for instance

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1. \end{cases} \tag{1.83}$$

Then $\int_{-1}^1 f(x) = 1$. Now scale it up and in by setting

$$f_N(x) = Nf(N^3x) = 0 \text{ if } |x| > N^{-3}. \tag{1.84}$$

So it vanishes outside $[-N^{-3}, N^{-3}]$ and has $\int_{-1}^1 f_N(x)dx = N^{-2}$. It follows that the sequence $\{f_N\}$ is absolutely summable with respect to the integral norm in (1.78) on $[-1,1]$. The pointwise series $\sum\limits_N f_N(x)$ converges everywhere except at $x = 0$ – since at each point $x \ne 0$, $f_N(x) = 0$ if $N^3|x| > 1$. The resulting function, even if we ignore the problem at $x = 0$, is not Riemann integrable because it is not bounded.

You might respond that the sum of the series is 'improperly Riemann integrable'. This is true but does not help much.

It is at this point that I start doing Lebesgue integration in the lectures. The following material is from later in the course but fits here quite reasonably.

8. Baire's theorem

At least once I wrote a version of the following material on the blackboard during the first mid-term test, in an an attempt to distract people. It did not work very well – its seems that MIT students have already been toughened up by this stage. Baire's theorem will be used later (it is also known as 'Baire category theory' although it has nothing to do with categories in the modern sense).

This is a theorem about complete metric spaces – it could be included in the earlier course 'Real Analysis' but the main applications are in Functional Analysis.

Theorem 2 (Baire). *If M is a non-empty complete metric space and $C_n \subset M$, $n \in \mathbb{N}$, are closed subsets such that*

$$M = \bigcup_n C_n \tag{1.85}$$

then at least one of the C_n's has an interior point.

Proof. We can assume that the first set $C_1 \ne \emptyset$ since they cannot all be empty and dropping any empty sets does no harm. Let's assume the contrary of the desired conclusion, namely that each of the C_n's has empty interior, hoping to arrive at a contradiction to (1.85) using the other properties. This means that an open ball $B(p,\epsilon)$ around a point of M (so it isn't empty) cannot be contained in any one of the C_n.

So, choose $p \in C_1$. Now, there must be a point $p_1 \in B(p, 1/3)$ which is not in C_1. Since C_1 is closed there exists $\epsilon_1 > 0$, and we can take $\epsilon_1 < 1/3$, such that $B(p_1, \epsilon_1) \cap C_1 = \emptyset$. Continue in this way, choose $p_2 \in B(p_1, \epsilon_1/3)$ which is not in C_2 and $\epsilon_2 > 0$, $\epsilon_2 < \epsilon_1/3$ such that $B(p_2, \epsilon_2) \cap C_2 = \emptyset$. Here we use both the fact that C_2 has empty interior and the fact that it is closed. So, inductively there is a sequence p_i, $i = 1, \dots, k$ and positive numbers $0 < \epsilon_k < \epsilon_{k-1}/3 < \epsilon_{k-2}/3^2 < \cdots < \epsilon_1/3^{k-1} < 3^{-k}$ such that $p_j \in B(p_{j-1}, \epsilon_{j-1}/3)$ and $B(p_j, \epsilon_j) \cap C_j = \emptyset$. Then we can add another p_{k+1} by using the properties of C_k – it has non-empty interior so there is some point in $B(p_k, \epsilon_k/3)$ which is not in C_{k+1} and then $B(p_{k+1}, \epsilon_{k+1}) \cap C_{k+1} = \emptyset$ where $\epsilon_{k+1} > 0$ but $\epsilon_{k+1} < \epsilon_k/3$. Thus, we have a sequence $\{p_k\}$ in M. Since $d(p_{k+1}, p_k) < \epsilon_k/3$ this is a Cauchy sequence, in fact

$$d(p_k, p_{k+l}) < \epsilon_k/3 + \cdots + \epsilon_{k+l-1}/3 < 3^{-k}. \tag{1.86}$$

Since M is complete the sequence converges to a limit, $q \in M$. Notice however that $p_l \in B(p_k, 2\epsilon_k/3)$ for all $k > l$ so $d(p_k, q) \le 2\epsilon_k/3$ which implies that $q \notin C_k$ for any k. This is the desired contradiction to (1.85).

Thus, at least one of the C_n must have non-empty interior. $\square$

In applications one might get a complete mentric space written as a countable union of subsets

$$M = \bigcup_n E_n,\ E_n \subset M \tag{1.87}$$

where the E_n are not necessarily closed. We can still apply Baire's theorem however, just take $C_n = \overline{E_n}$ to be the closures – then of course (1.85) holds since $E_n \subset C_n$. The conclusion is

(1.88) For at least one n the closure of E_n has non-empty interior.

9. Uniform boundedness

One application of Baire's theorem is often called the *uniform boundedness principle* or Banach-Steinhaus Theorem.

Theorem 3 (Uniform boundedness). *Let B be a Banach space and suppose that T_n is a sequence of bounded (i.e. continuous) linear operators $T_n : B \longrightarrow V$ where V is a normed space. Suppose that for each $b \in B$ the set $\{T_n(b)\} \subset V$ is bounded (in norm of course) then* $\sup_n \|T_n\| < \infty$.

Proof. This follows from a pretty direct application of Baire's theorem to B. Consider the sets

$$S_p = \{b \in B,\ \|b\| \le 1,\ \|T_n b\|_V \le p\ \forall\ n\},\ p \in \mathbb{N}. \tag{1.89}$$

Each S_p is closed because T_n is continuous, so if $b_k \to b$ is a convergent sequence in S_p then $\|b\| \le 1$ and $\|T_n(b)\| \le p$. The union of the S_p is the whole of the closed ball of radius one around the origin in B :

$$\{b \in B; d(b, 0) \le 1\} = \bigcup_p S_p \tag{1.90}$$

because of the assumption of 'pointwise boundedness' – each b with $\|b\| \le 1$ must be in one of the S_p's.

So, by Baire's theorem one of the sets S_p has non-empty interior, it therefore contains a closed ball of positive radius around some point. Thus for some p, some $v \in S_p$, and some $\delta > 0$,

$$w \in B,\ \|w\|_B \le \delta \Longrightarrow \|T_n(v+w)\|_V \le p \ \forall\ n. \tag{1.91}$$

Since $v \in S_p$ is fixed it follows that $\|T_n w\| \le \|T_n v\| + p \le 2p$ for all w with $\|w\| \le \delta$.

Moving v to $(1-\delta/2)v$ and halving δ as necessary it follows that this ball $B(v,\delta)$ is contained in the open ball around the origin of radius 1. Thus, using the triangle inequality, and the fact that $\|T_n(v)\|_V \le p$ this implies

$$w \in B,\ \|w\|_B \le \delta \Longrightarrow \|T_n(w)\|_V \le 2p \Longrightarrow \|T_n\| \le 2p/\delta. \tag{1.92}$$

The norm of the operator is $\sup\{\|Tw\|_V; \|w\|_B = 1\} = \frac{1}{\delta}\sup\{\|Tw\|_V; \|w\|_B = \delta\}$ so the norms are uniformly bounded:

$$\|T_n\| \le 2p/\delta \tag{1.93}$$

as claimed. □

10. Open mapping theorem

The second major application of Baire's theorem is to

THEOREM 4 (Open Mapping). *If $T : B_1 \longrightarrow B_2$ is a bounded and surjective linear map between two Banach spaces then T is open:*

$$T(O) \subset B_2 \text{ is open if } O \subset B_1 \text{ is open.} \tag{1.94}$$

This is 'wrong way continuity' and as such can be used to prove the continuity of inverse maps as we shall see. The proof uses Baire's theorem pretty directly, but then another similar sort of argument is needed to complete the proof. There are more direct but more computational proofs, see Problem 1. I prefer this one because I have a reasonable chance of remembering the steps.

PROOF. What we will try to show is that the image under T of the unit open ball around the origin, $B(0,1) \subset B_1$ contains an open ball around the origin in B_2. The first part, of the proof, using Baire's theorem shows that the *closure* of the image, so in B_2, has 0 as an interior point – i.e. it contains an open ball around the origin in B_2 :

$$\overline{T(B(0,1))} \supset B(0,\delta),\ \delta > 0. \tag{1.95}$$

To see this we apply Baire's theorem to the sets

$$C_p = \mathrm{cl}_{B_2} T(B(0,p)) \tag{1.96}$$

the closure of the image of the ball in B_1 of radius p. We know that

$$B_2 = \bigcup_p T(B(0,p)) \tag{1.97}$$

since that is what surjectivity means – every point is the image of something. Thus one of the closed sets C_p has an interior point, v. Since T is surjective, $v = Tu$ for some $u \in B_1$. The sets C_p increase with p so we can take a larger p and v is still an interior point, from which it follows that $0 = v - Tu$ is an interior point as well. Thus indeed

$$C_p \supset B(0,\delta) \tag{1.98}$$

for some $\delta > 0$. Rescaling by p, using the linearity of T, it follows that with δ replaced by δ/p, we get (1.95).

Having applied Baire's thereom, consider now what (1.95) means. It follows that each $v \in B_2$, with $\|v\| = \delta$, is the limit of a sequence Tu_n where $\|u_n\| \le 1$. What we want to find is such a sequence, u_n, which converges. To do so we need to choose the sequence more carefully. Certainly we can stop somewhere along the way and see that

$$v \in B_2,\ \|v\| = \delta \Longrightarrow \exists\ u \in B_1,\ \|u\| \le 1,\ \|v - Tu\| \le \frac{\delta}{2} = \frac{1}{2}\|v\| \tag{1.99}$$

where of course we could replace $\frac{\delta}{2}$ by any positive constant but the point is the last inequality is now relative to the norm of v. Scaling again, if we take any $v \neq 0$ in B_2 and apply (1.99) to $v/\|v\|$ we conclude that (for $C = p/\delta$ a fixed constant)

$$v \in B_2 \Longrightarrow\ \exists\ u \in B_1,\ \|u\| \le C\|v\|,\ \|v - Tu\| \le \frac{1}{2}\|v\| \tag{1.100}$$

where the size of u only depends on the size of v; of course this is also true for $v = 0$ by taking $u = 0$.

Using this we construct the desired *better* approximating sequence. Given $w \in B_1$, choose $u_1 = u$ according to (1.100) for $v = w = w_1$. Thus $\|u_1\| \le C$, and $w_2 = w_1 - Tu_1$ satisfies $\|w_2\| \le \frac{1}{2}\|w\|$. Now proceed by induction, supposing that we have constructed a sequence u_j, $j < n$, in B_1 with $\|u_j\| \le C2^{-j+1}\|w\|$ and $\|w_j\| \le 2^{-j+1}\|w\|$ for $j \le n$, where $w_j = w_{j-1} - Tu_{j-1}$ – which we have for $n = 1$. Then we can choose u_n, using (1.100), so $\|u_n\| \le C\|w_n\| \le C2^{-n+1}\|w\|$ and such that $w_{n+1} = w_n - Tu_n$ has $\|w_{n+1}\| \le \frac{1}{2}\|w_n\| \le 2^{-n}\|w\|$ to extend the induction. Thus we get a sequence u_n which is absolutely summable in B_1, since $\sum\limits_n \|u_n\| \le 2C\|w\|$, and hence converges by the assumed completeness of B_1 this time. Moreover

$$w - T(\sum_{j=1}^{n} u_j) = w_1 - \sum_{j=1}^{n}(w_j - w_{j+1}) = w_{n+1} \tag{1.101}$$

so $Tu = w$ and $\|u\| \le 2C\|w\|$.

Thus finally we have shown that each $w \in B(0,1)$ in B_2 is the image of some $u \in B_1$ with $\|u\| \le 2C$. Thus $T(B(0,3C)) \supset B(0,1)$. By scaling it follows that the image of any open ball around the origin contains an open ball around the origin.

Now, the linearity of T shows that the image $T(O)$ of any open set is open, since if $w \in T(O)$ then $w = Tu$ for some $u \in O$ and hence $u + B(0,\epsilon) \subset O$ for $\epsilon > 0$ and then $w + B(0,\delta) \subset T(O)$ for $\delta > 0$ sufficiently small. $\square$

One important corollary of this is something that seems like it should be obvious, but definitely needs completeness to be true.

Corollary 2. *If $T : B_1 \longrightarrow B_2$ is a bounded linear map between Banach spaces which is 1-1 and onto, i.e. is a bijection, then it is a homeomorphism – meaning its inverse, which is necessarily linear, is also bounded.*

Proof. The only confusing thing is the notation. Note that T^{-1} is generally used both for the inverse, when it exists, and also to denote the inverse map on sets even when there is no true inverse. The inverse of T, let's call it $S : B_2 \longrightarrow B_1$, is certainly linear. If $O \subset B_1$ is open then $S^{-1}(O) = T(O)$, since to say $v \in S^{-1}(O)$

means $S(v) \in O$ which is just $v \in T(O)$, is open by the Open Mapping theorem, so S is continuous. $\square$

11. Closed graph theorem

For the next application you should check, it is one of the problems, that the product of two Banach spaces, $B_1 \times B_2$, – which is just the linear space of all pairs (u, v), $u \in B_1$ and $v \in B_2$, is a Banach space with respect to the sum of the norms

$$\|(u, v)\| = \|u\|_1 + \|v\|_2. \tag{1.102}$$

THEOREM 5 (Closed Graph). *If $T : B_1 \longrightarrow B_2$ is a linear map between Banach spaces then it is bounded if and only if its graph*

$$\operatorname{Gr}(T) = \{(u, v) \in B_1 \times B_2; v = Tu\} \tag{1.103}$$

is a closed subset of the Banach space $B_1 \times B_2$.

PROOF. Suppose first that T is bounded, i.e. continuous. A sequence $(u_n, v_n) \in B_1 \times B_2$ is in $\operatorname{Gr}(T)$ if and only if $v_n = Tu_n$. So, if it converges, then $u_n \to u$ and $v_n = Tu_n \to Tv$ by the continuity of T, so the limit is in $\operatorname{Gr}(T)$ which is therefore closed.

Conversely, suppose the graph is closed. This means that viewed as a normed space in its own right it is complete. Given the graph we can reconstruct the map it comes from (whether linear or not) in a little diagram. From $B_1 \times B_2$ consider the two projections, $\pi_1(u, v) = u$ and $\pi_2(u, v) = v$. Both of them are continuous since the norm of either u or v is less than the norm in (1.102). Restricting them to $\operatorname{Gr}(T) \subset B_1 \times B_2$ gives

$$\begin{tikzcd} & \operatorname{Gr}(T) \arrow[dl, shift left, "\pi_1"] \arrow[dr, "\pi_2"] & \\ B_1 \arrow[ur, shift left, "S"] \arrow[rr, "T"] & & B_2. \end{tikzcd} \tag{1.104}$$

This little diagram commutes. Indeed there are two ways to map a point $(u, v) \in \operatorname{Gr}(T)$ to B_2, either directly, sending it to v or first sending it to $u \in B_1$ and then to Tu. Since $v = Tu$ these are the same.

Now, as already noted, $\operatorname{Gr}(T) \subset B_1 \times B_2$ is a closed subspace, so it too is a Banach space and π_1 and π_2 remain continuous when restricted to it. The map π_1 is 1-1 and onto, because each u occurs as the first element of precisely one pair, namely $(u, Tu) \in \operatorname{Gr}(T)$. Thus the Corollary above applies to π_1 to show that its inverse, S is continuous. But then $T = \pi_2 \circ S$, from the commutativity, is also continuous proving the theorem. $\square$

12. Hahn-Banach theorem

Now, there is always a little pressure to state and prove the Hahn-Banach Theorem. This is about extension of functionals. Stately starkly, the basic question is: Does a normed space have *any* non-trivial continuous linear functionals on it? That is, is the dual space always non-trivial (of course there is always the zero linear functional but that is not very amusing). We do not really encounter this problem since for a Hilbert space, or even a pre-Hilbert space, there is always the space itself, giving continuous linear functionals through the pairing – Riesz' Theorem says that in the case of a Hilbert space that is all there is. If you are following the course then at this point you should also see that the only continuous linear functionals on a pre-Hilbert space correspond to points in the completion. I could have used the Hahn-Banach Theorem to show that any normed space has a completion, but I gave a more direct argument for this, which was in any case much more relevant for the cases of $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ for which we wanted *concrete* completions.

THEOREM 6 (Hahn-Banach). *If $M \subset V$ is a linear subspace of a normed space and $u : M \longrightarrow \mathbb{C}$ is a linear map such that*

$$|u(t)| \le C\|t\|_V \;\forall\; t \in M \tag{1.105}$$

then there exists a bounded linear functional $U : V \longrightarrow \mathbb{C}$ with $\|U\| \le C$ and $U\big|_M = u$.

First, by computation, we show that we can extend any continuous linear functional 'a little bit' without increasing the norm.

LEMMA 5. *Suppose $M \subset V$ is a subspace of a normed linear space, $x \notin M$ and $u : M \longrightarrow \mathbb{C}$ is a bounded linear functional as in* (1.105) *then there exists $u' : M' \longrightarrow \mathbb{C}$, where $M' = \{t' \in V; t' = t + ax,\ a \in \mathbb{C}\}$, such that*

$$u'\big|_M = u,\ |u'(t+ax)| \le C\|t+ax\|_V,\ \forall\; t \in M,\ a \in \mathbb{C}. \tag{1.106}$$

PROOF. Note that the decompositon $t' = t + ax$ of a point in M' is unique, since $t + ax = \tilde{t} + \tilde{a}x$ implies $(a - \tilde{a})x \in M$ so $a = \tilde{a}$, since $x \notin M$ and hence $t = \tilde{t}$ as well. Thus

$$u'(t+ax) = u'(t) + au(x) = u(t) + \lambda a,\ \lambda = u'(x) \tag{1.107}$$

and all we have at our disposal is the choice of λ. Any choice will give a linear functional extending u, the problem of course is to arrange the continuity estimate without increasing the constant C. In fact if $C = 0$ then $u = 0$ and we can take the zero extension. So we might as well assume that $C = 1$ since dividing u by C arranges this and if u' extends u/C then Cu' extends u and the norm estimate in (1.106) follows. So we now assume that

$$|u(t)| \le \|t\|_V \;\forall\; t \in M. \tag{1.108}$$

We want to choose λ so that

$$|u(t) + a\lambda| \le \|t + ax\|_V \;\forall\; t \in M,\ a \in \mathbb{C}. \tag{1.109}$$

Certainly when $a = 0$ this represents no restriction on λ. For $a \neq 0$ we can divide through by $-a$ and (1.109) becomes

$$|a||u(-\frac{t}{a}) - \lambda| = |u(t) + a\lambda| \le \|t + ax\|_V = |a|\| - \frac{t}{a} - x\|_V \tag{1.110}$$

and since $-t/a \in M$ we only need to arrange that

$$|u(t) - \lambda| \le \|t - x\|_V \;\forall\; t \in M \tag{1.111}$$

and the general case will follow by reversing the scaling.

We will show that it is possible to choose λ to be real. A complex linear functional such as u can be recovered from its real part, as we see below, so set

$$w(t) = \mathrm{Re}(u(t)) \;\forall\; t \in M \tag{1.112}$$

and just try to extend w to a real functional – it is not linear over the complex numbers of course, just over the reals – satisfying the analogue of (1.111):

$$|w(t) - \lambda| \le \|t - x\|_V \;\forall\; t \in M \tag{1.113}$$

which anyway does not involve linearity. What we know about w is the norm estimate (1.108) which (using linearity) implies

$$|w(t_1) - w(t_2)| \le |u(t_1) - u(t_2)| \le \|t_1 - t_2\| \le \|t_1 - x\|_V + \|t_2 - x\|_V. \tag{1.114}$$

Writing this out using the reality we find

$$\begin{aligned} w(t_1) - w(t_2) \le \|t_1 - x\|_V + \|t_2 - x\|_V \Longrightarrow \\ w(t_1) - \|t_1 - x\| \le w(t_2) + \|t_2 - x\|_V \ \forall\ t_1,\ t_2 \in M. \end{aligned} \tag{1.115}$$

We can then take the supremum on the left and the infimum on the right and choose λ in between – namely we have shown that there exists $\lambda \in \mathbb{R}$ with

$$\begin{aligned} w(t) - \|t - x\|_V \le \sup_{t_2 \in M} (w(t_1) - \|t_1 - x\|) \le \lambda \\ \le \inf_{t_2 \in M} (w(t_1) + \|t_1 - x\|) \le w(t) + \|t - x\|_V \ \forall\ t \in M. \end{aligned} \tag{1.116}$$

This in turn implies that

$$-\|t - x\|_V \le -w(t) + \lambda \le \|t - x\|_V \Longrightarrow |w(t) - \lambda| \le \|t - x\|_V \ \forall\ t \in M. \tag{1.117}$$

This is what we wanted – we have extended the real part of u to

$$w'(t + ax) = w(t) - (\operatorname{Re} a)\lambda \text{ and } |w'(t + ax)| \le \|t + ax\|_V. \tag{1.118}$$

Now, finally we get the extension of u itself by 'complexifying' – defining

$$u'(t + ax) = w'(t + ax) - iw'(i(t + ax)). \tag{1.119}$$

This is linear over the complex numbers since

$$\begin{aligned} u'(z(t + ax)) = w'(z(t + ax)) - iw'(iz(t + ax) \\ = w'(\operatorname{Re} z(t + ax) + i \operatorname{Im} z(t + ax)) - iw'(i \operatorname{Re} z(t + ax)) + iw'(\operatorname{Im} z(t + ax)) \\ = (\operatorname{Re} z + i \operatorname{Im} z)w'(t + ax) - i(\operatorname{Re} z + i \operatorname{Im} z)(w'(i(t + ax)) = zu'(t + ax). \end{aligned} \tag{1.120}$$

It certainly extends u from M – since the same identity gives u in terms of its real part w.

Finally then, to see the norm estimate note that (as we did long ago) there exists a uniqe $\theta \in [0, 2\pi)$ such that

$$\begin{aligned} |u'(t + ax)| = \operatorname{Re} e^{i\theta} u'(t + ax) = \operatorname{Re} u'(e^{i\theta} t + e^{i\theta} ax) \\ = w'(e^{i\theta} u + e^{i\theta} ax) \le \|e^{i\theta}(t + ax)\|_V = \|t + ax\|_V. \end{aligned} \tag{1.121}$$

This completes the proof of the Lemma. □

Proof of Hahn-Banach. This is an application of Zorn's Lemma. I am not going to get into the derivation of Zorn's Lemma from the Axiom of Choice, but if you believe the latter – and you are advised to do so, at least before lunchtime – you should believe the former.

Zorn's Lemma is a statement about partially ordered sets. A partial order on a set E is a subset of $E \times E$, so a relation, where the condition that (e, f) be in the relation is written $e \prec f$ and it must satisfy

$$e \prec e,\ e \prec f \text{ and } f \prec e \Longrightarrow e = f,\ e \prec f \text{ and } f \prec g \Longrightarrow e \prec g. \tag{1.122}$$

So, the missing ingredient between this and an order is that two elements need not be related at all, either way.

A subsets of a partially ordered set inherits the partial order and such a subset is said to be a *chain* if each pair of its elements *is* related one way or the other.

An *upper bound* on a subset $D \subset E$ is an element $e \in E$ such that $d \prec e$ for all $d \in D$. A *maximal* element of E is one which is not majorized, that is $e \prec f$, $f \in E$, implies $e = f$.

LEMMA 6 (Zorn). *If every chain in a (non-empty) partially ordered set has an upper bound then the set contains at least one maximal element.*

Now, we are given a functional $u : M \longrightarrow \mathbb{C}$ defined on some linear subspace $M \subset V$ of a normed space where u is bounded with respect to the induced norm on M. We will apply Zorn's Lemma to the set E consisting of *all* extensions (v, N) of u with the same norm. That is,

$$V \supset N \supset M, \ v\big|_M = u \text{ and } \|v\|_N = \|u\|_M.$$

This is certainly non-empty since it contains (u, M) and has the natural partial order that $(v_1, N_1) \prec (v_2, N_2)$ if $N_1 \subset N_2$ and $v_2\big|_{N_1} = v_1$. You should check that this is a partial order.

Let C be a chain in this set of extensions. Thus for any two elements $(v_i, N_1) \in C$, either $(v_1, N_1) \prec (v_2, N_2)$ or the other way around. This means that

$$\tilde{N} = \bigcup \{N; (v, N) \in C \text{ for some } v\} \subset V \tag{1.123}$$

is a linear space. Note that this union need not be countable, or anything like that, but any two elements of $\tilde{N}$ are each in one of the N's and one of these must be contained in the other by the chain condition. Thus each pair of elements of $\tilde{N}$ is actually in a common N and hence so is their linear span. Similarly we can define an extension

$$\tilde{v} : \tilde{N} \longrightarrow \mathbb{C}, \ \tilde{v}(x) = v(x) \text{ if } x \in N, \ (v, N) \in C. \tag{1.124}$$

There may be many pairs $(v, N) \in C$ satisfying $x \in N$ for a given x but the chain condition implies that $v(x)$ is the same for all of them. Thus $\tilde{v}$ is well defined, and is clearly also linear, extends u and satisfies the norm condition $|\tilde{v}(x)| \le \|u\|_M \|v\|_V$. Thus $(\tilde{v}, \tilde{N})$ is an upper bound for the chain C.

So, the set of all extension E, with the norm condition, satisfies the hypothesis of Zorn's Lemma, so must – at least in the mornings – have a maximal element $(\tilde{u}, \tilde{M})$. If $\tilde{M} = V$ then we are done. However, in the contary case there exists $x \in V \setminus \tilde{M}$. This means we can apply our little lemma and construct an extension $(u', \tilde{M}')$ of $(\tilde{u}, \tilde{M})$ which is therefore also an element of E and satisfies $(\tilde{u}, \tilde{M}) \prec (u', \tilde{M}')$. This however contradicts the condition that $(\tilde{u}, \tilde{M})$ be maximal, so is forbidden by Zorn. □

There are many applications of the Hahn-Banach Theorem. As remarked earlier, one significant one is that the dual space of a non-trivial normed space is itself non-trivial.

PROPOSITION 7. *For any normed space V and element $0 \neq v \in V$ there is a continuous linear functional $f : V \longrightarrow \mathbb{C}$ with $f(v) = 1$ and $\|f\| = 1/\|v\|_V$.*

PROOF. Start with the one-dimensional space, M, spanned by v and define $u(zv) = z$. This has norm $1/\|v\|_V$. Extend it using the Hahn-Banach Theorem and you will get a continuous functional f as desired. □

13. Double dual

Let me give another application of the Hahn-Banach theorem, although I have never covered this in lectures. If V is a normed space, we know its dual space, V', to be a Banach space. Let $V'' = (V')'$ be the dual of the dual.

PROPOSITION 8. *If $v \in V$ then the linear map on V' :*

$$T_v : V' \longrightarrow \mathbb{C},\ T_v(v') = v'(v) \tag{1.125}$$

is continuous and this defines an isometric linear injection $V \hookrightarrow V''$, $\|T_v\| = \|v\|$.

PROOF. The definition of T_v is 'tautologous', meaning it is almost the definition of V'. First check T_v in (1.125) is linear. Indeed, if v_1', $v_2' \in V'$ and λ_1, $\lambda_2 \in \mathbb{C}$ then $T_v(\lambda_1 v_1' + \lambda_2 v_2') = (\lambda_1 v_1' + \lambda_2 v_2')(v) = \lambda_1 v_1'(v) + \lambda_2 v_2'(v) = \lambda_1 T_v(v_1') + \lambda_2 T_v(v_2')$. That $T_v \in V''$, i.e. is bounded, follows too since $|T_v(v')| = |v'(v)| \le \|v'\|_{V'}\|v\|_V$; this also shows that $\|T_v\|_{V''} \le \|v\|$. On the other hand, by Proposition 7 above, if $\|v\| = 1$ then there exists $v' \in V'$ such that $v'(v) = 1$ and $\|v'\|_{V'} = 1$. Then $T_v(v') = v'(v) = 1$ shows that $\|T_v\| = 1$ so in general $\|T_v\| = \|v\|$. It also needs to be checked that $V \ni v \longmapsto T_v \in V''$ is a linear map – this is clear from the definition. It is necessarily 1-1 since $\|T_v\| = \|v\|$. $\square$

Now, it is definitely not the case in general that $V'' = V$ in the sense that this injection is also a surjection. Since V'' is always a Banach space, one necessary condition is that V itself should be a Banach space. In fact the closure of the image of V in V'' is a completion of V. If the map to V'' is a bijection then V is said to be *reflexive*. It is pretty easy to find examples of non-reflexive Banach spaces, the most familiar is c_0 – the space of infinite sequences converging to 0. Its dual can be identified with l^1, the space of summable sequences. Its dual in turn, the bidual of c_0, is the space l^∞ of bounded sequences, into which the embedding is the obvious one, so c_0 is not reflexive. In fact l^1 is not reflexive either. There are useful characterizations of reflexive Banach spaces. You may be interested enough to look up James' Theorem:- A Banach space is reflexive if and only if every continuous linear functional on it attains its supremum on the unit ball.

14. Axioms of a vector space

In case you missed out on one of the basic linear algebra courses, or have a poor memory, here are the axioms of a vector space over a field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$ for us).

A *vector space structure* on a set V is a pair of maps

$$+ : V \times V \longrightarrow V,\ \cdot : \mathbb{K} \times V \longrightarrow V \tag{1.126}$$

satisfying the conditions listed below. These maps are written $+(v_1, v_2) = v_1 + v_2$ and $\cdot(\lambda, v) = \lambda v$, $\lambda \in \mathbb{K}$, V, v_1, $v_2 \in V$.

additive commutativity $v_1 + v_2 = v_2 + v_2$ for all v_1, $v_2 \in V$.
additive associativity $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$ for all v_1, v_2, $v_3 \in V$.
existence of zero There is an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.
additive invertibility For each $v \in V$ there exists $w \in V$ such that $v + w = 0$.
distributivity of scalar additivity $(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$ for all λ_1, $\lambda_2 \in \mathbb{K}$ and $v \in V$.
multiplicativity $\lambda_1(\lambda_2 v) = (\lambda_1\lambda_2)v$ for all λ_1, $\lambda_2 \in \mathbb{K}$ and $v \in V$.
action of multiplicative identity $1v = v$ for all $v \in V$.

distributivity of space additivity $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$ for all $\lambda \in \mathbb{K}$ v_1, $v_2 \in V$.

CHAPTER 2

The Lebesgue integral

This part of the course, on Lebesgue integration, has evolved the most. Initially I followed the book of Debnaith and Mikusinski, completing the space of step functions on the line under the L^1 norm. Since the 'Spring' semester of 2011, I have decided to circumvent the discussion of step functions, proceeding directly by completing the Riemann integral. Some of the older material resurfaces in later sections on step functions, which are there in part to give students an opportunity to see something closer to a traditional development of measure and integration.

The treatment of the Lebesgue integral here is intentionally compressed. In lectures everything is done for the real line but in such a way that the extension to higher dimensions – carried out partly in the text but mostly in the problems – is not much harder. Some further extensions are also discussed in the problems.

1. Integrable functions

Recall that the Riemann integral is defined for a certain class of bounded functions $u : [a, b] \longrightarrow \mathbb{C}$ (namely the Riemann integrable functions) which includes all continuous function. It depends on the compactness of the interval but can be extended to an 'improper integral', for which some of the good properties fail, on certain functions on the whole line. This is NOT what we will do. Rather we consider the space of continuous functions 'with compact support':

$$\mathcal{C}_{\mathrm{c}}(\mathbb{R}) = \{u : \mathbb{R} \longrightarrow \mathbb{C}; u \text{ is continuous and } \exists\ R \text{ such that } u(x) = 0 \text{ if } |x| > R\}. \tag{2.1}$$

Thus each element $u \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ vanishes outside an interval $[-R, R]$ where the R depends on the u. Note that the *support* of a continuous function is defined to be the complement of the largest open set on which it vanishes (not the set of points at which it is non-zero). Thus (2.1) says that the support, which is necessarily closed, is contained in some interval $[-R, R]$, which is equivalent to saying it is compact.

Lemma 7. *The Riemann integral defines a continuous linear functional on $\mathcal{C}_c(\mathbb{R})$ equipped with the L^1 norm*

$$\begin{gathered}\int_{\mathbb{R}} u = \lim_{R\to\infty} \int_{[-R,R]} u(x)dx,\\ \|u\|_{L^1} = \lim_{R\to\infty} \int_{[-R,R]} |u(x)|dx,\\ |\int_{\mathbb{R}} u| \le \|u\|_{L^1}.\end{gathered} \tag{2.2}$$

The limits here are trivial in the sense that the functions involved are constant for large R.

PROOF. These are basic properties of the Riemann integral see Rudin [**3**]. □

Note that $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$ is a normed space with respect to $\|u\|_{L^1}$ as defined above.

With this preamble we can directly define the 'space' of Lebesgue integrable functions on $\mathbb{R}$.

DEFINITION 5. A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is *Lebesgue integrable*, written $f \in \mathcal{L}^1(\mathbb{R})$, if there exists a series $w_n = \sum\limits_{j=1}^{n} f_j$, $f_j \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ which is absolutely summable,

$$\sum_j \int |f_j| < \infty \tag{2.3}$$

and such that

$$\sum_j |f_j(x)| < \infty \Longrightarrow \lim_{n\to\infty} w_n(x) = \sum_j f_j(x) = f(x). \tag{2.4}$$

This is a somewhat convoluted definition which you should think about a bit. Its virtue is that it is all there. The problem is that it takes a bit of unravelling. Before proceeding, let me give a simple example and check that this definition does include continuous functions defined on an interval and extended to be zero outside – so the theory we develop will include the usual Riemann integral.

LEMMA 8. *If $u \in \mathcal{C}([a,b])$ then*

$$\tilde{u}(x) = \begin{cases} u(x) & \textit{if } x \in [a,b] \\ 0 & \textit{otherwise} \end{cases} \tag{2.5}$$

is an integrable function.

PROOF. Just 'add legs' to $\tilde{u}$ by considering the sequence

$$g_n(x) = \begin{cases} 0 & \text{if } x < a - 1/n \text{ or } x > b + 1/n, \\ (1 + n(x-a))u(a) & \text{if } a - 1/n \le x < a, \\ (1 - n(x-b))u(b) & \text{if } b < x \le b + 1/n, \\ u(x) & \text{if } x \in [a,b]. \end{cases} \tag{2.6}$$

This is a continuous function on each of the open subintervals in the description with common limits at the endpoints, so $g_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$. By construction, $g_n(x) \to \tilde{u}(x)$ for each $x \in \mathbb{R}$. Define the sequence which has partial sums the g_n,

$$f_1 = g_1,\ f_n = g_n - g_{n-1},\ n > 1 \Longrightarrow g_n(x) = \sum_{k=1}^{n} f_k(x). \tag{2.7}$$

Then $f_n = 0$ in $[a,b]$ and it can be written in terms of the 'legs'

$$l_n = \begin{cases} 0 & \text{if } x < a - 1/n,\ x \ge a \\ (1 + n(x-a)) & \text{if } a - 1/n \le x < a, \end{cases}$$

$$r_n = \begin{cases} 0 & \text{if } x \le b,\ x\, b + 1/n \\ (1 - n(x-b)) & \text{if } b << x \le b + 1/n, \end{cases}$$

as

$$|f_n(x)| = (l_n - l_{n-1})|u(a)| + (r_n - r_{n-1})|u(b)|,\ n > 1. \tag{2.8}$$

It follows that

$$\int |f_n(x)| = \frac{(|u(a)| + |u(b)|)}{n(n-1)}$$

so $\{f_n\}$ is an absolutely summable series showing that $\tilde{u} \in \mathcal{L}^1(\mathbb{R})$. $\square$

Returning to the definition, notice that we only say 'there exists' an absolutely summable sequence and that it is required to converge to the function *only* at points at which the pointwise sequence is absolutely summable. At other points anything is permitted. So it is not immediately clear that there are any functions *not* satisfying this condition. Indeed if there was a sequence like f_j above with $\sum\limits_j |f_j(x)| = \infty$ always, then (2.4) would represent no restriction at all. So the point of the definition is that absolute summability – a condition on the integrals in (2.3) – does imply something about (absolute) convergence of the pointwise series. Let us enforce this idea with another definition:-

Definition 6. A set $E \subset \mathbb{R}$ is said to be *of measure zero* in the sense of Lebesgue (which is pretty much always the meaning here) if there is a series $w_n = \sum\limits_{j=1}^{n} h_j$, $h_j \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ which is absolutely summable, $\sum\limits_j \int |h_j| < \infty$, and such that

$$\sum_j |h_j(x)| = \infty \ \forall \ x \in E. \tag{2.9}$$

Notice that we do not require E to be precisely the set of points at which the series in (2.9) diverges, only that it does so at all points of E, so E is just a subset of the set on which some absolutely summable series of functions in $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$ does not converge absolutely. So any subset of a set of measure zero is automatically of measure zero. To introduce the little trickery we use to unwind the defintion above, consider first the following (important) result.

Lemma 9. *Any finite union of sets of measure zero is a set of measure zero.*

Proof. Since we can proceed in steps, it suffices to show that the union of two sets of measure zero has measure zero. So, let the two sets be E and F and two corresponding absolutely summable sequences, as in Definition 6, be h_j and g_j. Consider the alternating sequence

$$u_k = \begin{cases} h_j & \text{if } k = 2j-1 \text{ is odd} \\ g_j & \text{if } k = 2j \text{ is even.} \end{cases} \tag{2.10}$$

Thus $\{u_k\}$ simply interlaces the two sequences. It follows that u_k is absolutely summable, since

$$\sum_k \|u_k\|_{L^1} = \sum_j \|h_j\|_{L^1} + \sum_j \|g_j\|_{L^1}. \tag{2.11}$$

Moreover, the pointwise series $\sum\limits_k |u_k(x)|$ diverges precisely where one or other of the two series $\sum\limits_j |u_j(x)|$ or $\sum\limits_j |g_j(x)|$ diverges. In particular it must diverge on $E \cup F$ which is therefore, by definition, a set of measure zero. $\square$

The definition of $f \in \mathcal{L}^1(\mathbb{R})$ above certainly requires that the equality on the right in (2.4) should hold outside a set of measure zero, but in fact a specific one, the one on which the series on the left diverges. Using the same idea as in the lemma above we can get rid of this restriction.

PROPOSITION 9. *If $f : \mathbb{R} \longrightarrow \mathbb{C}$ and there exists a series $w_n = \sum\limits_{j=1}^{n} g_j$ with $g_j \in \mathcal{C}_c(\mathbb{R})$ which is absolutely summable, so $\sum\limits_j \int |g_j| < \infty$, and a set $E \subset \mathbb{R}$ of measure zero such that*

$$x \in \mathbb{R} \setminus E \Longrightarrow f(x) = \sum_{j=1}^{\infty} g_j(x) \tag{2.12}$$

then $f \in \mathcal{L}^1(\mathbb{R})$.

Recall that when one writes down an equality such as on the right in (2.12) one is implicitly saying that $\sum\limits_{j=1}^{\infty} g_j(x)$ converges *and* the inequality holds for the limit. We will call a sequence as the g_j above an 'approximating series' for $f \in \mathcal{L}^1(\mathbb{R})$. This is indeed a refinement of the definition since all $f \in \mathcal{L}^1(\mathbb{R})$ arise this way, taking E to be the set where $\sum\limits_j |f_j(x)| = \infty$ for a series as in the defintion.

PROOF. By definition of a set of measure zero there is some series h_j as in (2.9). Now, consider the series obtained by alternating the terms between g_j, h_j and $-h_j$. Explicitly, set

$$f_j = \begin{cases} g_k & \text{if } j = 3k-2 \\ h_k & \text{if } j = 3k-1 \\ -h_k(x) & \text{if } j = 3k. \end{cases} \tag{2.13}$$

This defines a series in $\mathcal{C}_c(\mathbb{R})$ which is absolutely summable, with

$$\sum_j \int |f_j(x)| = \sum_k \int |g_k| + 2 \sum_k \int |h_k|. \tag{2.14}$$

The same sort of identity is true for the pointwise series which shows that

$$\sum_j |f_j(x)| < \infty \text{ iff } \sum_k |g_k(x)| < \infty \text{ and } \sum_k |h_k(x)| < \infty. \tag{2.15}$$

So if the pointwise series on the left converges absolutely, then $x \notin E$, by definition and hence, by the assumption of the Proposition

$$f(x) = \sum_k g_k(x) \tag{2.16}$$

(including of course the requirement that the series itself converges). So in fact we find that

$$\sum_j |f_j(x)| < \infty \Longrightarrow f(x) = \sum_j f_j(x) \tag{2.17}$$

since the sequence of partial sums of the f_j cycles through w_n, $w_n(x) + h_n(x)$, then $w_n(x)$ and then to $w_{n+1}(x)$. Since $\sum\limits_k |h_k(x)| < \infty$ the sequence $|h_n(x)| \to 0$ so (2.17) follows from (2.12). $\square$

This is the trick at the heart of the definition of integrability above. Namely we can manipulate the series involved in this sort of way to prove things about the elements of $\mathcal{L}^1(\mathbb{R})$. One point to note is that if g_j is an absolutely summable series in $\mathcal{C}_c(\mathbb{R})$ then

$$F = \begin{cases} \sum\limits_j |g_j(x)| & \text{when this is finite} \\ 0 & \text{otherwise} \end{cases} \Longrightarrow F \in \mathcal{L}^1(\mathbb{R}). \tag{2.18}$$

The sort of property (2.12), where some condition holds on the complement of a set of measure zero is so commonly encountered in integration theory that we give it a simpler name.

Definition 7. A condition that holds on $\mathbb{R} \setminus E$ for some set of measure zero, E, is sais to hold *almost everywhere.* In particular we write

$$f = g \text{ } a.e. \text{ if } f(x) = g(x) \ \forall \ x \in \mathbb{R} \setminus E, \ E \text{ of measure zero.} \tag{2.19}$$

Of course as yet we are living dangerously because we have done nothing to show that sets of measure zero are 'small' let alone 'ignorable' as this definition seems to imply. Beware of the trap of 'proof by declaration'!

Now Proposition 9 can be paraphrased as 'A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is Lebesgue integrable if and only if it is the pointwise sum a.e. of an absolutely summable series in $\mathcal{C}_c(\mathbb{R})$.' Summable here remember means integrable.

2. Linearity of $\mathcal{L}^1$

The word 'space' is quoted in the definition of $\mathcal{L}^1(\mathbb{R})$ above, because it is not immediately obvious that $\mathcal{L}^1(\mathbb{R})$ is a linear space, even more importantly it is far from obvious that the integral of a function in $\mathcal{L}^1(\mathbb{R})$ is well defined (which is the point of the exercise after all). In fact we wish to define the integral to be

$$\int_{\mathbb{R}} f = \sum_n \int f_n \tag{2.20}$$

where $f_n \in \mathcal{C}(\mathbb{R})$ is any 'approximating series' meaning now as the g_j in Propsition 9. This is fine in so far as the series on the right (of complex numbers) does converge – since we demanded that $\sum\limits_n \int |f_n| < \infty$ so this series converges absolutely – but not fine in so far as the answer might well depend on *which* series we choose which 'approximates f' in the sense of the definition or Proposition 9.

So, the immediate problem is to prove these two things. First we will do a little more than prove the linearity of $\mathcal{L}^1(\mathbb{R})$. Recall that a function is 'positive' if it takes only non-negative values.

Proposition 10. *The space $\mathcal{L}^1(\mathbb{R})$ is linear (over $\mathbb{C}$) and if $f \in \mathcal{L}^1(\mathbb{R})$ the real and imaginary parts,* $\operatorname{Re} f$, $\operatorname{Im} f$ *are Lebesgue integrable as are their positive parts and as is also the absolute value, $|f|$. For a real function there is an approximating sequence as in Proposition 9 which is real and it can be chosen to be non-nagative if $f \geq 0$.*

Proof. We first consider the real part of a function $f \in \mathcal{L}^1(\mathbb{R})$. Suppose $f_n \in \mathcal{C}_c(\mathbb{R})$ is an approximating sequence as in Proposition 9. Then consider $g_n = \operatorname{Re} f_n$.

This is absolutely summable, since $\int |g_n| \le \int |f_n|$ and

$$\sum_n f_n(x) = f(x) \Longrightarrow \sum_n g_n(x) = \operatorname{Re} f(x). \tag{2.21}$$

Since the left identity holds a.e., so does the right and hence $\operatorname{Re} f \in \mathcal{L}^1(\mathbb{R})$ by Proposition 9. The same argument with the imaginary parts shows that $\operatorname{Im} f \in \mathcal{L}^1(\mathbb{R})$. This also shows that a real element has a real approximating sequence and taking positive parts that a positive function has a positive approximating sequence.

The fact that the sum of two integrable functions is integrable really is a simple consequence of Proposition 9 and Lemma 9. Indeed, if $f,\ g \in \mathcal{L}^1(\mathbb{R})$ have approximating series f_n and g_n as in Proposition 9 then $h_n = f_n + g_n$ is absolutely summable,

$$\sum_n \int |h_n| \le \sum_n \int |f_n| + \sum_n \int |g_n| \tag{2.22}$$

and

$$\sum_n f(x) = f(x),\ \sum_n g_n(x) = g(x) \Longrightarrow \sum_n h_n(x) = f(x) + g(x).$$

The first two conditions hold outside (probably different) sets of measure zero, E and F, so the conclusion holds outside $E \cup F$ which is of measure zero. Thus $f + g \in \mathcal{L}^1(\mathbb{R})$. The case of cf for $c \in \mathbb{C}$ is more obvious.

The proof that $|f| \in \mathcal{L}^1(\mathbb{R})$ if $f \in \mathcal{L}^1(\mathbb{R})$ is similar but perhaps a little trickier. Again, let $\{f_n\}$ be a sequence as in the definition showing that $f \in \mathcal{L}^1(\mathbb{R})$. To make a series for $|f|$ we can try the 'obvious' thing. Namely we know that

$$\sum_{j=1}^{n} f_j(x) \to f(x) \text{ if } \sum_j |f_j(x)| < \infty \tag{2.23}$$

so certainly it follows that

$$|\sum_{j=1}^{n} f_j(x)| \to |f(x)| \text{ if } \sum_j |f_j(x)| < \infty.$$

So, set

$$g_1(x) = |f_1(x)|,\ g_k(x) = |\sum_{j=1}^{k} f_j(x)| - |\sum_{j=1}^{k-1} f_j(x)|\ \forall\ x \in \mathbb{R}. \tag{2.24}$$

Then, for sure,

$$\sum_{k=1}^{N} g_k(x) = |\sum_{j=1}^{N} f_j(x)| \to |f(x)| \text{ if } \sum_j |f_j(x)| < \infty. \tag{2.25}$$

So equality holds off a set of measure zero and we only need to check that $\{g_j\}$ is an absolutely summable series.

The triangle inequality in the 'reverse' form $||v| - |w|| \le |v - w|$ shows that, for $k > 1$,

$$|g_k(x)| = ||\sum_{j=1}^{k} f_j(x)| - |\sum_{j=1}^{k-1} f_j(x)|| \le |f_k(x)|. \tag{2.26}$$

Thus

$$\sum_k \int |g_k| \le \sum_k \int |f_k| < \infty \tag{2.27}$$

so the g_k's do indeed form an absolutely summable series and (2.25) holds almost everywhere, so $|f| \in \mathcal{L}^1(\mathbb{R})$. □

By combining these result we can see again that if f, $g \in \mathcal{L}^1(\mathbb{R})$ are both real valued then

$$f_+ = \max(f, 0),\ \max(f, g),\ \min(f, g) \in \mathcal{L}^1(\mathbb{R}). \tag{2.28}$$

Indeed, the positive part, $f_+ = \frac{1}{2}(|f| + f)$, $\max(f, g) = g + (f - g)_+$, $\min(f, g) = -\max(-f, -g)$.

3. The integral on $\mathcal{L}^1$

Next we want to show that the integral is well defined via (2.20) for any approximating series. From Propostion 10 it is enough to consider only real functions. For this, recall a result concerning a case where uniform convergence of continuous functions follows from pointwise convergence, namely when the convergence is monotone, the limit is continuous, and the space is compact. It works on a general compact metric space but we can concentrate on the case at hand.

Lemma 10. *If $u_n \in \mathcal{C}_c(\mathbb{R})$ is a decreasing sequence of non-negative functions such that $\lim_{n\to\infty} u_n(x) = 0$ for each $x \in \mathbb{R}$ then $u_n \to 0$ uniformly on $\mathbb{R}$ and*

$$\lim_{n\to\infty} \int u_n = 0. \tag{2.29}$$

Proof. Since all the $u_n(x) \ge 0$ and they are decreasing (which means not increasing of course) if $u_1(x)$ vanishes at x then all the other $u_n(x)$ vanish there too. Thus there is one $R > 0$ such that $u_n(x) = 0$ if $|x| > R$ for all n, namely one that works for u_1. So we only need consider what happens on $[-R, R]$ which is compact. For any $\epsilon > 0$ look at the sets

$$S_n = \{x \in [-R, R]; u_n(x) \ge \epsilon\}.$$

This can also be written $S_n = u_n^{-1}([\epsilon, \infty)) \cap [-R, R]$ and since u_n is continuous it follows that S_n is closed and hence compact. Moreover the fact that the $u_n(x)$ are decreasing means that $S_{n+1} \subset S_n$ for all n. Finally,

$$\bigcap_n S_n = \emptyset$$

since, by assumption, $u_n(x) \to 0$ for each x. Now the property of compact sets in a metric space that we use is that if such a sequence of decreasing compact sets has empty intersection then the sets themselves are empty from some n onwards. This means that there exists N such that $\sup_x u_n(x) < \epsilon$ for all $n > N$. Since $\epsilon > 0$ was arbitrary, $u_n \to 0$ uniformly.

One of the basic properties of the Riemann integral is that the integral of the limit of a uniformly convergent sequence (even of Riemann integrable functions but here continuous) is the limit of the sequence of integrals, which is (2.29) in this case. □

We can easily extend this in a useful way – the direction of convergence is reversed really just to mentally distinquish this from the preceding lemma.

Lemma 11. *If $v_n \in \mathcal{C}_c(\mathbb{R})$ is any increasing sequence such that $\lim_{n\to\infty} v_n(x) \geq 0$ for each $x \in \mathbb{R}$ (where the possibility $v_n(x) \to \infty$ is included) then*

$$\lim_{n\to\infty} \int v_n dx \geq 0 \text{ including possibly } +\infty. \tag{2.30}$$

Proof. This is really a corollary of the preceding lemma. Consider the sequence of functions

$$w_n(x) = \begin{cases} 0 & \text{if } v_n(x) \geq 0 \\ -v_n(x) & \text{if } v_n(x) < 0. \end{cases} \tag{2.31}$$

Since this is the maximum of two continuous functions, namely $-v_n$ and 0, it is continuous and it vanishes for large x, so $w_n \in \mathcal{C}_c(\mathbb{R})$. Since $v_n(x)$ is increasing, w_n is decreasing and it follows that $\lim w_n(x) = 0$ for all x – either it gets there for some finite n and then stays 0 or the limit of $v_n(x)$ is zero. Thus Lemma 10 applies to w_n so

$$\lim_{n\to\infty} \int_{\mathbb{R}} w_n(x)dx = 0.$$

Now, $v_n(x) \geq -w_n(x)$ for all x, so for each n, $\int v_n \geq -\int w_n$. From properties of the Riemann integral, $v_{n+1} \geq v_n$ implies that $\int v_n dx$ is an increasing sequence and it is bounded below by one that converges to 0, so (2.30) is the only possibility. □

From this result applied carefully we see that the integral behaves sensibly for absolutely summable series.

Lemma 12. *Suppose $f_n \in \mathcal{C}_c(\mathbb{R})$ is an absolutely summable sequence of real-valued functions, so $\sum\limits_n \int |f_n| dx < \infty$, and also suppose that*

$$\sum_n f_n(x) = 0 \text{ a.e.} \tag{2.32}$$

then

$$\sum_n \int f_n dx = 0. \tag{2.33}$$

Proof. As already noted, the series (2.33) does converge, since the inequality $|\int f_n dx| \leq \int |f_n| dx$ shows that it is absolutely convergent (hence Cauchy, hence convergent).

If E is a set of measure zero such that (2.32) holds on the complement then we can modify f_n as in (2.13) by adding and subtracting a non-negative absolutely summable sequence g_k which diverges absolutely on E. For the new sequence f_n (2.32) is strengthened to

$$\sum_n |f_n(x)| < \infty \Longrightarrow \sum_n f_n(x) = 0 \tag{2.34}$$

and the conclusion (2.33) holds for the new sequence if and only if it holds for the old one.

Now, we need to get ourselves into a position to apply Lemma 11. To do this, just choose some integer N (large but it doesn't matter yet) and consider the sequence of functions – it depends on N but I will suppress this dependence –

$$F_1(x) = \sum_{n=1}^{N+1} f_n(x),\ F_j(x) = |f_{N+j}(x)|,\ j \geq 2. \tag{2.35}$$

This is a sequence in $\mathcal{C}_c(\mathbb{R})$ and it is absolutely summable – the convergence of $\sum\limits_j \int |F_j| dx$ only depends on the 'tail' which is the same as for f_n. For the same reason,

$$\sum_j |F_j(x)| < \infty \Longleftrightarrow \sum_n |f_n(x)| < \infty. \tag{2.36}$$

Now the sequence of partial sums

$$g_p(x) = \sum_{j=1}^{p} F_j(x) = \sum_{n=1}^{N+1} f_n(x) + \sum_{j=2}^{p} |f_{N+j}| \tag{2.37}$$

is increasing with p – since we are adding non-negative functions. If the two equivalent conditions in (2.36) hold then

$$\sum_n f_n(x) = 0 \Longrightarrow \sum_{n=1}^{N+1} f_n(x) + \sum_{j=2}^{\infty} |f_{N+j}(x)| \geq 0 \Longrightarrow \lim_{p\to\infty} g_p(x) \geq 0, \tag{2.38}$$

since we are only increasing each term. On the other hand if these conditions do not hold then the tail, any tail, sums to infinity so

$$\lim_{p\to\infty} g_p(x) = \infty. \tag{2.39}$$

Thus the conditions of Lemma 11 hold for g_p and hence

$$\sum_{n=1}^{N+1} \int f_n + \sum_{j\geq N+2} \int |f_j(x)| dx \geq 0. \tag{2.40}$$

Using the same inequality as before this implies that

$$\sum_{n=1}^{\infty} \int f_n \geq -2 \sum_{j\geq N+2} \int |f_j(x)| dx. \tag{2.41}$$

This is true for any N and as $N \to \infty$, $\lim_{N\to\infty} \sum\limits_{j\geq N+2} \int |f_j(x)| dx = 0$. So the fixed number on the left in (2.41), which is what we are interested in, must be non-negative. In fact the signs in the argument can be reversed, considering instead

$$h_1(x) = -\sum_{n=1}^{N+1} f_n(x),\ h_p(x) = |f_{N+p}(x)|,\ p \geq 2 \tag{2.42}$$

and the final conclusion is the opposite inequality in (2.41). That is, we conclude what we wanted to show, that

$$\sum_{n=1}^{\infty} \int f_n = 0. \tag{2.43}$$

$\square$

Finally then we are in a position to show that the integral of an element of $\mathcal{L}^1(\mathbb{R})$ is well-defined.

PROPOSITION 11. *If $f \in \mathcal{L}^1(\mathbb{R})$ then*

$$\int f = \lim_{n\to\infty} \sum_n \int f_n \tag{2.44}$$

is independent of the approximating sequence, f_n, used to define it. Moreover,

$$\begin{gathered}\int |f| = \lim_{N\to\infty} \int |\sum_{k=1}^N f_k|,\\ |\int f| \le \int |f| \text{ and}\\ \lim_{n\to\infty} \int |f - \sum_{j=1}^n f_j| = 0.\end{gathered} \tag{2.45}$$

So in some sense the definition of the Lebesgue integral 'involves no cancellations'. There are various extensions of the integral which do exploit cancellations – I invite you to look into the definition of the Henstock integral (and its relatives).

PROOF. The uniqueness of $\int f$ follows from Lemma 12. Namely, if f_n and f_n' are two sequences approximating f as in Proposition 9 then the real and imaginary parts of the difference $f_n' - f_n$ satisfy the hypothesis of Lemma 12 so it follows that

$$\sum_n \int f_n = \sum_n \int f_n'.$$

Then the first part of (2.45) follows from this definition of the integral applied to $|f|$ and the approximating series for $|f|$ devised in the proof of Proposition 10. The inequality

$$|\sum_n \int f_n| \le \sum_n \int |f_n|, \tag{2.46}$$

which follows from the finite inequalities for the Riemann integrals

$$|\sum_{n\le N} \int f_n| \le \sum_{n\le N} \int |f_n| \le \sum_n \int |f_n|$$

gives the second part.

The final part follows by applying the same arguments to the series $\{f_k\}_{k>n}$, as an absolutely summable series approximating $f - \sum\limits_{j=1}^n f_j$ and observing that the integral is bounded by

$$\int |f - \sum_{k=1}^n f_k| \le \sum_{k=n+1}^\infty \int |f_k| \to 0 \text{ as } n \to \infty. \tag{2.47}$$

□

4. Summable series in $\mathcal{L}^1(\mathbb{R})$

The next thing we want to know is when the 'norm', which is in fact only a seminorm, on $\mathcal{L}^1(\mathbb{R})$, vanishes. That is, when does $\int |f| = 0$? One way is fairly easy. The full result we are after is:-

Proposition 12. *For an integrable function $f \in \mathcal{L}^1(\mathbb{R})$, the vanishing of $\int |f|$ implies that f is a null function in the sense that*

$$f(x) = 0 \ \forall \ x \in \mathbb{R} \setminus E \text{ where } E \text{ is of measure zero.} \tag{2.48}$$

Conversely, if (2.48) *holds then $f \in \mathcal{L}^1(\mathbb{R})$ and $\int |f| = 0$.*

Proof. The main part of this is the first part, that the vanishing of $\int |f|$ implies that f is null. The converse is the easier direction in the sense that we have already done it.

Namely, if f is null in the sense of (2.48) then $|f|$ is the limit a.e. of the absolutely summable series with all terms 0. It follows from the definition of the integral above that $|f| \in \mathcal{L}^1(\mathbb{R})$ and $\int |f| = 0$. □

For the forward argument we will use the following more technical result, which is also closely related to the completeness of $L^1(\mathbb{R})$.

Proposition 13. *If $f_n \in \mathcal{L}^1(\mathbb{R})$ is an absolutely summable series, meaning that $\sum\limits_n \int |f_n| < \infty$, then*

$$E = \{x \in \mathbb{R}; \sum_n |f_n(x)| = \infty\} \text{ has measure zero.} \tag{2.49}$$

If $f : \mathbb{R} \longrightarrow \mathbb{C}$ satisfies

$$f(x) = \sum_n f_n(x) \text{ a.e.} \tag{2.50}$$

then $f \in \mathcal{L}^1(\mathbb{R})$,

$$\int f = \sum_n \int f_n,$$

$$|\int f| \le \int |f| = \lim_{n \to \infty} \int |\sum_{j=1}^n f_j| \le \sum_j \int |f_j| \text{ and} \tag{2.51}$$

$$\lim_{n\to\infty} \int |f - \sum_{j=1}^n f_j| = 0.$$

This basically says we can replace 'continuous function of compact support' by 'Lebesgue integrable function' in the definition and get the same result. Of course this makes no sense without the original definition, so what we are showing is that iterating it makes no difference – we do not get a bigger space.

Proof. The proof is very like the proof of completeness via absolutely summable series for a normed space outlined in the preceding chapter.

By assumption each $f_n \in \mathcal{L}^1(\mathbb{R})$, so there exists a sequence $f_{n,j} \ni \mathcal{C}_c(\mathbb{R})$ with $\sum\limits_j \int |f_{n,j}| < \infty$ and

$$\sum_j |f_{n,j}(x)| < \infty \Longrightarrow f_n(x) = \sum_j f_{n,j}(x). \tag{2.52}$$

We can expect $f(x)$ to be given by the sum of the $f_{n,j}(x)$ over both n and j, but in general, this double series is not absolutely summable. However we can replace it by one that is. For each n choose N_n so that

$$\sum_{j>N_n} \int |f_{n,j}| < 2^{-n}. \tag{2.53}$$

This is possible by the assumed absolute summability – the tail of the series therefore being small. Having done this, we replace the series $f_{n,j}$ by

$$f'_{n,1} = \sum_{j\le N_n} f_{n,j}(x),\ \ f'_{n,j}(x) = f_{n,N_n+j-1}(x)\ \forall\ j \ge 2, \tag{2.54}$$

summing the first N_n terms. This still converges to f_n on the same set as in (2.52). So in fact we can simply replace $f_{n,j}$ by $f'_{n,j}$ and we have in addition the estimate

$$\sum_j \int |f'_{n,j}| \le \int |f_n| + 2^{-n+1}\ \forall\ n. \tag{2.55}$$

This follows from the triangle inequality since, using (2.53),

$$\int |f'_{n,1} + \sum_{j=2}^{N} f'_{n,j}| \ge \int |f'_{n,1}| - \sum_{j\ge 2} \int |f'_{n,j}| \ge \int |f'_{n,1}| - 2^{-n} \tag{2.56}$$

and the left side converges to $\int |f_n|$ by (2.45) as $N \to \infty$. Using (2.53) again gives (2.55).

Dropping the primes from the notation and using the new series as $f_{n,j}$ we can let g_k be some enumeration of the $f_{n,j}$ – using the standard diagonalization procedure. This gives a new series of continuous functions which is absolutely summable since

$$\sum_{k=1}^{N} \int |g_k| \le \sum_{n,j} \int |f_{n,j}| \le \sum_n (\int |f_n| + 2^{-n+1}) < \infty. \tag{2.57}$$

Using the freedom to rearrange absolutely convergent series we see that

$$\sum_{n,j} |f_{n,j}(x)| < \infty \Longrightarrow f(x) = \sum_k g_k(x) = \sum_n \sum_j f_{n,j}(x). \tag{2.58}$$

The set where (2.58) fails is a set of measure zero, by definition. Thus $f \in \mathcal{L}^1(\mathbb{R})$ and (2.49) also follows. To get the final result (2.51), rearrange the double series for the integral (which is also absolutely convergent). □

For the moment we only need the weakest part, (2.49), of this. To paraphrase this, for any absolutely summable series of integrable functions the absolute pointwise series converges off a set of measure zero – it can only diverge on a set of measure zero. It is rather shocking but this allows us to prove the rest of Proposition 12! Namely, suppose $f \in \mathcal{L}^1(\mathbb{R})$ and $\int |f| = 0$. Then Proposition 13 applies to the series with each term being $|f|$. This is absolutely summable since all the integrals are zero. So it must converge pointwise except on a set of measure zero. Clearly it diverges whenever $f(x) \neq 0$,

$$\int |f| = 0 \Longrightarrow \{x; f(x) \neq 0\} \text{ has measure zero} \tag{2.59}$$

which is what we wanted to show to finally complete the proof of Proposition 12.

5. The space $L^1(\mathbb{R})$

Finally this allows us to define the standard Lebesgue space

$$L^1(\mathbb{R}) = \mathcal{L}^1(\mathbb{R})/\mathcal{N},\ \mathcal{N} = \{\text{null functions}\} \tag{2.60}$$

and to check that it is a Banach space with the norm (arising from, to be pedantic) $\int |f|$.

THEOREM 7. *The quotient space* $L^1(\mathbb{R})$ *defined by* (2.60) *is a Banach space in which the continuous functions of compact support form a dense subspace.*

The elements of $L^1(\mathbb{R})$ are *equivalence classes of functions*

$$[f] = f + \mathcal{N},\ f \in \mathcal{L}^1(\mathbb{R}). \tag{2.61}$$

That is, we 'identify' two elements of $\mathcal{L}^1(\mathbb{R})$ if (and only if) their difference is null, which is to say they are equal off a set of measure zero. Note that the set which is ignored here is not fixed, but can depend on the functions.

PROOF. For an element of $L^1(\mathbb{R})$ the integral of the absolute value is well-defined by Propositions 10 and 12

$$\|[f]\|_{L^1} = \int |f|,\ f \in [f] \tag{2.62}$$

and gives a *semi-norm* on $\mathcal{L}^1(\mathbb{R})$. It follows from Proposition 5 that on the quotient, $\|[f]\|$ is indeed a norm.

The completeness of $L^1(\mathbb{R})$ is a direct consequence of Proposition 13. Namely, to show a normed space is complete it is enough to check that any absolutely summable series converges. So if $[f_j]$ is an absolutely summable series in $L^1(\mathbb{R})$ then f_j is absolutely summable in $\mathcal{L}^1(\mathbb{R})$ and by Proposition 13 the sum of the series exists so we can use (2.50) to define f off the set E and take it to be zero on E. Then, $f \in \mathcal{L}^1(\mathbb{R})$ and the last part of (2.51) means precisely that

$$\lim_{n\to\infty} \|[f] - \sum_{j<n}[f_j]\|_{L^1} = \lim_{n\to\infty} \int |f - \sum_{j<n} f_j| = 0 \tag{2.63}$$

showing the desired completeness. □

Note that despite the fact that it is technically incorrect, everyone says '$L^1(\mathbb{R})$ is the space of Lebesgue integrable functions' even though it is really the space of equivalence classes of these functions modulo equality almost everywhere. Not much harm can come from this mild abuse of language.

Another consequence of Proposition 13 and the proof above is an extension of Lemma 9.

PROPOSITION 14. *Any countable union of sets of measure zero is a set of measure zero.*

PROOF. If E is a set of measure zero then any function f which is defined on $\mathbb{R}$ and vanishes outside E is a null function – is in $\mathcal{L}^1(\mathbb{R})$ and has $\int |f| = 0$. Conversely if the characteristic function of E, the function equal to 1 on E and zero in $\mathbb{R} \setminus E$ is integrable and has integral zero then E has measure zero. This is the characterization of null functions above. Now, if E_j is a sequence of sets of measure zero and χ_k is the characteristic function of

$$\bigcup_{j\le k} E_j \tag{2.64}$$

then $\int|\chi_k|=0$ so this is an absolutely summable series with sum, the characteristic function of the union, integrable and of integral zero. □

6. The three integration theorems

Even though we now 'know' which functions are Lebesgue integrable, it is often quite tricky to use the definitions to actually show that a particular function has this property. There are three standard results on convergence of sequences of integrable functions which are powerful enough to cover most situations that arise in practice – a Monotonicity Lemma, Fatou's Lemma and Lebesgue's Dominated Convergence theorem.

LEMMA 13 (Montonicity). *If $f_j\in\mathcal{L}^1(\mathbb{R})$ is a monotone sequence, either $f_j(x)\ge f_{j+1}(x)$ for all $x\in\mathbb{R}$ and all j or $f_j(x)\le f_{j+1}(x)$ for all $x\in\mathbb{R}$ and all j, and $\int f_j$ is bounded then*

$$\{x\in\mathbb{R};\lim_{j\to\infty}f_j(x)\ \textit{is finite}\}=\mathbb{R}\setminus E \tag{2.65}$$

where E has measure zero and

$$\begin{gathered}f=\lim_{j\to\infty}f_j(x)\ \textit{a.e. is an element of}\ \mathcal{L}^1(\mathbb{R})\\ \textit{with}\ \int f=\lim_{j\to\infty}\int f_j\ \textit{and}\ \lim_{j\to\infty}\int|f-f_j|=0.\end{gathered} \tag{2.66}$$

In the usual approach through measure one has the concept of a measureable, non-negative, function for which the integral 'exists but is infinite' – we do not have this (but we could easily do it, or rather you could). Using this one can drop the assumption about the finiteness of the integral but the result is not significantly stronger.

PROOF. Since we can change the sign of the f_i it suffices to assume that the f_i are monotonically increasing. The sequence of integrals is therefore also montonic increasing and, being bounded, converges. Turning the sequence into a series, by setting $g_1=f_1$ and $g_j=f_j-f_{j-1}$ for $j\ge1$ the g_j are non-negative for $j\ge1$ and

$$\sum_{j\ge2}\int|g_j|=\sum_{j\ge2}\int g_j=\lim_{n\to\infty}\int f_n-\int f_1 \tag{2.67}$$

converges. So this is indeed an absolutely summable series. We therefore know from Proposition 13 that it converges absolutely a.e., that the limit, f, is integrable and that

$$\int f=\sum_j\int g_j=\lim_{n\to\infty}\int f_j. \tag{2.68}$$

The second part, corresponding to convergence for the equivalence classes in $L^1(\mathbb{R})$ follows from the fact established earlier about $|f|$ but here it also follows from the monotonicity since $f(x) \geq f_j(x)$ a.e. so

$$\int |f - f_j| = \int f - \int f_j \to 0 \text{ as } j \to \infty. \tag{2.69}$$

□

Now, to Fatou's Lemma. This really just takes the monotonicity result and applies it to a sequence of integrable functions with bounded integral. You should recall that the max and min of two real-valued integrable functions is integrable and that

$$\int \min(f, g) \leq \min(\int f, \int g). \tag{2.70}$$

This follows from the identities

$$2\max(f, g) = |f - g| + f + g,\ 2\min(f, g) = -|f - g| + f + g. \tag{2.71}$$

LEMMA 14 (Fatou). *Let $f_j \in \mathcal{L}^1(\mathbb{R})$ be a sequence of real-valued integrable and non-negative functions such that $\int f_j$ is bounded above then*

$$\begin{gathered} f(x) = \liminf_{n\to\infty} f_n(x) \text{ exists a.e., } f \in \mathcal{L}^1(\mathbb{R}) \text{ and} \\ \int \liminf f_n = \int f \leq \liminf \int f_n. \end{gathered} \tag{2.72}$$

PROOF. You should remind yourself of the properties of lim inf as necessary! Fix k and consider

$$F_{k,n} = \min_{k \leq p \leq k+n} f_p(x) \in \mathcal{L}^1(\mathbb{R}). \tag{2.73}$$

As discussed above this is integrable. Moreover, this is a decreasing sequence, as n increases, because the minimum is over an increasing set of functions. Furthermore the $F_{k,n}$ are non-negative so Lemma 13 applies and shows that

$$g_k(x) = \inf_{p \geq k} f_p(x) \in \mathcal{L}^1(\mathbb{R}),\ \int g_k \leq \int f_n\ \forall\ n \geq k. \tag{2.74}$$

Note that for a decreasing sequence of non-negative numbers the limit exists and is indeed the infimum. Thus in fact,

$$\int g_k \leq \liminf \int f_n\ \forall\ k. \tag{2.75}$$

Now, let k vary. Then, the infimum in (2.74) is over a set which decreases as k increases. Thus the $g_k(x)$ are increasing. The integrals of this sequence are bounded above in view of (2.75) since we assumed a bound on the $\int f_n$'s. So, we can apply the monotonicity result again to see that

$$\begin{gathered} f(x) = \lim_{k\to\infty} g_k(x) \text{ exists a.e and } f \in \mathcal{L}^1(\mathbb{R}) \text{ has} \\ \int f \leq \liminf \int f_n. \end{gathered} \tag{2.76}$$

Since $f(x) = \liminf f_n(x)$, by definition of the latter, we have proved the Lemma.

□

Now, we apply Fatou's Lemma to prove what we are really after:-

Theorem 8 (Dominated convergence). *Suppose $f_j \in \mathcal{L}^1(\mathbb{R})$ is a sequence of integrable functions such that*

$$\begin{gathered}\exists\ h \in \mathcal{L}^1(\mathbb{R}) \text{ with } |f_j(x)| \le h(x) \text{ a.e. and} \\ f(x) = \lim_{j\to\infty} f_j(x) \text{ exists a.e.}\end{gathered} \tag{2.77}$$

then $f \in \mathcal{L}^1(\mathbb{R})$ and $[f_j] \to [f]$ in $L^1(\mathbb{R})$, so $\int f = \lim_{n\to\infty} \int f_n$ (including the assertion that this limit exists).

Proof. First, we can assume that the f_j are real since the hypotheses hold for the real and imaginary parts of the sequence and together give the desired result. Moroever, we can change all the f_j's to make them zero on the set on which the initial estimate in (2.77) does not hold. Then this bound on the f_j's becomes

$$-h(x) \le f_j(x) \le h(x)\ \forall\ x \in \mathbb{R}. \tag{2.78}$$

In particular this means that $g_j = h - f_j$ is a non-negative sequence of integrable functions and the sequence of integrals is also bounded, since (2.77) also implies that $\int |f_j| \le \int h$, so $\int g_j \le 2\int h$. Thus Fatou's Lemma applies to the g_j. Since we have *assumed* that the sequence $g_j(x)$ converges a.e. to $h - f$ we know that

$$\begin{gathered}h - f(x) = \liminf g_j(x) \text{ a.e. and} \\ \int h - \int f \le \liminf \int (h - f_j) = \int h - \limsup \int f_j.\end{gathered} \tag{2.79}$$

Notice the change on the right from liminf to limsup because of the sign.

Now we can apply the same argument to $g_j'(x) = h(x) + f_j(x)$ since this is also non-negative and has integrals bounded above. This converges a.e. to $h(x) + f(x)$ so this time we conclude that

$$\int h + \int f \le \liminf \int (h + f_j) = \int h + \liminf \int f_j. \tag{2.80}$$

In both inequalities (2.79) and (2.80) we can cancel an $\int h$ and combining them we find

$$\limsup \int f_j \le \int f \le \liminf \int f_j. \tag{2.81}$$

In particular the limsup on the left is smaller than, or equal to, the liminf on the right, for the same real sequence. This however implies that they are equal and that the sequence $\int f_j$ converges. Thus indeed

$$\int f = \lim_{n\to\infty} \int f_n. \tag{2.82}$$

Convergence of f_j to f in $L^1(\mathbb{R})$ follows by applying the results proved so far to $f - f_j$, converging almost everywhere to 0. □

Generally in applications it is Lebesgue's dominated convergence which is used to prove that some function is integrable. Of course, since we deduced it from Fatou's lemma, and the latter from the Monotonicity lemma, you might say that Lebesgue's theorem is the weakest of the three! However, it is very handy.

7. Notions of convergence

We have been dealing with two basic notions of convergence, but really there are more. Let us pause to clarify the relationships between these different concepts.

(1) Convergence of a sequence in $L^1(\mathbb{R})$ (or by slight abuse of language in $\mathcal{L}^1(\mathbb{R})$) – f and $f_n \in L^1(\mathbb{R})$ and

$$\|f - f_n\|_{L^1} \to 0 \text{ as } n \to \infty. \tag{2.83}$$

(2) Convergence almost every where:- For some sequence of functions f_n and function f,

$$f_n(x) \to f(x) \text{ as } n \to \infty \text{ for } x \in \mathbb{R} \setminus E \tag{2.84}$$

where $E \subset \mathbb{R}$ is of measure zero.

(3) Dominated convergence:- For $f_j \in L^1(\mathbb{R})$ (or representatives in $\mathcal{L}^1(\mathbb{R})$) such that $|f_j| \le F$ (a.e.) for some $F \in L^1(\mathbb{R})$ and (2.84) holds.

(4) What we might call 'absolutely summable convergence'. Thus $f_n \in L^1(\mathbb{R})$ are such that $f_n = \sum\limits_{j=1}^{n} g_j$ where $g_j \in L^1(\mathbb{R})$ and $\sum\limits_j \int |g_j| < \infty$. Then (2.84) holds for some f.

(5) Monotone convergence. For $f_j \in \mathcal{L}^1(\mathbb{R})$, real valued and montonic, we require that $\int f_j$ is bounded and it then follows that $f_j \to f$ almost everywhere, with $f \in \mathcal{L}^1(\mathbb{R})$ and that the convergence is $\mathcal{L}^1$ and also that $\int f = \lim_j \int f_j$.

So, one important point to know is that 1 does not imply 2. Nor conversely does 2 imply 1 even if we assume that all the f_j and f are in $L^1(\mathbb{R})$.

However, Montone convergence implies Dominated convergence. Namely if f is the limit then $|f_j| \le |f| + 2|f_1|$ and $f_j \to f$ almost everywhere. Also, Monotone convergence implies convergence with absolute summability simply by taking the sequence to have first term f_1 and subsequence terms $f_j - f_{j-1}$ (assuming that f_j is monotonic increasing) one gets an absolutely summable series with sequence of finite sums converging to f. Similarly absolutely summable convergence implies dominated convergence for the sequence of partial sums; by montone convergence the series $\sum\limits_n |f_n(x)|$ converges a.e. and in L^1 to some function F which dominates the partial sums which in turn converge pointwise.

8. Measurable functions

From our original definition of $\mathcal{L}^1(\mathbb{R})$ we know that $\mathcal{C}_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$. We also know that elements of $\mathcal{C}_c(\mathbb{R})$ can be approximated uniformly, and hence in $L^1(\mathbb{R})$ by step functions – finite linear combinations of the characteristic functions of intervals. It is usual in measure theory to consider the somewhat large class of functions which contains the step functions:

Definition 8. A *simple* function on $\mathbb{R}$ is a finite linear combination (generally with complex coefficients) of characteristic functions of subsets of finite measure:

$$f = \sum_{j=1}^{N} c_j \chi(B_j), \ \chi(B_j) \in \mathcal{L}^1(\mathbb{R}). \tag{2.85}$$

The real and imaginary parts of a simple function are simple and the positive and negative parts of a real simple function are simple. Since step functions are simple, we know that simple functions are dense in $\mathcal{L}^1(\mathbb{R})$ and that if $0 \le F \in \mathcal{L}^1(\mathbb{R})$ then there exists a sequence of simple functions (take them to be a summable sequence of step functions) $f_n \ge 0$ such that $f_n \to F$ almost everywhere and $f_n \le G$ for some other $G \in \mathcal{L}^1(\mathbb{R})$.

We elevate a special case of the second notion of convergence above to a definition.

Definition 9. A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is *(Lebesgue) measurable* if it is the pointwise limit almost everywhere of a sequence of simple functions.

The measurable functions form a linear space since if f and g are measurable and f_n, g_n are sequences of simple functions as required by the definition then $c_1 f_n(x) + c_2 f_2(x) \to c_1 f(x) + c_2 g(x)$ on the intersection of the sets where $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ which is the complement of a set of measure zero.

Now, from the discussion above, we know that each element of $\mathcal{L}^1(\mathbb{R})$ is measurable. Conversely:

Lemma 15. *A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is an element of $\mathcal{L}^1(\mathbb{R})$ if and only if it is measurable and there exists $F \in \mathcal{L}^1(\mathbb{R})$ such that $|f| \le F$ almost everywhere.*

Proof. If f is measurable there exists a sequence of simple functions f_n such that $f_n \to f$ almost everywhere. The real part, $\operatorname{Re} f$, is also measurable as the limit almost everywhere of $\operatorname{Re} f_n$ and from the hypothesis $|\operatorname{Re} f| \le F$. We know that there exists a sequence of simple functions g_n, $g_n \to F$ almost everywhere and $0 \le g_n \le G$ for another element $G \in \mathcal{L}^1(\mathbb{R})$. Then set

$$u_n(x) = \begin{cases} g_n(x) & \text{if } \operatorname{Re} f_n(x) > g_n(x) \\ \operatorname{Re} f_n(x) & \text{if } -g_n(x) \le \operatorname{Re} f_n(x) \le g_n(x) \\ -g_n(x) & \text{if } \operatorname{Re} f_n(x) < -g_n(x). \end{cases} \tag{2.86}$$

Thus $u_n = \max(v_n, -g_n)$ where $v_n = \min(\operatorname{Re} f_n, g_n)$ so u_n is simple and $u_n \to f$ almost everywhere. Since $|u_n| \le G$ it follows from Lebesgue Dominated Convergence that $\operatorname{Re} f \in \mathcal{L}^1(\mathbb{R})$. The same argument shows $\operatorname{Im} f = -\operatorname{Re}(if) \in \mathcal{L}^1(\mathbb{R})$ so $f \in \mathcal{L}^1(\mathbb{R})$ as claimed. □

9. The spaces $L^p(\mathbb{R})$

We use Lemma 15 as a model:

Definition 10. For $1 \le p < \infty$ we set

$$\mathcal{L}^p(\mathbb{R}) = \{f : \mathbb{R} \longrightarrow \mathbb{C}; f \text{ is measurable and } |f|^p \in \mathcal{L}^1(\mathbb{R})\}. \tag{2.87}$$

Proposition 15. *For each $1 \le p < \infty$,*

$$\|u\|_{L^p} = \left(\int |u|^p\right)^{\frac{1}{p}} \tag{2.88}$$

is a seminorm on the linear space $\mathcal{L}^p(\mathbb{R})$ vanishing only on the null functions and making the quotient $L^p(\mathbb{R}) = \mathcal{L}^p(\mathbb{R})/\mathcal{N}$ into a Banach space.

Proof. The real part of an element of $\mathcal{L}^p(\mathbb{R})$ is in $\mathcal{L}^p(\mathbb{R})$ since it is measurable and $|\operatorname{Re} f|^p \le |f|^p$ so $|\operatorname{Re} f|^p \in \mathcal{L}^1(\mathbb{R})$. Similarly, $\mathcal{L}^p(\mathbb{R})$ is linear; it is clear that $cf \in \mathcal{L}^p(\mathbb{R})$ if $f \in \mathcal{L}^p(\mathbb{R})$ and $c \in \mathbb{C}$ and the sum of two elements, f, g, is measurable and satisfies $|f+g|^p \le 2^p(|f|^p + |g|^p)$ so $|f+g|^p \in \mathcal{L}^1(\mathbb{R})$.

We next strengthen (2.87) to the approximation condition that there exists a sequence of simple functions v_n such that

$$v_n \to f \text{ a.e. and } |v_n|^p \le F \in \mathcal{L}^1(\mathbb{R}) \text{ a.e.} \tag{2.89}$$

which certainly implies (2.87). As in the proof of Lemma 15, suppose $f \in \mathcal{L}^p(\mathbb{R})$ is real and choose f_n real-valued simple functions and converging to f almost everywhere. Since $|f|^p \in \mathcal{L}^1(\mathbb{R})$ there is a sequence of simple functions $0 \le h_n$ such that $|h_n| \le F$ for some $F \in \mathcal{L}^1(\mathbb{R})$ and $h_n \to |f|^p$ almost everywhere. Then set $g_n = h_n^{\frac{1}{p}}$ which is also a sequence of simple functions and define v_n by (2.86). It follows that (2.89) holds for the real part of f but combining sequences for real and imaginary parts such a sequence exists in general.

The advantage of the approximation condition (2.89) is that it allows us to conclude that the triangle inequality holds for $\|u\|_{L^p}$ defined by (2.88) since we know it for simple functions and from (2.89) it follows that $|v_n|^p \to |f|^p$ in $\mathcal{L}^1(\mathbb{R})$ so $\|v_n\|_{L^p} \to \|f\|_{L^p}$. Then if w_n is a similar sequence for $g \in \mathcal{L}^p(\mathbb{R})$

$$\|f+g\|_{L^p} \le \limsup_n \|v_n+w_n\|_{L^p} \le \limsup_n \|v_n\|_{L^p} + \limsup_n \|w_n\|_{L^p} = \|f\|_{L^p} + \|g\|_{L^p}. \tag{2.90}$$

The other two conditions being clear it follows that $\|u\|_{L^p}$ is a seminorm on $\mathcal{L}^p(\mathbb{R})$.

The vanishing of $\|u\|_{L^p}$ implies that $|u|^p$ and hence $u \in \mathcal{N}$ and the converse follows immediately. Thus $L^p(\mathbb{R}) = \mathcal{L}^p(\mathbb{R})/\mathcal{N}$ is a normed space and it only remains to check completeness.

□

10. The space $L^2(\mathbb{R})$

So far we have discussed the Banach space $L^1(\mathbb{R})$. The real aim is to get a good hold on the (Hilbert) space $L^2(\mathbb{R})$. This can be approached in several ways. We could start off as for $L^1(\mathbb{R})$ and define $L^2(\mathbb{R})$ as the completion of $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$ with respect to the norm

$$\|f\|_{L^2} = \left(\int |f|^2\right)^{\frac{1}{2}}. \tag{2.91}$$

This would be rather repetitious; instead we adopt an approach based on Dominated convergence. You might think, by the way, that it is enough just to ask that $|f|^2 \in \mathcal{L}^1(\mathbb{R})$. This does not work, since even if real the sign of f could jump around and make it non-integrable. This approach would not even work for $L^1(\mathbb{R})$.

Definition 11. A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is said to be 'Lebesgue square integrable', written $f \in \mathcal{L}^2(\mathbb{R})$, if there exists a sequence $u_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ such that

$$u_n(x) \to f(x) \text{ a.e. and } |u_n(x)|^2 \le F(x) \text{ for some } F \in \mathcal{L}^1(\mathbb{R}). \tag{2.92}$$

Proposition 16. *The space $\mathcal{L}^2(\mathbb{R})$ is linear, $f \in \mathcal{L}^2(\mathbb{R})$ implies $|f|^2 \in \mathcal{L}^1(\mathbb{R})$ and* (2.91) *defines a seminorm on $\mathcal{L}^2(\mathbb{R})$ which vanishes precisely on the null functions $\mathcal{N} \subset \mathcal{L}^2(\mathbb{R})$.*

After going through this result I normally move on to the next chapter on Hilbert spaces with this as important motivation.

PROOF. First to see the linearity of $\mathcal{L}^2(\mathbb{R})$ note that $f \in \mathcal{L}^2(\mathbb{R})$ and $c \in \mathbb{C}$ then $cf \in \mathcal{L}^2(\mathbb{R})$ where if u_n is a sequence as in the definition for f then cu_n is such a sequence for cf.

Similarly if $f,\, g \in \mathcal{L}^2(\mathbb{R})$ with sequences u_n and v_n then $w_n = u_n + v_n$ has the first property – since we know that the union of two sets of measure zero is a set of measure zero and the second follows from the estimate

$$|w_n(x)|^2 = |u_n(x) + v_n(x)|^2 \le 2|u_n(x)|^2 + 2|v_n(x)|^2 \le 2(F+G)(x) \tag{2.93}$$

where $|u_n(x)|^2 \le F(x)$ and $|v_n(x)|^2 \le G(x)$ with $F,\, G \in \mathcal{L}^1(\mathbb{R})$.

Moreover, if $f \in \mathcal{L}^2(\mathbb{R})$ then the sequence $|u_n(x)|^2$ converges pointwise almost everywhere to $|f(x)|^2$ so by Lebesgue's Dominated Convergence, $|f|^2 \in \mathcal{L}^1(\mathbb{R})$. Thus $\|f\|_{L^2}$ is well-defined. It vanishes if and only if $|f|^2 \in \mathcal{N}$ but this is equivalent to $f \in \mathcal{N}$ – conversely $\mathcal{N} \subset \mathcal{L}^2(\mathbb{R})$ since the zero sequence works in the definition above.

So we only need to check the triangle inquality, absolutely homogeneity being clear, to deduce that $L^2 = \mathcal{L}^2/\mathcal{N}$ is at least a normed space. In fact we checked this earlier on $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$ and the general case follows by continuity:-

$$\begin{multline} \|u_n + v_n\|_{L^2} \le \|u_n\|_{L^2} + \|v_n\|_{L^2}\ \forall\ n \Longrightarrow \\ \|f+g\|_{L^2} = \lim_{n\to\infty} \|u_n + v_n\|_{L^2} \le \|u\|_{L^2} + \|v\|_{L^2}. \end{multline} \tag{2.94}$$

□

We will get a direct proof of the triangle inequality as soon as we start talking about (pre-Hilbert) spaces.

So it only remains to check the completeness of $L^2(\mathbb{R})$, which is really the whole point of the discussion of Lebesgue integration.

THEOREM 9. *The space $L^2(\mathbb{R})$ is complete with respect to $\|\cdot\|_{L^2}$ and is a completion of $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$ with respect to this norm.*

PROOF. That $\mathcal{C}_{\mathrm{c}}(\mathbb{R}) \subset \mathcal{L}^2(\mathbb{R})$ follows directly from the definition and in fact this is a dense subset. Indeed, if $f \in \mathcal{L}^2(\mathbb{R})$ a sequence $u_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ as in Definition 11 satisfies

$$|u_n(x) - u_m(x)|^2 \le 4F(x)\ \forall\ n,\ m, \tag{2.95}$$

and converges almost everwhere to $|f(x) - u_m(x)|^2$ as $n \to \infty$. Thus, by Dominated Convergence, $|f(x) - u_m(x)|^2 \in \mathcal{L}^1(\mathbb{R})$. Moreover, as $m \to \infty$ $|f(x) - u_m(x)|^2 \to 0$ almost everywhere and $|f(x) - u_m(x)|^2 \le 4F(x)$ so again by dominated convergence

$$\|f - u_m\|_{L^2} = \left(\|(|f-u_m|^2)\|_{L^1}\right)^{\frac{1}{2}} \to 0. \tag{2.96}$$

This shows the density of $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$ in $L^2(\mathbb{R})$, the quotient by the null functions.

To prove completeness, we only need show that any absolutely L^2-summable sequence in $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$ converges in L^2 and the general case follows by density. So, suppose $\phi_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ is such a sequence:

$$\sum_n \|\phi_n\|_{L^2} < \infty.$$

Consider $F_k(x) = \left(\sum\limits_{n\le k} |\phi_k(x)|\right)^2$. This is an increasing sequence in $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$ and its L^1 norm is bounded:

$$\|F_k\|_{L^1} = \|\sum_{n\le k} |\phi_n|\|_{L^2}^2 \le \left(\sum_{n\le k} \|\phi_n\|_{L^2}\right)^2 \le C^2 < \infty \tag{2.97}$$

using the triangle inequality and absolutely L^2 summability. Thus, by Monotone Convergence, $F_k \to F \in \mathcal{L}^1(\mathbb{R})$ and $F_k(x) \le F(x)$ for all x.

Thus the sequence of partial sums $u_k(x) = \sum\limits_{n\le k} \phi_n(x)$ satisfies $|u_k|^2 \le F_k \le F$. Moreover, on any finite interval the Cauchy-Schwarz inequality gives

$$\sum_{n\le k} \|\chi_R\phi_n\|_{L^1} \le (2R)^{\frac{1}{2}} \sum_{n\le k} \|\phi_n\|_{L^2} \tag{2.98}$$

so the sequence $\chi_R\phi_n$ is absolutely summable in L^1. It therefore converges almost everywhere and hence (using the fact a countable union of sets of measure zero is of measure zero)

$$\sum_n \phi(x) \to f(x) \text{ exists } a.e. \tag{2.99}$$

By the definition above the function $f \in \mathcal{L}^2(\mathbb{R})$ and the preceding discussion shows that

$$\|f - \sum_{n\le k} \phi_k\|_{L^2} \to 0. \tag{2.100}$$

Thus in fact $L^2(\mathbb{R})$ is complete. □

Now, at this point we will pass to the discussion of abstract Hilbert spaces, of which $L^2(\mathbb{R})$ is our second important example (after l^2).

Observe that if f, $g \in \mathcal{L}^2(\mathbb{R})$ have approximating sequences u_n, v_n as in Definition 11, so $|u_n(x)|^2 \le F(x)$ and $|v_n(x)|^2 \le G(x)$ with F, $G \in \mathcal{L}^1(\mathbb{R})$ then

$$u_n(x)v_n(x) \to f(x)g(x) \text{ a.e. and } |u_n(x)v_n(x)| \le F(x) + G(x) \tag{2.101}$$

shows that $fg \in \mathcal{L}^1(\mathbb{R})$ by Dominated Convergence. This leads to the basic property of the norm on a (pre)-Hilbert space – that it comes from an inner product. In this case

$$\langle f, g\rangle_{L^2} = \int f(x)\overline{g(x)},\ \|f\|_{L^2} = \langle f, f\rangle^{\frac{1}{2}}. \tag{2.102}$$

11. The spaces $L^p(\mathbb{R})$

Local integrablility of a function is introduced above. Thus $f : \mathbb{R} \longrightarrow \mathbb{C}$ is locally integrable if

$$F_{[-N,N]} = \begin{cases} f(x) & x \in [-N, N] \\ 0 & x \text{ if } |x| > N \end{cases} \Longrightarrow F_{[-N,N]} \in \mathcal{L}^1(\mathbb{R})\ \forall\ N. \tag{2.103}$$

For example any continuous function on $\mathbb{R}$ is locally integrable as is any element of $\mathcal{L}^1(\mathbb{R})$.

Lemma 16. *The locally integrable functions form a linear space, $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R})$.*

PROOF. Follows from the linearity of $\mathcal{L}^1(\mathbb{R})$. $\square$

DEFINITION 12. The space $\mathcal{L}^p(\mathbb{R})$ for any $1 \le p < \infty$ consists of those functions in $\mathcal{L}^1_{\text{loc}}$ such that $|f|^p \in \mathcal{L}^1(\mathbb{R})$; for $p = \infty$,

$$\mathcal{L}^\infty(\mathbb{R}) = \{f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}); \sup_{\mathbb{R}\setminus E} |f(x)| < \infty \text{ for some } E \text{ of measure zero.} \tag{2.104}$$

It is important to note that $|f|^p \in \mathcal{L}^1(\mathbb{R})$ is not, on its own, enough to show that $f \in \mathcal{L}^p(\mathbb{R})$ – it does not in general imply the local integrability of f.

What are some examples of elements of $\mathcal{L}^p(\mathbb{R})$? One class, which we use below, comes from cutting off elements of $\mathcal{L}^1_{\text{loc}}(\mathbb{R})$. Namely, we can cut off outside $[-R, R]$ and for a real function we can cut off 'at height R' (it doesn't have to be the same R but I am saving letters)

$$f^{(R)}(x) = \begin{cases} 0 & \text{if } |x| > R \\ R & \text{if } |x| \le R,\ |f(x)| > R \\ f(x) & \text{if } |x| \le R,\ |f(x)| \le R \\ -R & \text{if } |x| \le R,\ f(x) < -R. \end{cases} \tag{2.105}$$

For a complex function apply this separately to the real and imaginary parts. Now, $f^{(R)} \in \mathcal{L}^1(\mathbb{R})$ since cutting off outside $[-R, R]$ gives an integrable function and then we are taking min and max successively with $\pm R\chi_{[-R,R]}$. If we go back to the definition of $\mathcal{L}^1(\mathbb{R})$ but use the insight we have gained from there, we know that there is an absolutely summable sequence of continuous functions of compact support, f_j, with sum converging a.e. to $f^{(R)}$. The absolute summability means that $|f_j|$ is also an absolutely summable series, and hence its sum a.e., denoted g, is an integrable function by the Monotonicity Lemma above – it is increasing with bounded integral. Thus if we let F_n be the partial sum of the series

$$F_n \to f^{(R)} \text{ a.e.},\ |F_n| \le g \tag{2.106}$$

and we are in the setting of Dominated convergence – except of course we already know that the limit is in $\mathcal{L}^1(\mathbb{R})$. However, we can replace F_n by the sequence of cut-off continuous functions $F_n^{(R)}$ without changing the convergence a.e. or the bound. Now,

$$|F_n^{(R)}|^p \to |f^{(R)}|^p \text{ a.e.}, |F_n^{(R)}|^p \le R^p\chi_{[-R,R]} \tag{2.107}$$

and we see that $|f^{(R)}| \in \mathcal{L}^p(\mathbb{R})$ by Lebesgue Dominated convergence.

We can encapsulate this in a lemma:-

LEMMA 17. *If $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R})$ then with the definition from (2.105), $f^{(R)} \in \mathcal{L}^p(\mathbb{R})$, $1 \le p < \infty$ and there exists a sequence s_n of continuous functions of compact support converging a.e. to $f^{(R)}$ with $|s_n| \le R\chi_{[-R,R]}$.*

THEOREM 10. *The spaces $\mathcal{L}^p(\mathbb{R})$ are linear, the function*

$$\|f\|_{L^p} = \left(\int |f|^p\right)^{1/p} \tag{2.108}$$

is a seminorm on it with null space $\mathcal{N}$, the space of null functions on $\mathbb{R}$, and $L^p(\mathbb{R}) = \mathcal{L}^p(\mathbb{R})/\mathcal{N}$ is a Banach space in which the continuous functions of compact support and step functions include as dense subspaces.

PROOF. First we need to check the linearity of $\mathcal{L}^p(\mathbb{R})$. Clearly $\lambda f \in \mathcal{L}^p(\mathbb{R})$ if $f \in \mathcal{L}^p(\mathbb{R})$ and $\lambda \in \mathbb{C}$ so we only need consider the sum. Then however, we can use Lemma 17. Thus, if f and g are in $\mathcal{L}^p(\mathbb{R})$ then $f^{(R)}$ and $g^{(R)}$ are in $\mathcal{L}^p(\mathbb{R})$ for any $R > 0$. Now, the approximation by continuous functions in the Lemma shows that $f^{(R)} + g^{(R)} \in \mathcal{L}^p(\mathbb{R})$ since it is in $\mathcal{L}^1(\mathbb{R})$ and $|f^{(R)} + g^{(R)}|^p \in \mathcal{L}^1(\mathbb{R})$ by dominated convergence (the model functions being bounded). Now, letting $R \to \infty$ we see that

$$
\begin{gathered}
f^{(R)}(x) + g^{(R)}(x) \to f(x) + g(x) \ \forall \ x \in \mathbb{R} \\
|f^{(R)} + g^{(R)}|^p \le ||f^{(R)}| + |g^{(R)}||^p \le 2^p(|f|^p + |g|^p)
\end{gathered} \tag{2.109}
$$

so by Dominated Convergence, $f + g \in \mathcal{L}^p(\mathbb{R})$.

That $\|f\|_{L^p}$ is a seminorm on $\mathcal{L}^p(\mathbb{R})$ is an integral form of Minkowski's inequality. In fact we can deduce if from the finite form. Namely, for two step functions f and g we can always find a finite collection of intervals on which they are both constant and outside which they both vanish, so the same is true of the sum. Thus

$$
\begin{gathered}
\|f\|_{L^p} = \left(\sum_{j=1}^n |c_i|^p (b_i - a_i)\right)^{\frac{1}{p}}, \ \|g\|_{L^p} = \left(\sum_{j=1}^n |d_i|^p (b_i - a_i)\right)^{\frac{1}{p}}, \\
\|f + g\|_{L^p} = \left(\sum_{j=1}^n |c_i + d_i|^p (b_i - a_i)\right)^{\frac{1}{p}}.
\end{gathered} \tag{2.110}
$$

Absorbing the lengths into the constants, by setting $c_i' = c_i(b_i - a_i)^{\frac{1}{p}}$ and $d_i' = d_i(b_i - a_i)^{\frac{1}{p}}$, Minkowski's inequality now gives

$$
\|f + g\|_{L^p} = \left(\sum_i |c_i' + d_i'|^p\right)^{\frac{1}{p}} \le \|f\|_{L^p} + \|g\|_{L^p} \tag{2.111}
$$

which is the integral form for step functions. Thus indeed, $\|f\|_{L^p}$ is a *norm* on the step functions.

For general elements $f,\ g \in \mathcal{L}^p(\mathbb{R})$ we can use the approximation by step functions in Lemma 17. Thus for any R, there exist sequences of step functions $s_n \to f^{(R)}$, $t_n \to g^{(R)}$ a.e. and bounded by R on $[-R, R]$ so by Dominated Convergence, $\int |f^{(R)}|^p = \lim \int |s_n|^p$, $\int |g^{(R)}|^p$ and $\int |f^{(R)} + g^{(R)}|^p = \lim \int |s_n + t_n|^p$. Thus the triangle inequality holds for $f^{(R)}$ and $g^{(R)}$. Then again applying dominated convergence as $R \to \infty$ gives the general case. The other conditions for a seminorm are clear.

Then the space of functions with $\int |f|^p = 0$ is again just $\mathcal{N}$, independent of p, is clear since $f \in \mathcal{N} \Longleftrightarrow |f|^p \in \mathcal{N}$. The fact that $L^p(\mathbb{R}) = \mathcal{L}^p(\mathbb{R})/\mathcal{N}$ is a normed space follows from the earlier general discussion, or as in the proof above for $L^1(\mathbb{R})$.

So, only the comleteness of $L^p(\mathbb{R})$ remains to be checked and we know this is equivalent to the convergence of any absolutely summable series. So, we can suppose $f_n \in \mathcal{L}^p(\mathbb{R})$ have

$$
\sum_n \left(\int |f_n|^p\right)^{\frac{1}{p}} < \infty. \tag{2.112}
$$

Consider the sequence $g_n = f_n\chi_{[-R,R]}$ for some fixed $R > 0$. This is in $\mathcal{L}^1(\mathbb{R})$ and

$$\|g_n\|_{L^1} \le (2R)^{\frac{1}{q}}\|f_n\|_{L^p} \tag{2.113}$$

by the integral form of Hölder's inequality
(2.114)

$$f \in \mathcal{L}^p(\mathbb{R}),\ g \in \mathcal{L}^q(\mathbb{R}),\ \frac{1}{p}+\frac{1}{q} = 1 \Longrightarrow fg \in \mathcal{L}^1(\mathbb{R}) \text{ and } |\int fg| \le \|f\|_{L^p}\||g\|_{L^q}$$

which can be proved by the same approximation argument as above, see Problem 4. Thus the series g_n is absolutely summable in L^1 and so converges absolutely almost everywhere. It follows that the series $\sum\limits_n f_n(x)$ converges absolutely almost everywhere – since it is just $\sum\limits_n g_n(x)$ on $[-R,R]$. The limit, f, of this series is therefore in $\mathcal{L}^1_{\text{loc}}(\mathbb{R})$.

So, we only need show that $f \in \mathcal{L}^p(\mathbb{R})$ and that $\int |f - F_n|^p \to 0$ as $n \to \infty$ where $F_n = \sum\limits_{k=1}^{n} f_k$. By Minkowski's inequality we know that $h_n = (\sum\limits_{k=1}^{n} |f_k|)^p$ has bounded L^1 norm, since

$$\||h_n|\|^{\frac{1}{p}}_{L^1} = \|\sum_{k=1}^{n}|f_k|\|_{L^p}. \le \sum_k \|f_k\|_{L^p}. \tag{2.115}$$

Thus, h_n is an increasing sequence of functions in $\mathcal{L}^1(\mathbb{R})$ with bounded integral, so by the Monotonicity Lemma it converges a.e. to a function $h \in \mathcal{L}^1(\mathbb{R})$. Since $|F_n|^p \le h$ and $|F_n|^p \to |f|^p$ a.e. it follows by Dominated convergence that

$$|f|^p \in \mathcal{L}^1(\mathbb{R}),\ \||f|^p\|^{\frac{1}{p}}_{L^1} \le \sum_n \|f_n\|_{L^p} \tag{2.116}$$

and hence $f \in \mathcal{L}^p(\mathbb{R})$. Applying the same reasoning to $f - F_n$ which is the sum of the series starting at term $n+1$ gives the norm convergence:

$$\|f - F_n\|_{L^p} \le \sum_{k>n} \|f_k\|_{L^p} \to 0 \text{ as } n \to \infty. \tag{2.117}$$

□

12. Lebesgue measure

In case anyone is interested in how to define Lebesgue measure from where we are now we can just use the integral.

Definition 13. A set $A \subset \mathbb{R}$ is *measurable* if its characteristic function χ_A is locally integrable. A measurable set A has finite measure if $\chi_A \in \mathcal{L}^1(\mathbb{R})$ and then

$$\mu(A) = \int \chi_A \tag{2.118}$$

is the Lebesgue measure of A. If A is measurable but not of finite measure then $\mu(A) = \infty$ by definition.

Functions which are the finite sums of constant multiples of the characteristic functions of measurable sets of finite measure are called 'simple functions' and behave rather like our step functions. One of the standard approaches to Lebesgue integration, but starting from some knowledge of a measure, is to 'complete' the space of simple functions with respect to the integral.

We know immediately that any interval (a, b) is measurable (indeed whether open, semi-open or closed) and has finite measure if and only if it is bounded – then the measure is $b - a$. Some things to check:-

Proposition 17. *The complement of a measurable set is measurable and any* countable *union or countable intersection of measurable sets is measurable.*

Proof. The first part follows from the fact that the constant function 1 is locally integrable and hence $\chi_{\mathbb{R}\setminus A} = 1 - \chi_A$ is locally integrable if and only if χ_A is locally integrable.

Notice the relationship between a characteristic function and the set it defines:-

$$\chi_{A\cup B} = \max(\chi_A, \chi_B),\ \chi_{A\cap B} = \min(\chi_A, \chi_B). \tag{2.119}$$

If we have a sequence of sets A_n then $B_n = \bigcup_{k\le n} A_k$ is clearly an increasing sequence of sets and

$$\chi_{B_n} \to \chi_B,\ B = \sum_n A_n \tag{2.120}$$

is an increasing sequence which converges pointwise (at each point it jumps to 1 somewhere and then stays or else stays at 0.) Now, if we multiply by $\chi_{[-N,N]}$ then

$$f_n = \chi_{[-N,N]}\chi_{B_n} \to \chi_{B\cap[-N,N]} \tag{2.121}$$

is an increasing sequence of integrable functions – assuming that is that the A_k's are measurable – with integral bounded above, by $2N$. Thus by the monotonicity lemma the limit is integrable so χ_B is locally integrable and hence $\bigcup_n A_n$ is measurable.

For countable intersections the argument is similar, with the sequence of characteristic functions decreasing. □

Corollary 3. *The (Lebesgue) measurable subsets of $\mathbb{R}$ form a collection, $\mathcal{M}$, of the power set of $\mathbb{R}$, including $\emptyset$ and $\mathbb{R}$ which is closed under complements, countable unions and countable intersections.*

Such a collection of subsets of a set X is called a 'σ-algebra' – so a σ-algebra Σ in a set X is a collection of subsets of X containing X, $\emptyset$, the complement of any element and countable unions and intersections of any element. A (positive) measure is usually defined as a map $\mu : \Sigma \longrightarrow [0, \infty]$ with $\mu(\emptyset) = 0$ and such that

$$\mu(\bigcup_n E_n) = \sum_n \mu(E_n) \tag{2.122}$$

for any sequence $\{E_m\}$ of sets in Σ which are disjoint (in pairs).

As for Lebesgue measure a set $A \in \Sigma$ is 'measurable' and if $\mu(A)$ is not of finite measure it is said to have infinite measure – for instance $\mathbb{R}$ is of infinite measure in this sense. Since the measure of a set is always non-negative (or undefined if it isn't measurable) this does not cause any problems and in fact Lebesgue measure is countably additive provided as in (2.122) provided we allow ∞ as a value of the measure. It is a good exercise to prove this!

13. Density of step functions

You can skip this section, since it is inserted here to connect the approach via continuous functions and the Riemann integral, in Section 1, to the more usual approach via step functions starting in Section **??** (which does not use the Riemann integral). We prove the 'density' of step functions in $\mathcal{L}^1(\mathbb{R})$ and this leads below to the proof that Definition 5 is equivalent to Definition ?? so that one can use either.

A step function $h : \mathbb{R} \longrightarrow \mathbb{C}$ is by definition a function which is the sum of multiples of characteristic functions of (finite) intervals. Mainly for reasons of consistency we use half-open intervals here, we define $\chi_{(a,b]} = 1$ when $x \in (a,b]$ (which if you like is empty when $a \geq b$) and vanishes otherwise. So a step function is a finite sum

$$h = \sum_{i=1}^{M} c_i \chi_{(a_i,b_i]} \tag{2.123}$$

where it doesn't matter if the intervals overlap since we can cut them up. Anyway, that is the definition.

Proposition 18. *The linear space of step functions is a subspace of $\mathcal{L}^1(\mathbb{R})$, on which $\int |h|$ is a norm, and for any element $f \in \mathcal{L}^1(\mathbb{R})$ there is an absolutely summable series of step functions $\{h_i\}$, $\sum\limits_i \int |h_i| < \infty$ such that*

$$f(x) = \sum_i h_i(x) \ \ a.e. \tag{2.124}$$

Proof. First we show that the characteristic function $\chi_{(a,b]} \in \mathcal{L}^1(\mathbb{R})$. To see this, take a decreasing sequence of continuous functions such as

$$u_n(x) = \begin{cases} 0 & \text{if } x < a - 1/n \\ n(x - a + 1/n) & \text{if } a - 1/n \leq x \leq a \\ 1 & \text{if } a < x \leq b \\ 1 - n(x - b) & \text{if } b < x \leq b + 1/n \\ 0 & \text{if } x > b + 1/n. \end{cases} \tag{2.125}$$

This is continuous because each piece is continuous and the limits from the two sides at the switching points are the same. This is clearly a decreasing sequence of continuous functions which converges pointwise to $\chi_{(a,b]}$ (not uniformly of course). It follows that detelescoping, setting $f_1 = u_1$ and $f_j = u_j - u_{j-1}$ for $j \geq 2$, gives a series of continuous functions which converges pointwise and to $\chi_{(a,b]}$. It follows from the fact that u_j is decreasing that series is absolutely summable, so $\chi_{(a,b]} \in \mathcal{L}^1(\mathbb{R})$.

Now, conversely, each element $f \in \mathcal{C}(\mathbb{R})$ is the uniform limit of step functions – this follows directly from the uniform continuity of continuous functions on compact sets. It suffices to suppose that f is real and then combine the real and imaginary parts. Suppose $f = 0$ outside $[-R, R]$. Take the subdivision of $(-R, R]$ into $2n$ equal intervals of length R/n and let h_n be the step function which is $\sup f$ for the closure of that interval. Choosing n large enough, $\sup f - \inf f < \epsilon$ on each such interval, by uniform continuity, and so $\sup |f - h_n| < \epsilon$. Again this is a decreasing sequence of step functions with integral bounded below so in fact it is the sequence of partial sums of the absolutely summable series obtained by detelescoping.

Certainly then for each element $f \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ there is a sequence of step functions with $\int |f - h_n| \to 0$. The same is therefore true of any element $g \in \mathcal{L}^1(\mathbb{R})$ since then there is a sequence $f_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ such that $\|f - f_n\|_{L^1} \to 0$. So just choosing a step function h_n with $\|f_n - h_n\| < 1/n$ ensures that $\|f - h_n\|_{L^1} \to 0$.

To get an absolutely summable series of step function $\{g_n\}$ with $\|f - \sum_{n=1}^{N} g_n\| \to 0$ we just have to drop elements of the approximating sequence to speed up the convergence and then detelescope the sequence. For the moment I do not say that

$$f(x) = \sum_n g_n(x) \text{ a.e.} \tag{2.126}$$

although it is true! It follows from the fact that the right side does define an element of $\mathcal{L}^1(\mathbb{R})$ and by the triangle inequality the difference of the two sides has vanishing L^1 norm, i.e. is a null function. So we just need to check that null functions vanish outside a set of measure zero. This is Proposition 12 below, which uses Proposition 13. Taking a little out of the proof of that proposition proves (2.126) directly. □

14. Measures on the line

Going back to starting point for Lebesgue measure and the Lebesgue integral, the discussion can be generalized, even in the one-dimensional case, by replacing the measure of an interval by a more general function. As for the Stieltjes integral this can be given by an increasing (meaning of course non-decreasing) function $m : \mathbb{R} \longrightarrow \mathbb{R}$. For the discussion in this chapter to go through with only minor changes we need to require that

$$\begin{gathered} m : \mathbb{R} \longrightarrow \mathbb{R} \text{ is non-decreasing and continuous from below} \\ \lim x \uparrow y m(x) = m(y) \ \forall \ y \in \mathbb{R}. \end{gathered} \tag{2.127}$$

Then we can define

$$\mu([a, b)) = m(b) - m(a). \tag{2.128}$$

For open and closed intervals we will expect that

$$\mu((a, b)) = \lim_{x \downarrow a} m(x) - m(b), \ \ \mu([a, b]) = m(a) - \lim_{x \downarrow b} m(x). \tag{2.129}$$

To pursue this, the first thing to check is that the analogue of Proposition **??** holds in this case – namely if $[a, b)$ is decomposed into a finite number of such semi-open intervals by choice of interior points then

$$\mu([a, b)) = \sum_i \mu([a_i, b_i)). \tag{2.130}$$

Of course this follows from (2.128). Similarly, $\mu([a, b)) \geq \mu([A, B))$ if $A \leq a$ and $b \leq B$, i.e. if $[a, b) \subset [A, B)$. From this it follows that the analogue of Lemma **??** also holds with μ in place of Lebesgue length.

Then we can define the μ-integral, $\int f d\mu$, of a step function, we do not get Proposition **??** since we might have intervals of μ length zero. Still, $\int |f| d\mu$ is a seminorm. The definition of a μ-Lebesgue-integrable function (just called μ-integrable usually), in terms of absolutely summable series with respect to this seminorm, can be carried over as in Definition **??**.

So far we have not used the continuity condition in (2.129), but now consider the covering result Proposition **??**. The first part has the same proof. For the second part, the proof proceeds by making the intervals a little longer at the closed end – to make them open. The continuity condition (2.129) ensures that this can be done in such a way as to make the difference $\mu(b_i) - m(a_i - \epsilon_i) < \mu([a_i, b_i)) + \delta 2^{-i}$

for any $\delta > 0$ by choosing $\epsilon_i > 0$ small enough. This covers $[a, b - \epsilon]$ for $\epsilon > 0$ and this allows the finite cover result to be applied to see that

$$\mu(b - \epsilon) - \mu(a) \le \sum_i \mu([a_i, b_i)) + 2\delta \tag{2.131}$$

for any $\delta > 0$ and $\epsilon > 0$. Then taking the limits as $\epsilon \downarrow 0$ and $\delta \downarrow 0$ gives the 'outer' intequality. So Proposition **??** carries over.

From this point the discussion of the μ integral proceeds in the same way with a few minor exceptions – Corollary **??** doesn't work again because there may be intervals of length zero. Otherwise things proceed pretty smoothly right through. The construction of Lebesgue measure, as in § 12, leasds to a σ-algebra Σ_μ, of subsets of $\mathbb{R}$ which contains all the intervals, whether open, closed or mixed and all the compact sets. You can check that the resulting countably additive measure is a 'Radon measure' in that

$$\begin{gathered}\mu(B) = \inf\{\sum_i \mu((a_i b_i));\ B \subset \bigcup_i (a_i, b_i)\},\ \forall\ B \in \Sigma_\mu,\\ \mu((a, b)) = \sup\{\mu(K); K \subset (a, b),\ K \text{ compact}\}.\end{gathered} \tag{2.132}$$

Conversely, every such positive Radon measure arises this way. Continuous functions are locally μ-integrable and if $\mu(\mathbb{R}) < \infty$ (which corresponds to a choice of m which is bounded) then $\int f d\mu < \infty$ for every bounded continuous function which vanishes at infinity.

Theorem 11. *[Riesz' other representation theorem] For any $f \in (C_0(\mathbb{R}))$ there are four uniquely determined (positive) Radon measures, μ_i, $i = 1, \dots, 4$ such that $\mu_i(\mathbb{R}) < \infty$ and*

$$f(u) = \int f d\mu_1 - \int f d\mu_2 + i \int f d\mu_3 - i \int f d\mu_4. \tag{2.133}$$

How hard is this to prove? Well, a proof is outlined in the problems.

15. Higher dimensions

I do not actually plan to cover this in lectures, but put it in here in case someone is interested (which you should be) or if I have time at the end of the course to cover a problem in two or more dimensions (I have the Dirichlet problem in mind).

So, we want – with the advantage of a little more experience – to go back to the beginning and define $\mathcal{L}^1(\mathbb{R}^n)$, $L^1(\mathbb{R}^n)$, $\mathcal{L}^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. In fact relatively little changes but there are some things that one needs to check a little carefully.

The first hurdle is that I am not assuming that you have covered the Riemann integral in higher dimensions. Fortunately we do not reall need that since we can just iterated the one-dimensional Riemann integral for continuous functions. So, define

$$\mathcal{C}_c(\mathbb{R}^n) = \{u : \mathbb{R}^n \longrightarrow \mathbb{C};\ \text{continuous and such that } u(x) = 0 \text{ for } |x| > R\} \tag{2.134}$$

where of course the R can depend on the element. Now, if we hold say the last $n-1$ variables fixed, we get a continuous function of 1 variable which vanishes when $|x| > R$:

$$u(\cdot, x_2, \dots, x_n) \in \mathcal{C}_c(\mathbb{R}) \text{ for each } (x_2, \dots, x_n) \in \mathbb{R}^{n-1}. \tag{2.135}$$

So we can integrate it and get a function

$$I_1(x_2, \dots, x_n) = \int_{\mathbb{R}} u(x, x_1, \dots, x_n),\ I_1 : \mathbb{R}^{n-1} \longrightarrow \mathbb{C}. \tag{2.136}$$

LEMMA 18. *For each $u \in \mathcal{C}_c(\mathbb{R}^n)$, $I_1 \in \mathcal{C}_c(\mathbb{R}^{n-1})$.*

PROOF. Certainly if $|(x_2, \dots, x_n)| > R$ then $u(\cdot, x_2, \dots, x_n) \equiv 0$ as a function of the first variable and hence $I_1 = 0$ in $|(x_2, \dots, x_n)| > R$. The continuity follows from the uniform continuity of a function on the compact set $|x| \le R$, $|(x_2, \dots, x_n) \le R$ of $\mathbb{R}^n$. Thus given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - x'| < \delta,\ |y - y'|_{\mathbb{R}^{n-1}} < \delta \Longrightarrow |u(x, y) - u(x', y')| < \epsilon. \tag{2.137}$$

From the standard estimate for the Riemann integral,

$$|I_1(y) - I_1(y')| \le \int_{-R}^{R} |u(x, y) - u(x, y')| dx \le 2R\epsilon \tag{2.138}$$

if $|y - y'| < \delta$. This implies the (uniform) continuity of I_1. Thus $I_1 \in \mathcal{C}_c(\mathbb{R}^{n-2})$ □

The upshot of this lemma is that we can integrate again, and hence a total of n times and so define the (iterated) Riemann integral as

$$\int_{\mathbb{R}^n} u(z) dz = \int_{-R}^{R} \int_{-R}^{R} \dots \int_{-R}^{R} u(x_1, x_2, x_3, \dots, x_n) dx_1 dx_2 \dots dx_n \in \mathbb{C}. \tag{2.139}$$

LEMMA 19. *The interated Riemann integral is a well-defined linear map*

$$\mathcal{C}_c(\mathbb{R}^n) \longrightarrow \mathbb{C} \tag{2.140}$$

which satisfies

$$|\int u| \le \int |u| \le (2R)^n \sup |u| \ \textit{if}\ u \in \mathcal{C}_c(\mathbb{R}^n)\ \textit{and}\ u(z) = 0\ \textit{in}\ |z| > R. \tag{2.141}$$

PROOF. This follows from the standard estimate in one dimension. □

Now, one annoying thing is to check that the integral is independent of the order of integration (although be careful with the signs here!) Fortunately we can do this later and not have to worry.

LEMMA 20. *The iterated integral*

$$\|u\|_{L^1} = \int_{\mathbb{R}^n} |u| \tag{2.142}$$

is a norm on $\mathcal{C}_c(\mathbb{R}^n)$.

PROOF. Straightforward. □

DEFINITION 14. The space $\mathcal{L}^1(\mathbb{R}^n)$ (resp. $\mathcal{L}^2(\mathbb{R}^n)$) is defined to consist of those functions $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ such that there exists a sequence $\{f_n\}$ which is absolutely summable with respect to the L^1 norm (resp. the L^2 norm) such that

$$\sum_n |f_n(x)| < \infty \Longrightarrow \sum_n f_n(x) = f(x). \tag{2.143}$$

PROPOSITION 19. *If* $f \in \mathcal{L}^1(\mathbb{R}^n)$ *then* $|f| \in \mathcal{L}^1(\mathbb{R}^n)$, $\operatorname{Re} f \in \mathcal{L}^1(\mathbb{R}^n)$ *and the space* $\mathcal{L}^1(\mathbb{R}^n)$ *is lienar. Moreover if* $\{f_j\}$ *is an absolutely summable sequence in* $\mathcal{C}_c(\mathbb{R}^n)$ *with respect to* L^1 *such that*

$$\sum_n |f_n(x)| < \infty \Longrightarrow \sum_n f_n(x) = 0 \tag{2.144}$$

then $\int f_n \to 0$ *and in consequence the limit*

$$\int_{\mathbb{R}^n} f = \sum_{n\to\infty} \int f_n \tag{2.145}$$

is well-defined on $\mathcal{L}^1(\mathbb{R}^n)$.

PROOF. Remarkably enough, nothing new is involved here. For the first part this is pretty clear, but also holds for the second part. There is a lot to work through, but it is all pretty much as in the one-dimensional case. □

Removed material

Here is a narrative for a later reading:- If you can go through this item by item, reconstruct the definitions and results as you go and see how thing fit together then you are doing well!

- Intervals and length.
- Covering lemma.
- Step functions and the integral.
- Monotonicity lemma.
- $\mathcal{L}^1(\mathbb{R})$ and absolutely summable approximation.
- $\mathcal{L}^1(\mathbb{R})$ is a linear space.
- $\int : \mathcal{L}^1(\mathbb{R}) \longrightarrow \mathbb{C}$ is well defined.
- If $f \in \mathcal{L}^1(\mathbb{R})$ then $|f| \in \mathcal{L}^1(\mathbb{R})$ and

$$\int |f| = \lim_{n\to\infty} \int |\sum_{j=1}^n f_j|, \ \lim_{n\to\infty} \int |f - \sum_{j=1}^n f_j| = 0 \tag{2.146}$$

  for any absolutely summable approximation.
- Sets of measure zero.
- Convergence a.e.
- If $\{g_j\}$ in $\mathcal{L}^1(\mathbb{R})$ is absolutely summable then

$$\begin{gathered} g = \sum_j g_j \text{ a.e. } \Longrightarrow g \in \mathcal{L}^1(\mathbb{R}), \\ \{x \in \mathbb{R}; \sum_j |g_j(x)| = \infty\} \text{ is of measure zero} \\ \int g = \sum_j \int g_j, \ \int |g| = \lim_{n\to\infty} \int |\sum_{j=1}^n g_j|, \ \lim_{n\to\infty} \int |g - \sum_{j=1}^n g_j| = 0. \end{gathered} \tag{2.147}$$

- The space of null functions $\mathcal{N} = \{f \in \mathcal{L}^1(\mathbb{R}); \int |f| = 0\}$ consists precisely of the functions vanishing almost everywhere, $\mathcal{N} = \{f : \mathbb{R} \longrightarrow \mathbb{C}; f = 0 \ a.e.\}$.
- $L^1(\mathbb{R}) = \mathcal{L}^1(\mathbb{R})/\mathcal{N}$ is a Banach space with L^1 norm.
- Montonicity for Lebesgue functions.
- Fatou's Lemma.

- Dominated convergence.
- The Banach spaces $L^p(\mathbb{R}) = \mathcal{L}^p(\mathbb{R})/\mathcal{N}$, $1 \leq p < \infty$.
- Measurable sets.

CHAPTER 3

Hilbert spaces

There are really three 'types' of Hilbert spaces (over $\mathbb{C}$). The finite dimensional ones, essentially just $\mathbb{C}^n$, with which you are pretty familiar and two infinite dimensional cases corresponding to being separable (having a countable dense subset) or not. As we shall see, there is really only one separable infinite-dimensional Hilbert space and that is what we are mostly interested in. Nevertheless some proofs (usually the nicest ones) work in the non-separable case too.

I will first discuss the definition of pre-Hilbert and Hilbert spaces and prove Cauchy's inequality and the parallelogram law. This can be found in all the lecture notes listed earlier and many other places so the discussion here will be kept succinct. Another nice source is the book of G.F. Simmons, "Introduction to topology and modern analysis". I like it – but I think it is out of print.

1. pre-Hilbert spaces

A pre-Hilbert space, H, is a vector space (usually over the complex numbers but there is a real version as well) with a Hermitian inner product

$$\begin{gathered}(,) : H \times H \longrightarrow \mathbb{C},\\ (\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 (v_1, w) + \lambda_2 (v_2, w),\\ (w, v) = \overline{(v, w)}\end{gathered} \tag{3.1}$$

for any v_1, v_2, v and $w \in H$ and λ_1, $\lambda_2 \in \mathbb{C}$ which is positive-definite

$$(v, v) \geq 0,\ (v, v) = 0 \Longrightarrow v = 0. \tag{3.2}$$

Note that the reality of (v, v) follows from the second condition in (3.1), the positivity is an additional assumption as is the positive-definiteness.

The combination of the two conditions in (3.1) implies 'anti-linearity' in the second variable

$$(v, \lambda_1 w_1 + \lambda_2 w_2) = \overline{\lambda_1}(v, w_1) + \overline{\lambda_2}(v, w_2) \tag{3.3}$$

which is used without comment below.

The notion of 'definiteness' for such an Hermitian inner product exists without the need for positivity – it just means

$$(u, v) = 0\ \forall\ v \in H \Longrightarrow u = 0. \tag{3.4}$$

Lemma 21. *If H is a pre-Hilbert space with Hermitian inner product $(,)$ then*

$$\|u\| = (u, u)^{\frac{1}{2}} \tag{3.5}$$

is a norm on H.

PROOF. The first condition on a norm follows from (3.2). Absolute homogeneity follows from (3.1) since

$$\|\lambda u\|^2 = (\lambda u, \lambda u) = |\lambda|^2 \|u\|^2. \tag{3.6}$$

So, it is only the triangle inequality we need. This follows from the next lemma, which is the Cauchy-Schwarz inequality in this setting – (3.8). Indeed, using the 'sesqui-linearity' to expand out the norm

$$\begin{aligned} \|u+v\|^2 &= (u+v, u+v) \\ &= \|u\|^2 + (u,v) + (v,u) + \|v\|^2 \le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2. \end{aligned} \tag{3.7}$$

□

LEMMA 22. *The Cauchy-Schwarz inequality,*

$$|(u,v)| \le \|u\|\|v\| \ \forall \ u, v \in H \tag{3.8}$$

holds in any pre-Hilbert space.

PROOF. For any non-zero u, $v \in H$ and $s \in \mathbb{R}$ positivity of the norm shows that

$$0 \le \|u + sv\|^2 = \|u\|^2 + 2s \operatorname{Re}(u,v) + s^2 \|v\|^2. \tag{3.9}$$

This quadratic polynomial is non-zero for s large so can have only a single minimum at which point the derivative vanishes, i.e. it is where

$$2s\|v\|^2 + 2\operatorname{Re}(u,v) = 0. \tag{3.10}$$

Substituting this into (3.9) gives

$$\|u\|^2 - (\operatorname{Re}(u,v))^2/\|v\|^2 \ge 0 \Longrightarrow |\operatorname{Re}(u,v)| \le \|u\|\|v\| \tag{3.11}$$

which is what we want except that it is only the real part. However, we know that, for some $z \in \mathbb{C}$ with $|z| = 1$, $\operatorname{Re}(zu, v) = \operatorname{Re} z(u,v) = |(u,v)|$ and applying (3.11) with u replaced by zu gives (3.8). □

2. Hilbert spaces

DEFINITION 15. A Hilbert space H is a pre-Hilbert space which is complete with respect to the norm induced by the inner product.

As examples we know that $\mathbb{C}^n$ with the usual inner product

$$(z, z') = \sum_{j=1}^{n} z_j \overline{z'_j} \tag{3.12}$$

is a Hilbert space – since any finite dimensional normed space is complete. The example we had from the beginning of the course is l^2 with the extension of (3.12)

$$(a, b) = \sum_{j=1}^{\infty} a_j \overline{b_j}, \ a, b \in l^2. \tag{3.13}$$

Completeness was shown earlier.

The whole outing into Lebesgue integration was so that we could have the 'standard example' at our disposal, namely

$$L^2(\mathbb{R}) = \{u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}); |u|^2 \in \mathcal{L}^1(\mathbb{R})\}/\mathcal{N} \tag{3.14}$$

where $\mathcal{N}$ is the space of null functions. and the inner product is

$$(u,v)=\int u\overline{v}. \tag{3.15}$$

Note that we showed that if u, $v\in\mathcal{L}^2(\mathbb{R})$ then $uv\in\mathcal{L}^1(\mathbb{R})$.

3. Orthonormal sets

Two elements of a pre-Hilbert space H are said to be orthogonal if

$$(u,v)=0\Longleftrightarrow u\perp v. \tag{3.16}$$

A sequence of elements $e_i\in H$, (finite or infinite) is said to be *orthonormal* if $\|e_i\|=1$ for all i and $(e_i,e_j)=0$ for all $i\neq j$.

Proposition 20 (Bessel's inequality). *If e_i, $i\in\mathbb{N}$, is an orthonormal sequence in a pre-Hilbert space H, then*

$$\sum_i |(u,e_i)|^2\leq\|u\|^2\ \forall\ u\in H. \tag{3.17}$$

Proof. Start with the finite case, $i=1,\ldots,N$. Then, for any $u\in H$ set

$$v=\sum_{i=1}^N(u,e_i)e_i. \tag{3.18}$$

This is supposed to be 'the projection of u onto the span of the e_i'. Anyway, computing away we see that

$$(v,e_j)=\sum_{i=1}^N(u,e_i)(e_i,e_j)=(u,e_j) \tag{3.19}$$

using orthonormality. Thus, $u-v\perp e_j$ for all j so $u-v\perp v$ and hence

$$0=(u-v,v)=(u,v)-\|v\|^2. \tag{3.20}$$

Thus $\|v\|^2=|(u,v)|$ and applying the Cauchy-Schwarz inequality we conclude that $\|v\|^2\leq\|v\|\|u\|$ so either $v=0$ or $\|v\|\leq\|u\|$. Expanding out the norm (and observing that all cross-terms vanish)

$$\|v\|^2=\sum_{i=1}^N|(u,e_i)|^2\leq\|u\|^2$$

which is (3.17).

In case the sequence is infinite this argument applies to any finite subsequence, e_i, $i=1,\ldots,N$ since it just uses orthonormality, so (3.17) follows by taking the supremum over N. □

4. Gram-Schmidt procedure

Definition 16. An orthonormal sequence, $\{e_i\}$, (finite or infinite) in a pre-Hilbert space is said to be *maximal* if

$$u\in H,\ (u,e_i)=0\ \forall\ i\Longrightarrow u=0. \tag{3.21}$$

Theorem 12. *Every separable pre-Hilbert space contains a maximal orthonormal set.*

Proof. Take a countable dense subset – which can be arranged as a sequence $\{v_j\}$ and the existence of which is the definition of separability – and orthonormalize it. Thus if $v_1 \neq 0$ set $e_i = v_1/\|v_1\|$. Proceeding by induction we can suppose to have found for a given integer n elements e_i, $i = 1, \dots, m$, where $m \leq n$, which are orthonormal and such that the linear span

$$\operatorname{sp}(e_1, \dots, e_m) = \operatorname{sp}(v_1, \dots, v_n). \tag{3.22}$$

To show the inductive step observe that if v_{n+1} is in the span(s) in (3.22) then the same e_i's work for $n+1$. So we may as well assume that the next element, v_{n+1} is not in the span in (3.22). It follows that

$$w = v_{n+1} - \sum_{j=1}^{n} (v_{n+1}, e_j)e_j \neq 0 \text{ so } e_{m+1} = \frac{w}{\|w\|} \tag{3.23}$$

makes sense. By construction it is orthogonal to all the earlier e_i's so adding e_{m+1} gives the equality of the spans for $n+1$.

Thus we may continue indefinitely, since in fact the only way the dense set could be finite is if we were dealing with the space with one element, 0, in the first place. There are only two possibilities, either we get a finite set of e_i's or an infinite sequence. In either case this must be a maximal orthonormal sequence. That is, we claim

$$H \ni u \perp e_j \ \forall\ j \Longrightarrow u = 0. \tag{3.24}$$

This uses the density of the v_n's. There must exist a sequence w_j where each w_j is a v_n, such that $w_j \to u$ in H, assumed to satisfy (3.24). Now, each v_n, and hence each w_j, is a finite linear combination of e_k's so, by Bessel's inequality

$$\|w_j\|^2 = \sum_k |(w_j, e_k)|^2 = \sum_k |(u - w_j, e_k)|^2 \leq \|u - w_j\|^2 \tag{3.25}$$

where $(u, e_j) = 0$ for all j has been used. Thus $\|w_j\| \to 0$ and $u = 0$. □

Now, although a non-complete but separable pre-Hilbert space has maximal orthonormal sets, these are not much use without completeness.

5. Complete orthonormal bases

Definition 17. A maximal orthonormal sequence in a separable Hilbert space is called a complete orthonormal basis.

This notion of basis is not quite the same as in the finite dimensional case (although it is a legitimate extension of it).

Theorem 13. *If $\{e_i\}$ is a complete orthonormal basis in a Hilbert space then for any element $u \in H$ the 'Fourier-Bessel series' converges to u :*

$$u = \sum_{i=1}^{\infty} (u, e_i)e_i. \tag{3.26}$$

Proof. The sequence of partial sums of the Fourier-Bessel series

$$u_N = \sum_{i=1}^{N} (u, e_i)e_i \tag{3.27}$$

is Cauchy. Indeed, if $m < m'$ then

$$\|u_{m'} - u_m\|^2 = \sum_{i=m+1}^{m'} |(u, e_i)|^2 \le \sum_{i>m} |(u, e_i)|^2 \tag{3.28}$$

which is small for large m by Bessel's inequality. Since we are now assuming completeness, $u_m \to w$ in H. However, $(u_m, e_i) = (u, e_i)$ as soon as $m > i$ and $|(w - u_n, e_i)| \le \|w - u_n\|$ so in fact

$$(w, e_i) = \lim_{m\to\infty} (u_m, e_i) = (u, e_i) \tag{3.29}$$

for each i. Thus in fact $u - w$ is orthogonal to all the e_i so by the assumed completeness of the orthonormal basis must vanish. Thus indeed (3.26) holds. $\square$

6. Isomorphism to l^2

A finite dimensional Hilbert space is isomorphic to $\mathbb{C}^n$ with its standard inner product. Similarly from the result above

Proposition 21. *Any infinite-dimensional separable Hilbert space (over the complex numbers) is isomorphic to l^2, that is there exists a linear map*

$$T : H \longrightarrow l^2 \tag{3.30}$$

which is 1-1, onto and satisfies $(Tu, Tv)_{l^2} = (u, v)_H$ and $\|Tu\|_{l^2} = \|u\|_H$ for all u, $v \in H$.

Proof. Choose an orthonormal basis – which exists by the discussion above and set

$$Tu = \{(u, e_j)\}_{j=1}^{\infty}. \tag{3.31}$$

This maps H into l^2 by Bessel's inequality. Moreover, it is linear since the entries in the sequence are linear in u. It is 1-1 since $Tu = 0$ implies $(u, e_j) = 0$ for all j implies $u = 0$ by the assumed completeness of the orthonormal basis. It is surjective since if $\{c_j\}_{j=1}^{\infty} \in l^2$ then

$$u = \sum_{j=1}^{\infty} c_j e_j \tag{3.32}$$

converges in H. This is the same argument as above – the sequence of partial sums is Cauchy since if $n > m$,

$$\| \sum_{j=m+1}^{n} c_j e_j \|_H^2 = \sum_{j=m+1}^{n} |c_|^2. \tag{3.33}$$

Again by continuity of the inner product, $Tu = \{c_j\}$ so T is surjective.

The equality of the norms follows from equality of the inner products and the latter follows by computation for finite linear combinations of the e_j and then in general by continuity. $\square$

7. Parallelogram law

What exactly is the difference between a general Banach space and a Hilbert space? It is of course the existence of the inner product defining the norm. In fact it is possible to formulate this condition intrinsically in terms of the norm itself.

PROPOSITION 22. *In any pre-Hilbert space the parallelogram law holds –*

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2,\ \forall\ v, w \in H. \tag{3.34}$$

PROOF. Just expand out using the inner product

$$\|v+w\|^2 = \|v\|^2 + (v,w) + (w,v) + \|w\|^2 \tag{3.35}$$

and the same for $\|v-w\|^2$ and see the cancellation. □

PROPOSITION 23. *Any normed space where the norm satisfies the parallelogram law,* (3.34)*, is a pre-Hilbert space in the sense that*

$$(v,w) = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2 + i\|v+iw\|^2 - i\|v-iw\|^2\right) \tag{3.36}$$

is a positive-definite Hermitian inner product which reproduces the norm.

PROOF. A problem below. □

So, when we use the parallelogram law and completeness we are using the essence of the Hilbert space.

8. Convex sets and length minimizer

The following result does not need the hypothesis of separability of the Hilbert space and allows us to prove the subsequent results – especially Riesz' theorem – in full generality.

PROPOSITION 24. *If $C \subset H$ is a subset of a Hilbert space which is*

(1) *Non-empty*
(2) *Closed*
(3) *Convex, in the sense that $v_1, v_1 \in C$ implies $\frac{1}{2}(v_1+v_2) \in C$*

then there exists a unique element $v \in C$ closest to the origin, i.e. such that

$$\|v\|_H = \inf_{u \in C} \|u\|_H. \tag{3.37}$$

PROOF. By definition of inf there must exist a sequence $\{v_n\}$ in C such that $\|v_n\| \to d = \inf_{u \in C} \|u\|_H$. We show that v_n converges and that the limit is the point we want. The parallelogram law can be written

$$\|v_n - v_m\|^2 = 2\|v_n\|^2 + 2\|v_m\|^2 - 4\|(v_n+v_m)/2\|^2. \tag{3.38}$$

Since $\|v_n\| \to d$, given $\epsilon > 0$ if N is large enough then $n > N$ implies $2\|v_n\|^2 < 2d^2 + \epsilon^2/2$. By convexity, $(v_n+v_m)/2 \in C$ so $\|(v_n+v_m)/2\|^2 \geq d^2$. Combining these estimates gives

$$n, m > N \Longrightarrow \|v_n - v_m\|^2 \leq 4d^2 + \epsilon^2 - 4d^2 = \epsilon^2 \tag{3.39}$$

so $\{v_n\}$ is Cauchy. Since H is complete, $v_n \to v \in C$, since C is closed. Moreover, the distance is continuous so $\|v\|_H = \lim_{n\to\infty} \|v_n\| = d$.

Thus v exists and uniqueness follows again from the parallelogram law. If v and v' are two points in C with $\|v\| = \|v'\| = d$ then $(v + v')/2 \in C$ so

$$\|v - v'\|^2 = 2\|v\|^2 + 2\|v'\|^2 - 4\|(v + v')/2\|^2 \leq 0 \Longrightarrow v = v'. \tag{3.40}$$

□

9. Orthocomplements and projections

PROPOSITION 25. *If $W \subset H$ is a linear subspace of a Hilbert space then*

$$W^\perp = \{u \in H; (u, w) = 0 \ \forall \ w \in W\} \tag{3.41}$$

is a closed linear subspace and $W \cap W^\perp = \{0\}$. If W is also closed then

$$H = W \oplus W^\perp \tag{3.42}$$

meaning that any $u \in H$ has a unique decomposition $u = w + w^\perp$ where $w \in W$ and $w^\perp \in W^\perp$.

PROOF. That $W^\perp$ defined by (3.41) is a linear subspace follows from the linearity of the condition defining it. If $u \in W^\perp$ and $u \in W$ then $u \perp u$ by the definition so $(u, u) = \|u\|^2 = 0$ and $u = 0$. Since the map $H \ni u \longrightarrow (u, w) \in \mathbb{C}$ is continuous for each $w \in H$ its null space, the inverse image of 0, is closed. Thus

$$W^\perp = \bigcap_{w \in W} \{(u, w) = 0\} \tag{3.43}$$

is closed.

Now, suppose W is closed. If $W = H$ then $W^\perp = \{0\}$ and there is nothing to show. So consider $u \in H$, $u \notin W$ and set

$$C = u + W = \{u' \in H; u' = u + w, \ w \in W\}. \tag{3.44}$$

Then C is closed, since a sequence in it is of the form $u'_n = u + w_n$ where w_n is a sequence in W and u'_n converges if and only if w_n converges. Also, C is non-empty, since $u \in C$ and it is convex since $u' = u + w'$ and $u'' = u + w''$ in C implies $(u' + u'')/2 = u + (w' + w'')/2 \in C$.

Thus the length minimization result above applies and there exists a unique $v \in C$ such that $\|v\| = \inf_{u' \in C} \|u'\|$. The claim is that this v is perpendicular to W – draw a picture in two real dimensions! To see this consider an aritrary point $w \in W$ and $\lambda \in \mathbb{C}$ then $v + \lambda w \in C$ and

$$\|v + \lambda w\|^2 = \|v\|^2 + 2\operatorname{Re}(\lambda(v, w)) + |\lambda|^2\|w\|^2. \tag{3.45}$$

Choose $\lambda = te^{i\theta}$ where t is real and the phase is chosen so that $e^{i\theta}(v, w) = |(v, w)| \geq 0$. Then the fact that $\|v\|$ is minimal means that

$$\begin{gathered} \|v\|^2 + 2t|(v, w))| + t^2\|w\|^2 \geq \|v\|^2 \Longrightarrow \\ t(2|(v, w)| + t\|w\|^2) \geq 0 \ \forall \ t \in \mathbb{R} \Longrightarrow |(v, w)| = 0 \end{gathered} \tag{3.46}$$

which is what we wanted to show.

Thus indeed, given $u \in H \setminus W$ we have constructed $v \in W^\perp$ such that $u = v + w$, $w \in W$. This is (3.42) with the uniqueness of the decomposition already shown since it reduces to 0 having only the decomposition $0 + 0$ and this in turn is $W \cap W^\perp = \{0\}$. □

Since the construction in the preceding proof associates a unique element in W, a closed linear subspace, to each $u \in H$, it defines a map

$$\Pi_W : H \longrightarrow W. \tag{3.47}$$

This map is linear, by the uniqueness since if $u_i = v_i + w_i$, $w_i \in W$, $(v_i, w_i) = 0$ are the decompositions of two elements then

$$\lambda_1 u_1 + \lambda_2 u_2 = (\lambda_1 v_1 + \lambda_2 v_2) + (\lambda_1 w_1 + \lambda_2 w_2) \tag{3.48}$$

must be the corresponding decomposition. Moreover $\Pi_W w = w$ for any $w \in W$ and $\|u\|^2 = \|v\|^2 + \|w\|^2$, Pythagoras' Theorem, shows that

$$\Pi_W^2 = \Pi_W,\ \|\Pi_W u\| \le \|u\| \Longrightarrow \|\Pi_W\| \le 1. \tag{3.49}$$

Thus, projection onto W is an operator of norm 1 (unless $W = \{0\}$) equal to its own square. Such an operator is called a projection or sometimes an idempotent (which sounds fancier).

LEMMA 23. *If $\{e_j\}$ is any finite or countable orthonormal set in a Hilbert space then the orthogonal projection onto the closure of the span of these elements is*

$$Pu = \sum (u, e_k) e_k. \tag{3.50}$$

PROOF. We know that the series in (3.50) converges and defines a bounded linear operator of norm at most one by Bessel's inequality. Clearly $P^2 = P$ by the same argument. If W is the closure of the span then $(u - Pu) \perp W$ since $(u - Pu) \perp e_k$ for each k and the inner product is continuous. Thus $u = (u - Pu) + Pu$ is the orthogonal decomposition with respect to W. $\square$

10. Riesz' theorem

The most important application of these results is to prove Riesz' representation theorem (for Hilbert space, there is another one to do with measures).

THEOREM 14. *If H is a Hilbert space then for any continuous linear functional $T : H \longrightarrow \mathbb{C}$ there exists a unique element $\phi \in H$ such that*

$$T(u) = (u, \phi)\ \forall\ u \in H. \tag{3.51}$$

PROOF. If T is the zero functional then $\phi = 0$ gives (3.51). Otherwise there exists some $u' \in H$ such that $T(u') \ne 0$ and then there is some $u \in H$, namely $u = u'/T(u')$ will work, such that $T(u) = 1$. Thus

$$C = \{u \in H; T(u) = 1\} = T^{-1}(\{1\}) \ne \emptyset. \tag{3.52}$$

The continuity of T and the second form shows that C is closed, as the inverse image of a closed set under a continuous map. Moreover C is convex since

$$T((u + u')/2) = (T(u) + T(u'))/2. \tag{3.53}$$

Thus, by Proposition 24, there exists an element $v \in C$ of minimal length.

Notice that $C = \{v + w; w \in N\}$ where $N = T^{-1}(\{0\})$ is the null space of T. Thus, as in Proposition 25 above, v is orthogonal to N. In this case it is the unique element orthogonal to N with $T(v) = 1$.

Now, for any $u \in H$,

$$u - T(u)v \text{ satisfies } T(u - T(u)v) = T(u) - T(u)T(v) = 0 \Longrightarrow u = w + T(u)v,\ w \in N. \tag{3.54}$$

Then, $(u,v) = T(u)\|v\|^2$ since $(w,v) = 0$. Thus if $\phi = v/\|v\|^2$ then

$$u = w + (u,\phi)v \Longrightarrow T(u) = (u,\phi)T(v) = (u,\phi). \tag{3.55}$$

□

11. Adjoints of bounded operators

As an application of Riesz' we can see that to any bounded linear operator on a Hilbert space

$$A : H \longrightarrow H,\ \|Au\|_H \le C\|u\|_H\ \forall\ u \in H \tag{3.56}$$

there corresponds a unique adjoint operator.

Proposition 26. *For any bounded linear operator $A : H \longrightarrow H$ on a Hilbert space there is a unique bounded linear operator $A^* : H \longrightarrow H$ such that*

$$(Au,v)_H = (u,A^*v)_H\ \forall\ u,v \in H\ \text{and}\ \|A\| = \|A^*\|. \tag{3.57}$$

Proof. To see the existence of A^*v we need to work out what $A^*v \in H$ should be for each fixed $v \in H$. So, fix v in the desired identity (3.57), which is to say consider

$$H \ni u \longrightarrow (Au,v) \in \mathbb{C}. \tag{3.58}$$

This is a linear map and it is clearly bounded, since

$$|(Au,v)| \le \|Au\|_H\|v\|_H \le (\|A\|\|v\|_H)\|u\|_H. \tag{3.59}$$

Thus it is a continuous linear functional on H which depends on v. In fact it is just the composite of two continuous linear maps

$$H \stackrel{u\longmapsto Au}{\longrightarrow} H \stackrel{w\longmapsto (w,v)}{\longrightarrow} \mathbb{C}. \tag{3.60}$$

By Riesz' theorem there is a unique element in H, which we can denote A^*v (since it only depends on v) such that

$$(Au,v) = (u,A^*v)\ \forall\ u \in H. \tag{3.61}$$

Now this defines the map $A^* : H \longrightarrow H$ but we need to check that it is linear and continuous. Linearity follows from the uniqueness part of Riesz' theorem. Thus if $v_1,\ v_2 \in H$ and $c_1,\ c_2 \in \mathbb{C}$ then

$$\begin{aligned}(Au, c_1v_1 + c_2v_2) &= \overline{c_1}(Au,v_1) + \overline{c_2}(Au,v_2)\\ &= \overline{c_1}(u,A^*v_1) + \overline{c_2}(u,A^*v_2) = (u, c_1A^*v_2 + c_2A^*v_2)\end{aligned} \tag{3.62}$$

where we have used the definitions of A^*v_1 and A^*v_2 – by uniqueness we must have $A^*(c_1v_1 + c_2v_2) = c_1A^*v_1 + c_2A^*v_2$.

Since we know the optimality of Cauchy's inequality

$$\|v\|_H = \sup_{\|u\|=1} |(u,v)| \tag{3.63}$$

it follows that

$$\|A^*v\| = \sup_{\|u\|=1} |(u,A^*v)| = \sup_{\|u\|=1} |(Au,v)| \le \|A\|\|v\|. \tag{3.64}$$

So in fact

$$\|A^*\| \le \|A\| \tag{3.65}$$

which shows that A^* is bounded.

The defining identity (3.57) also shows that $(A^*)^* = A$ so the reverse equality in (3.65) also holds and so

$$\|A^*\| = \|A\|. \tag{3.66}$$

□

12. Compactness and equi-small tails

A compact subset in a general metric space is one with the property that any sequence in it has a convergent subsequence, with its limit in the set. You will recall, with pleasure no doubt, the equivalence of this condition to the (more general since it makes good sense in an arbitrary topological space) covering condition, that *any* open cover of the set has a finite subcover. So, in a separable Hilbert space the notion of a compact set is already fixed. We want to characterize it, actually in several ways.

A general result in a metric space is that any compact set is both closed and bounded, so this must be true in a Hilbert space. The Heine-Borel theorem gives a converse to this, for $\mathbb{R}^n$ or $\mathbb{C}^n$ (and hence in any finite dimensional normed space) in which any closed and bounded set is compact. Also recall that the convergence of a sequence in $\mathbb{C}^n$ is equivalent to the convergence of the n sequences given by its components and this is what is used to pass first from $\mathbb{R}$ to $\mathbb{C}$ and then to $\mathbb{C}^n$. All of this fails in infinite dimensions and we need some condition in addition to being bounded and closed for a set to be compact.

To see where this might come from, observe that

LEMMA 24. *In any metric space a set, S, consisting of the points of a convergent sequence, $s : \mathbb{N} \longrightarrow M$, together with its limit, s, is compact.*

PROOF. The set here is the image of the sequence, thought of as a map from the integers into the metric space, together with the limit (which might or might not already be in the image of the sequence). Certainly this set is bounded, since the distance from the intial point is bounded. Moreover it is closed. Indeed, the complement $M \setminus S$ is open – if $p \in M \setminus S$ then it is not the limit of the sequence, so for some $\epsilon > 0$, and some N, if $n > N$ then $s(n) \notin B(p, \epsilon)$. Shrinking ϵ further if necessary, we can make sure that all the $s(k)$ for $k \le N$ are not in the ball either – since they are each at a positive distance from p. Thus $B(p, \epsilon) \subset M \setminus S$.

Finally, S is compact since any sequence in S has a convergent subsequence. To see this, observe that a sequence $\{t_j\}$ in S either has a subsequence converging to the limit s of the original sequence or it does not. So we only need consider the latter case, but this means that, for some $\epsilon > 0$, $d(t_j, s) > \epsilon$; but then t_j takes values in a finite set, since $S \setminus B(s, \epsilon)$ is finite – hence some value is repeated infinitely often and there is a convergent subsequence. □

LEMMA 25. *The image of a convergent sequence in a Hilbert space is a set with equi-small tails with respect to any orthonormal sequence, i.e. if e_k is an othonormal sequence and $u_n \to u$ is a convergent sequence then given $\epsilon > 0$ there exists N such that*

$$\sum_{k>N} |(u_n, e_k)|^2 < \epsilon^2 \ \forall \ n. \tag{3.67}$$

PROOF. Bessel's inequality shows that for any $u \in \mathcal{H}$,

$$\sum_k |(u, e_k)|^2 \le \|u\|^2. \tag{3.68}$$

The convergence of this series means that (3.67) can be arranged for any single element u_n or the limit u by choosing N large enough, thus given $\epsilon > 0$ we can choose N' so that

$$\sum_{k>N'} |(u, e_k)|^2 < \epsilon^2/2. \tag{3.69}$$

Consider the closure of the subspace spanned by the e_k with $k > N$. The orthogonal projection onto this space (see Lemma 23) is

$$P_N u = \sum_{k>N} (u, e_k) e_k. \tag{3.70}$$

Then the convergence $u_n \to u$ implies the convergence in norm $\|P_N u_n\| \to \|P_N u\|$, so

$$\|P_N u_n\|^2 = \sum_{k>N} |(u_n, e_k)|^2 < \epsilon^2, \ n > n'. \tag{3.71}$$

So, we have arranged (3.67) for $n > n'$ for some N. This estimate remains valid if N is increased – since the tails get smaller – and we may arrange it for $n \le n'$ by chossing N large enough. Thus indeed (3.67) holds for all n if N is chosen large enough. □

This suggests one useful characterization of compact sets in a separable Hilbert space.

PROPOSITION 27. *A set $K \subset \mathcal{H}$ in a separable Hilbert space is compact if and only if it is bounded, closed and the Fourier-Bessel sequence with respect to any (one) complete orthonormal basis converges* uniformly *on it.*

PROOF. We already know that a compact set in a metric space is closed and bounded. Suppose the equi-smallness of tails condition fails with respect to some orthonormal basis e_k. This means that for some $\epsilon > 0$ and all p there is an element $u_p \in K$, such that

$$\sum_{k>p} |(u_p, e_k)|^2 \ge \epsilon^2. \tag{3.72}$$

Consider the subsequence $\{u_p\}$ generated this way. No subsequence of it can have equi-small tails (recalling that the tail decreases with p). Thus, by Lemma 25, it cannot have a convergent subsequence, so K cannot be compact if the equi-smallness condition fails.

Thus we have proved the equi-smallness of tails condition to be necessary for the compactness of a closed, bounded set. It remains to show that it is sufficient.

So, suppose K is closed, bounded and satisfies the equi-small tails condition with respect to an orthonormal basis e_k and $\{u_n\}$ is a sequence in K. We only need show that $\{u_n\}$ has a Cauchy subsequence, since this will converge ($\mathcal{H}$ being complete) and the limit will be in K (since it is closed). Consider each of the sequences of coefficients (u_n, e_k) in $\mathbb{C}$. Here k is fixed. This sequence is bounded:

$$|(u_n, e_k)| \le \|u_n\| \le C \tag{3.73}$$

by the boundedness of K. So, by the Heine-Borel theorem, there is a subsequence u_{n_l} such that (u_{n_l}, e_k) converges as $l \to \infty$.

We can apply this argument for each $k = 1, 2, \dots$. First extract a subsequence $\{u_{n,1}\}$ of $\{u_n\}$ so that the sequence $(u_{n,1}, e_1)$ converges. Then extract a subsequence $u_{n,2}$ of $u_{n,1}$ so that $(u_{n,2}, e_2)$ *also* converges. Then continue inductively. Now pass to the 'diagonal' subsequence v_n of $\{u_n\}$ which has kth entry the kth term, $u_{k,k}$ in the kth subsequence. It is 'eventually' a subsequence of each of the subsequences previously constructed – meaning it coincides with a subsequence from some point onward (namely the kth term onward for the kth subsquence). Thus, for this subsequence *each* of the (v_n, e_k) converges.

Consider the identity (the orthonormal set e_k is complete by assumption) for the difference

$$
\begin{aligned}
\|v_n - v_{n+l}\|^2 &= \sum_{k \le N} |(v_n - v_{n+l}, e_k)|^2 + \sum_{k>N} |(v_n - v_{n+l}, e_k)|^2 \\
&\le \sum_{k \le N} |(v_n - v_{n+l}, e_k)|^2 + 2\sum_{k>N} |(v_n, e_k)|^2 + 2\sum_{k>N} |(v_{n+l}, e_k)|^2
\end{aligned} \tag{3.74}
$$

where the parallelogram law on $\mathbb{C}$ has been used. To make this sum less than ϵ^2 we may choose N so large that the last two terms are less than $\epsilon^2/2$ and this may be done for all n and l by the equi-smallness of the tails. Now, choose n so large that each of the terms in the first sum is less than $\epsilon^2/2N$, for all $l > 0$ using the Cauchy condition on each of the finite number of sequence (v_n, e_k). Thus, $\{v_n\}$ is a Cauchy subsequence of $\{u_n\}$ and hence as already noted convergent in K. Thus K is indeed compact. □

13. Finite rank operators

Now, we need to starting thinking a little more seriously about operators on a Hilbert space, remember that an operator is just a continuous linear map $T : \mathcal{H} \longrightarrow \mathcal{H}$ and the space of them (a Banach space) is denoted $\mathcal{B}(\mathcal{H})$ (rather than the more cumbersome $\mathcal{B}(\mathcal{H}, \mathcal{H})$ which is needed when the domain and target spaces are different).

Definition 18. An operator $T \in \mathcal{B}(\mathcal{H})$ is of *finite rank* if its range has finite dimension (and that dimension is called the rank of T); the set of finite rank operators will be denoted $\mathcal{R}(\mathcal{H})$.

Why not $\mathcal{F}(\mathcal{H})$? Because we want to use this for the *Fredholm operators*.

Clearly the sum of two operators of finite rank has finite rank, since the range is contained in the sum of the ranges (but is often smaller):

$$
(T_1 + T_2)u \in \operatorname{Ran}(T_1) + \operatorname{Ran}(T_2)\ \forall\ u \in \mathcal{H}. \tag{3.75}
$$

Since the range of a constant multiple of T is contained in the range of T it follows that the finite rank operators form a linear subspace of $\mathcal{B}(\mathcal{H})$.

What does a finite rank operator look like? It really looks like a matrix.

Lemma 26. *If $T : H \longrightarrow H$ has finite rank then there is a finite orthonormal set $\{e_k\}_{k=1}^L$ in H such that*

$$
Tu = \sum_{i,j=1}^{L} c_{ij}(u, e_j)e_i. \tag{3.76}
$$

Proof. By definition, the range of T, $R = T(H)$ is a finite dimensional subspace. So, it has a basis which we can diagonalize in H to get an orthonormal basis, e_i, $i = 1, \ldots, p$. Now, since this is a basis of the range, Tu can be expanded relative to it for any $u \in H$:

$$Tu = \sum_{i=1}^{p} (Tu, e_i)e_i. \tag{3.77}$$

On the other hand, the map $u \longrightarrow (Tu, e_i)$ is a continuous linear functional on H, so $(Tu, e_i) = (u, v_i)$ for some $v_i \in H$; notice in fact that $v_i = T^* e_i$. This means the formula (3.77) becomes

$$Tu = \sum_{i=1}^{p} (u, v_i)e_i. \tag{3.78}$$

Now, the Gram-Schmidt procedure can be applied to orthonormalize the sequence $e_1, \ldots, e_p, v_1 \ldots, v_p$ resulting in $e_1, \ldots, e_L$. This means that each v_i is a linear combination which we can write as

$$v_i = \sum_{j=1}^{L} \overline{c_{ij}} e_j. \tag{3.79}$$

Inserting this into (3.78) gives (3.76) (where the constants for $i > p$ are zero). □

It is clear that

$$B \in \mathcal{B}(\mathcal{H}) \text{ and } T \in \mathcal{R}(\mathcal{H}) \text{ then } BT \in \mathcal{R}(\mathcal{H}). \tag{3.80}$$

Indeed, the range of BT is the range of B restricted to the range of T and this is certainly finite dimensional since it is spanned by the image of a basis of $\operatorname{Ran}(T)$. Similalry $TB \in \mathcal{R}(\mathcal{H})$ since the range of TB is contained in the range of T. Thus we have in fact proved most of

Proposition 28. *The finite rank operators form a $*$-closed two-sided ideal in $\mathcal{B}(\mathcal{H})$, which is to say a linear subspace such that*

$$B_1,\ B_2 \in \mathcal{B}(\mathcal{H}),\ T \in \mathcal{R}(\mathcal{H}) \Longrightarrow B_1 T B_2,\ T^* \in \mathcal{R}(\mathcal{H}). \tag{3.81}$$

Proof. It is only left to show that T^* is of finite rank if T is, but this is an immediate consequence of Lemma 26 since if T is given by (3.76) then

$$T^* u = \sum_{i,j=1}^{N} \overline{c_{ij}} (u, e_i) e_j \tag{3.82}$$

is also of finite rank. □

Lemma 27 (Row rank=Colum rank). *For any finite rank operator on a Hilbert space, the dimension of the range of T is equal to the dimension of the range of T^*.*

Proof. From the formula (3.78) for a finite rank operator, it follows that the v_i, $i = 1, \ldots, p$ must be linearly independent – since the e_i form a basis for the range and a linear relation between the v_i would show the range had dimension less than p. Thus in fact the null space of T is precisely the orthocomplement of the span of the v_i – the space of vectors orthogonal to each v_i. Since

$$
\begin{aligned}
(Tu, w) &= \sum_{i=1}^{p} (u, v_i)(e_i, w) \Longrightarrow \\
(w, Tu) &= \sum_{i=1}^{p} (v_i, u)(w, e_i) \Longrightarrow \\
T^*w &= \sum_{i=1}^{p} (w, e_i)v_i
\end{aligned}
\tag{3.83}
$$

the range of T^* is the span of the v_i, so is also of dimension p. □

14. Compact operators

Definition 19. An element $K \in \mathcal{B}(\mathcal{H})$, the bounded operators on a separable Hilbert space, is said to be *compact* (the old terminology was 'totally bounded' or 'completely continuous') if the image of the unit ball is precompact, i.e. has compact closure – that is if the closure of $K\{u \in \mathcal{H}; \|u\|_{\mathcal{H}} \le 1\}$ is compact in $\mathcal{H}$.

Notice that in a metric space, to say that a set has compact closure is the same as saying it is contained in a compact set.

Proposition 29. *An operator $K \in \mathcal{B}(\mathcal{H})$, bounded on a separable Hilbert space, is compact if and only if it is the limit of a norm-convergent sequence of finite rank operators.*

Proof. So, we need to show that a compact operator is the limit of a convergent sequence of finite rank operators. To do this we use the characterizations of compact subsets of a separable Hilbert space discussed earlier. Namely, if $\{e_i\}$ is an orthonormal basis of $\mathcal{H}$ then a subset $I \subset \mathcal{H}$ is compact if and only if it is closed and bounded and has equi-small tails with respect to $\{e_i\}$, meaning given $\epsilon > 0$ there exits N such that

$$\sum_{i>N} |(v, e_i)|^2 < \epsilon^2 \ \forall\ v \in I. \tag{3.84}$$

Now we shall apply this to the set $K(B(0,1))$ where we assume that K is compact (as an operator, don't be confused by the double usage, in the end it turns out to be constructive) – so this set is *contained* in a compact set. Hence (3.84) applies to it. Namely this means that for any $\epsilon > 0$ there exists n such that

$$\sum_{i>n} |(Ku, e_i)|^2 < \epsilon^2 \ \forall\ u \in \mathcal{H},\ \|u\|_{\mathcal{H}} \le 1. \tag{3.85}$$

For each n consider the first part of these sequences and define

$$K_n u = \sum_{k \le n} (Ku, e_i)e_i. \tag{3.86}$$

This is clearly a linear operator and has finite rank – since its range is contained in the span of the first n elements of $\{e_i\}$. Since this is an orthonormal basis,

$$\|Ku - K_n u\|^2_{\mathcal{H}} = \sum_{i>n} |(Ku, e_i)|^2 \tag{3.87}$$

Thus (3.85) shows that $\|Ku - K_n u\|_{\mathcal{H}} \le \epsilon$. Now, increasing n makes $\|Ku - K_n u\|$ smaller, so given $\epsilon > 0$ there exists n such that for all $N \ge n$,

$$\|K - K_N\|_{\mathcal{B}} = \sup_{\|u\| \le 1} \|Ku - K_n u\|_{\mathcal{H}} \le \epsilon. \tag{3.88}$$

Thus indeed, $K_n \to K$ in norm and we have shown that the compact operators are contained in the norm closure of the finite rank operators.

For the converse we assume that $T_n \to K$ is a norm convergent sequence in $\mathcal{B}(\mathcal{H})$ where each of the T_n is of finite rank – of course we know nothing about the rank except that it is finite. We want to conclude that K is compact, so we need to show that $K(B(0,1))$ is precompact. It is certainly bounded, by the norm of K. By a result above on compactness of sets in a separable Hilbert space we know that it suffices to prove that the closure of the image of the unit ball has uniformly small tails. Let Π_N be the orthogonal projection *off* the first N elements of a complete orthonormal basis $\{e_k\}$ – so

$$u = \sum_{k \le N} (u, e_k) e_k + \Pi_N u. \tag{3.89}$$

Then we know that $\|\Pi_N\| = 1$ (assuming the Hilbert space is infinite dimensional) and $\|\Pi_N u\|$ is the 'tail'. So what we need to show is that given $\epsilon > 0$ there exists n such that

$$\|u\| \le 1 \Longrightarrow \|\Pi_N K u\| < \epsilon. \tag{3.90}$$

Now,

$$\|\Pi_N K u\| \le \|\Pi_N (K - T_n) u\| + \|\Pi_N T_n u\| \tag{3.91}$$

so choosing n large enough that $\|K - T_n\| < \epsilon/2$ and then using the compactness of T_n (which is finite rank) to choose N so large that

$$\|u\| \le 1 \Longrightarrow \|\Pi_N T_n u\| \le \epsilon/2 \tag{3.92}$$

shows that (3.90) holds and hence K is compact. □

Proposition 30. *For any separable Hilbert space, the compact operators form a closed and $*$-closed two-sided ideal in $\mathcal{B}(H)$.*

Proof. In any metric space (applied to $\mathcal{B}(H)$) the closure of a set is closed, so the compact operators are closed being the closure of the finite rank operators. Similarly the fact that it is closed under passage to adjoints follows from the same fact for finite rank operators. The ideal properties also follow from the corresponding properties for the finite rank operators, or we can prove them directly anyway. Namely if B is bounded and T is compact then for some $c > 0$ (namely $1/\|B\|$ unless it is zero) cB maps $B(0,1)$ into itself. Thus $cTB = TcB$ is compact since the image of the unit ball under it is contained in the image of the unit ball under T; hence TB is also compact. Similarly BT is compact since B is continuous and then

$$BT(B(0,1)) \subset B(\overline{T(B(0,1))}) \text{ is compact} \tag{3.93}$$

since it is the image under a continuous map of a compact set. □

15. Weak convergence

It is convenient to formalize the idea that a sequence be bounded and that each of the (u_n, e_k), the sequence of coefficients of some particular Fourier-Bessel series, should converge.

Definition 20. A sequence, $\{u_n\}$, in a Hilbert space, $\mathcal{H}$, is said to *converge weakly* to an element $u \in \mathcal{H}$ if it is bounded in norm and $(u_j, v) \to (u, v)$ converges in $\mathbb{C}$ for each $v \in \mathcal{H}$. This relationship is written

$$u_n \rightharpoonup u. \tag{3.94}$$

In fact as we shall see below, the assumption that $\|u_n\|$ is bounded and that u exists are both unnecessary. That is, a sequence converges weakly if and only if (u_n, v) converges in $\mathbb{C}$ for each $v \in \mathcal{H}$. Conversely, there is no harm in assuming it is bounded and that the 'weak limit' $u \in \mathcal{H}$ exists. Note that the weak limit is unique since if u and u' both have this property then $(u - u', v) = \lim_{n\to\infty}(u_n, v) - \lim_{n\to\infty}(u_n, v) = 0$ for all $v \in \mathcal{H}$ and setting $v = u - u'$ it follows that $u = u'$.

Lemma 28. *A (strongly) convergent sequence is weakly convergent with the same limit.*

Proof. This is the continuity of the inner product. If $u_n \to u$ then

$$|(u_n, v) - (u, v)| \le \|u_n - u\|\|v\| \to 0 \tag{3.95}$$

for each $v \in \mathcal{H}$ shows weak convergence. □

Lemma 29. *For a bounded sequence in a separable Hilbert space, weak convergence is equivalent to component convergence with respect to an orthonormal basis.*

Proof. Let e_k be an orthonormal basis. Then if u_n is weakly convergent it follows immediately that $(u_n, e_k) \to (u, e_k)$ converges for each k. Conversely, suppose this is true for a bounded sequence, just that $(u_n, e_k) \to c_k$ in $\mathbb{C}$ for each k. The norm boundedness and Bessel's inequality show that

$$\sum_{k\le p} |c_k|^2 = \lim_{n\to\infty} \sum_{k\le p} |(u_n, e_k)|^2 \le C^2 \sup_n \|u_n\|^2 \tag{3.96}$$

for all p. Thus in fact $\{c_k\} \in l^2$ and hence

$$u = \sum_k c_k e_k \in \mathcal{H} \tag{3.97}$$

by the completeness of $\mathcal{H}$. Clearly $(u_n, e_k) \to (u, e_k)$ for each k. It remains to show that $(u_n, v) \to (u, v)$ for all $v \in \mathcal{H}$. This is certainly true for any finite linear combination of the e_k and for a general v we can write

$$\begin{gathered}(u_n, v) - (u, v) = (u_n, v_p) - (u, v_p) + (u_n, v - v_p) - (u, v - v_p) \Longrightarrow \\ |(u_n, v) - (u, v)| \le |(u_n, v_p) - (u, v_p)| + 2C\|v - v_p\|\end{gathered} \tag{3.98}$$

where $v_p = \sum\limits_{k\le p} (v, e_k)e_k$ is a finite part of the Fourier-Bessel series for v and C is a bound for $\|u_n\|$. Now the convergence $v_p \to v$ implies that the last term in (3.98) can be made small by choosing p large, independent of n. Then the second last term can be made small by choosing n large since v_p is a finite linear combination of the

e_k. Thus indeed, $(u_n, v) \to (u, v)$ for all $v \in \mathcal{H}$ and it follows that u_n converges weakly to u. □

PROPOSITION 31. *Any bounded sequence $\{u_n\}$ in a separable Hilbert space has a weakly convergent subsequence.*

This can be thought of as an analogue in infinite dimensions of the Heine-Borel theorem if you say 'a bounded closed subset of a separable Hilbert space is *weakly* compact'.

PROOF. Choose an orthonormal basis $\{e_k\}$ and apply the procedure in the proof of Proposition 27 to extract a subsequence of the given bounded sequence such that (u_{n_p}, e_k) converges for each k. Now apply the preceeding Lemma to conclude that this subsequence converges weakly. □

LEMMA 30. *For a weakly convergent sequence $u_n \rightharpoonup u$*

$$\|u\| \le \liminf \|u_n\|. \tag{3.99}$$

PROOF. Choose an orthonormal basis e_k and observe that

$$\sum_{k\le p} |(u, e_k)|^2 = \lim_{n\to\infty} \sum_{k\le p} |(u_n, e_k)|^2. \tag{3.100}$$

The sum on the right is bounded by $\|u_n\|^2$ independently of p so

$$\sum_{k\le p} \|u, e_k\|^2 \le \liminf_n \|u_n\|^2 \tag{3.101}$$

by the definition of $\liminf$. Then let $p \to \infty$ to conclude that

$$\|u\|^2 \le \liminf_n \|u_n\|^2 \tag{3.102}$$

from which (3.99) follows. □

LEMMA 31. *An operator $K \in \mathcal{B}(\mathcal{H})$ is compact if and only if the image Ku_n of any weakly convergent sequence $\{u_n\}$ in $\mathcal{H}$ is strongly, i.e. norm, convergent.*

This is the origin of the old name 'completely continuous' for compact operators, since they turn even weakly convergent into strongly convergent sequences.

PROOF. First suppose that $u_n \rightharpoonup u$ is a weakly convergent sequence in $\mathcal{H}$ and that K is compact. We know that $\|u_n\| < C$ is bounded so the sequence Ku_n is contained in $CK(B(0,1))$ and hence in a compact set (clearly if D is compact then so is cD for any constant c.) Thus, any subsequence of Ku_n has a convergent subseqeunce and the limit is necessarily Ku since $Ku_n \rightharpoonup Ku$ (true for any bounded operator by computing

$$(Ku_n, v) = (u_n, K^*v) \to (u, K^*v) = (Ku, v).) \tag{3.103}$$

But the condition on a sequence in a metric space that every subsequence of it has a subsequence which converges to a fixed limit implies convergence. (If you don't remember this, reconstruct the proof: To say a sequence v_n *does not* converge to v is to say that for some $\epsilon > 0$ there is a subsequence along which $d(v_{n_k}, v) \ge \epsilon$. This is impossible given the subsequence of subsequence condition (converging to the fixed limit v.))

Conversely, suppose that K has this property of turning weakly convergent into strongly convergent sequences. We want to show that $K(B(0,1))$ has compact closure. This just means that any sequence in $K(B(0,1))$ has a (strongly) convergent subsequence – where we do not have to worry about whether the limit is in the set or not. Such a sequence is of the form Ku_n where u_n is a sequence in $B(0,1)$. However we know that the ball is weakly compact, that is we can pass to a subsequence which converges weakly, $u_{n_j} \rightharpoonup u$. Then, by the assumption of the Lemma, $Ku_{n_j} \to Ku$ converges strongly. Thus u_n does indeed have a convergent subsequence and hence $K(B(0,1))$ must have compact closure. $\square$

As noted above, it is not really necessary to assume that a sequence in a Hilbert space is bounded, provided one has the Uniform Boundedness Principle, Theorem 3, at the ready.

Proposition 32. *If $u_n \in H$ is a sequence in a Hilbert space and for all $v \in H$*

$$(u_n, v) \to F(v) \text{ converges in } \mathbb{C} \tag{3.104}$$

then $\|u_n\|_H$ is bounded and there exists $w \in H$ such that $u_n \rightharpoonup w$ (converges weakly).

Proof. Apply the Uniform Boundedness Theorem to the continuous functionals

$$T_n(u) = (u, u_n),\ T_n : H \longrightarrow \mathbb{C} \tag{3.105}$$

where we reverse the order to make them linear rather than anti-linear. Thus, each set $|T_n(u)|$ is bounded in $\mathbb{C}$ since it is convergent. It follows from the Uniform Boundedness Principle that there is a bound

$$\|T_n\| \le C. \tag{3.106}$$

However, this norm as a functional is just $\|T_n\| = \|u_n\|_H$ so the original sequence must be bounded in H. Define $T : H \longrightarrow \mathbb{C}$ as the limit for each u :

$$T(u) = \lim_{n\to\infty} T_n(u) = \lim_{n\to\infty} (u, u_n). \tag{3.107}$$

This exists for each u by hypothesis. It is a linear map and from (3.106) it is bounded, $\|T\| \le C$. Thus by the Riesz Representation theorem, there exists $w \in H$ such that

$$T(u) = (u, w)\ \forall\ u \in H. \tag{3.108}$$

Thus $(u_n, u) \to (w, u)$ for all $u \in H$ so $u_n \rightharpoonup w$ as claimed. $\square$

16. The algebra $\mathcal{B}(H)$

Recall the basic properties of the Banach space, and algebra, of bounded operators $\mathcal{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$. In particular that it is a Banach space with respect to the norm

$$\|A\| = \sup_{\|u\|_{\mathcal{H}}=1} \|Au\|_{\mathcal{H}} \tag{3.109}$$

and that the norm satisfies

$$\|AB\| \le \|A\|\|B\| \tag{3.110}$$

as follows from the fact that

$$\|ABu\| \le \|A\|\|Bu\| \le \|A\|\|B\|\|u\|.$$

Consider the set of invertible elements:

$$\mathrm{GL}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}); \exists\ B \in \mathcal{B}(\mathcal{H}),\ BA = AB = \mathrm{Id}\}. \tag{3.111}$$

Note that this is equivalent to saying A is 1-1 and onto in view of the Open Mapping Theorem, Theorem 4.

This set is open, to see this consider a neighbourhood of the identity.

LEMMA 32. *If $A \in \mathcal{B}(\mathcal{H})$ and $\|A\| < 1$ then*

$$\mathrm{Id} - A \in \mathrm{GL}(\mathcal{H}). \tag{3.112}$$

PROOF. This follows from the convergence of the Neumann series. If $\|A\| < 1$ then $\|A^j\| \le \|A\|^j$, from (3.110), and it follows that

$$B = \sum_{j=0}^{\infty} A^j \tag{3.113}$$

(where $A^0 = \mathrm{Id}$ by definition) is absolutely summable in $\mathcal{B}(\mathcal{H})$ since $\sum\limits_{j=0}^{\infty} \|A^j\|$ converges. Since $\mathcal{B}(H)$ is a Banach space, the sum converges. Moreover by the continuity of the product with respect to the norm

$$AB = A \lim_{n\to\infty} \sum_{j=0}^{n} A^j = \lim_{n\to\infty} \sum_{j=1}^{n+1} A^j = B - \mathrm{Id} \tag{3.114}$$

and similarly $BA = B - \mathrm{Id}$. Thus $(\mathrm{Id} - A)B = B(\mathrm{Id} - A) = \mathrm{Id}$ shows that B is a (and hence *the*) 2-sided inverse of $\mathrm{Id} - A$. □

PROPOSITION 33. *The invertible elements form an open subset* $\mathrm{GL}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$.

PROOF. Suppose $G \in \mathrm{GL}(\mathcal{H})$, meaning it has a two-sided (and unique) inverse $G^{-1} \in \mathcal{B}(\mathcal{H})$:

$$G^{-1}G = GG^{-1} = \mathrm{Id}. \tag{3.115}$$

Then we wish to show that $B(G; \epsilon) \subset \mathrm{GL}(\mathcal{H})$ for some $\epsilon > 0$. In fact we shall see that we can take $\epsilon = \|G^{-1}\|^{-1}$. To show that $G + B$ is invertible set

$$E = -G^{-1}B \Longrightarrow G + B = G(\mathrm{Id} + G^{-1}B) = G(\mathrm{Id} - E) \tag{3.116}$$

From Lemma 32 we know that

$$\|B\| < 1/\|G^{-1}\| \Longrightarrow \|G^{-1}B\| < 1 \Longrightarrow \mathrm{Id} - E \text{ is invertible.} \tag{3.117}$$

Then $(\mathrm{Id} - E)^{-1}G^{-1}$ satisfies

$$(\mathrm{Id} - E)^{-1}G^{-1}(G + B) = (\mathrm{Id} - E)^{-1}(\mathrm{Id} - E) = \mathrm{Id}. \tag{3.118}$$

Moreover $E' = -BG^{-1}$ also satisfies $\|E'\| \le \|B\|\|G^{-1}\| < 1$ and

$$(G + B)G^{-1}(\mathrm{Id} - E')^{-1} = (\mathrm{Id} - E')(\mathrm{Id} - E')^{-1} = \mathrm{Id}. \tag{3.119}$$

Thus $G + B$ has both a 'left' and a 'right' inverse. The associtivity of the operator product (that $A(BC) = (AB)C$) then shows that

$$G^{-1}(\mathrm{Id} - E')^{-1} = (\mathrm{Id} - E)^{-1}G^{-1}(G + B)G^{-1}(\mathrm{Id} - E')^{-1} = (\mathrm{Id} - E)^{-1}G^{-1} \tag{3.120}$$

so the left and right inverses are equal and hence $G + B$ is invertible. □

Thus $\mathrm{GL}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$, the set of invertible elements, is open. It is also a group – since the inverse of G_1G_2 if $G_1,\ G_2 \in \mathrm{GL}(\mathcal{H})$ is $G_2^{-1}G_1^{-1}$.

This group of invertible elements has a smaller subgroup, $\mathrm{U}(\mathcal{H})$, the unitary group, defined by

$$\mathrm{U}(\mathcal{H}) = \{U \in \mathrm{GL}(\mathcal{H}); U^{-1} = U^*\}. \tag{3.121}$$

The unitary group consists of the linear isometric isomorphisms of $\mathcal{H}$ onto itself – thus

$$(Uu, Uv) = (u, v),\ \|Uu\| = \|u\|\ \forall\ u, v \in \mathcal{H},\ U \in \mathrm{U}(\mathcal{H}). \tag{3.122}$$

This is an important object and we will use it a little bit later on.

The groups $\mathrm{GL}(H)$ and $\mathrm{U}(H)$ for a separable Hilbert space may seem very similar to the familiar groups of invertible and unitary $n \times n$ matrices, $\mathrm{GL}(n)$ and $\mathrm{U}(n)$, but this is somewhat deceptive. For one thing they are much bigger. In fact there are other important qualitative differences – you can find some of this in the problems. One important fact that you should know, even though we will not try prove it here, is that both $\mathrm{GL}(H)$ and $\mathrm{U}(\mathcal{H})$ are contractible as a metric spaces – they have no significant topology. This is to be constrasted with the $\mathrm{GL}(n)$ and $\mathrm{U}(n)$ which have a lot of topology, and are not at all simple spaces – especially for large n. One upshot of this is that $\mathrm{U}(\mathcal{H})$ does not look much like the limit of the $\mathrm{U}(n)$ as $n \to \infty$. Another important fact that we will show is that $\mathrm{GL}(H)$ is *not* dense in $\mathcal{B}(H)$, in contrast to the finite dimensional case.

17. Spectrum of an operator

Another direct application of Lemma 32, the convergence of the Neumann series, is that if $A \in \mathcal{B}(H)$ and $\lambda \in \mathbb{C}$ has $|\lambda| > \|A\|$ then $\|\lambda^{-1}A\| < 1$ so $(\mathrm{Id} - \lambda^{-1}A)^{-1}$ exists and satisfies

$$(\lambda \mathrm{Id} - A)\lambda^{-1}(\mathrm{Id} - \lambda^{-1}A)^{-1} = \mathrm{Id} = \lambda^{-1}(\mathrm{Id} - \lambda^{-1}A)^{-1}(\lambda - A). \tag{3.123}$$

Thus, $\lambda - A \in \mathrm{GL}(H)$ has inverse $(\lambda - A)^{-1} = \lambda^{-1}(\mathrm{Id} - \lambda^{-1}A)^{-1}$. The set of λ for which this operator is invertible,

$$\{\lambda \in \mathbb{C}; (\lambda \mathrm{Id} - A) \in \mathrm{GL}(H)\} \subset \mathbb{C} \tag{3.124}$$

is an open, and non-empty, set called the *resolvent set* (usually $(A - \lambda)^{-1}$ is called the resolvent). The complement of the resolvent set is called the spectrum of A

$$\mathrm{Spec}(A) = \{\lambda \in \mathbb{C}; \lambda \mathrm{Id} - A \notin \mathrm{GL}(H)\}. \tag{3.125}$$

As follows from the discussion above it is a compact set – it cannot be empty. You should resist the temptation to think that this is the set of eigenvalues of A, that is not really true.

For a bounded self-adjoint operator we can say more quite a bit more.

Proposition 34. *If $A : H \longrightarrow H$ is a bounded operator on a Hilbert space and $A^* = A$ then $A - \lambda \mathrm{Id}$ is invertible for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and at least one of $A - \|A\| \mathrm{Id}$ and $A + \|A\| \mathrm{Id}$ is not invertible.*

The proof of the last part depends on a different characterization of the norm in the self-adjoint case.

LEMMA 33. *If $A^* = A$ then*

$$\|A\| = \sup_{\|u\|=1} |\langle Au, u\rangle|. \tag{3.126}$$

PROOF. Certainly, $|\langle Au, u\rangle| \le \|A\|\|u\|^2$ so the right side can only be smaller than or equal to the left. Suppose that

$$\sup_{\|u\|=1} |\langle Au, u\rangle| = a.$$

Then for any $u, v \in H$, $|\langle Au, v\rangle| = \langle Ae^{i\theta}u, v\rangle$ for some $\theta \in [0, 2\pi)$, so we can arrange that $\langle Au, v\rangle = |\langle Au', v\rangle|$ is non-negative and $\|u'\| = 1 = \|u\| = \|v\|$. Dropping the primes and computing using the polarization identity (really just the parallelogram law)

$$4\langle Au, v\rangle = \langle A(u+v), u+v\rangle - \langle A(u-v), u-v\rangle + i\langle A(u+iv), u+iv\rangle - i\langle A(u-iv), u-iv\rangle. \tag{3.127}$$

By the reality of the left side we can drop the last two terms and use the bound to see that

$$4\langle Au, v\rangle \le a(\|u+v\|^2 + \|u-v\|^2) = 2a(\|u\|^2 + \|v\|^2) = 4a \tag{3.128}$$

Thus, $\|A\| = \sup_{\|u\|=\|v\|=1} |\langle Au, v\rangle| \le a$ and hence $\|A\| = a$. □

PROOF OF PROPOSITION 34. If $\lambda = s+it$ where $t \ne 0$ then $A-\lambda = (A-s)-it$ and $A-s$ is bounded and selfadjoint, so it is enough to consider the special case that $\lambda = it$. Then for any $u \in H$,

$$\operatorname{Im}\langle (A-it)u, u\rangle = -t\|u\|^2. \tag{3.129}$$

So, certainly $A - it$ is injective, since $(A-it)u = 0$ implies $u = 0$ if $t \ne 0$. The adjoint of $A - it$ is $A + it$ so the adjoint is injective too. It follows that the range of $A - it$ is dense in H. Indeed, if $v \in H$ and $v \perp (A-it)u$ for all $u \in H$, so v is orthogonal to the range, then

$$0 = \operatorname{Im}\langle (A-it)v, v\rangle = -t\|v\|^2. \tag{3.130}$$

By this density of the range, if $w \in H$ there exists a sequence u_n in H with $(A-it)u_n \to w$. But this implies that $\|u_n\|$ is bounded, since $t\|u_n\|^2 = -\operatorname{Im}\langle (A-it)u_n, u_n\rangle$ and hence we can pass to a weakly convergent subsequence, $u_n \rightharpoonup u$. Then $(A-it)u_n \rightharpoonup (A-it)u = w$ so $A - it$ is 1-1 and onto. From the Open Mapping Theorem, $(A - it)$ is invertible.

Finally then we need to show that one of $A \pm \|A\| \operatorname{Id}$ is NOT invertible. This follows from (3.126). Indeed, by the definition of sup there is a sequence $u_n \in H$ with $\|u_n\| = 1$ such that either $\langle Au_n, u_n\rangle \to \|A\|$ or $\langle Au_n, u_n\rangle \to -\|A\|$. We may pass to a weakly convergent subsequence and so assume $u_n \rightharpoonup u$. Assume we are in the first case, so this means $\langle (A - \|A\|)u_n, u_n\rangle \to 0$. Then

$$\begin{aligned} \|(A-\|A\|)u_n\|^2 &= \|Au_n\|^2 - 2\|A\|\rangle Au_n, u_n\rangle + \|A\|^2\|u_n\|^2 \\ &\|Au_n\|^2 - 2\|A\|\rangle (A-\|A\|)u_n, u_n\rangle - \|A\|^2\|u_n\|^2. \end{aligned} \tag{3.131}$$

The second two terms here have limit $-\|A\|^2$ by assumption and the first term is less than or equal to $\|A\|^2$. Since the sequence is positive it follows that $\|(A - \|A\|)^2u_n\| \to 0$. This means that $A - \|A\| \operatorname{Id}$ is not invertible, since if it had a bounded inverse B then $1 = \|u_n\| \le \|B\|\|(A - \|A\|)^2u_n\|$ which is impossible.

The other case is similar (or you can replace A by $-A$) so one of $A \pm \|A\|$ is not invertible. $\square$

18. Spectral theorem for compact self-adjoint operators

One of the important differences between a general bounded self-adjoint operator and a compact self-adjoint operator is that the latter has eigenvalues and eigenvectors – lots of them.

THEOREM 15. *If $A \in \mathcal{K}(\mathcal{H})$ is a self-adjoint, compact operator on a separable Hilbert space, so $A^* = A$, then H has an orthonormal basis consisting of eigenvectors of A, u_j such that*

$$Au_j = \lambda_j u_j,\ \ \lambda_j \in \mathbb{R} \setminus \{0\}, \tag{3.132}$$

consisting of an orthonormal basis for the possibly infinite-dimensional (closed) null space and eigenvectors with non-zero eigenvalues which can be arranged into a sequence such that $|\lambda_j|$ is a non-increasing and $\lambda_j \to 0$ as $j \to \infty$ (in case $\mathrm{Nul}(A)^\perp$ *is finite dimensional, this sequence is finite).*

The operator A maps $\mathrm{Nul}(A)^\perp$ into itself so it may be clearer to first split off the null space and then look at the operator acting on $\mathrm{Nul}(A)^\perp$ which has an orthonormal basis of eigenvectors with non-vanishing eigenvalues.

Before going to the proof, let's notice some useful conclusions. One is that we have 'Fredholm's alternative' in this case.

COROLLARY 4. *If $A \in \mathcal{K}(\mathcal{H})$ is a compact self-adjoint operator on a separable Hilbert space then the equation*

$$u - Au = f \tag{3.133}$$

either has a unique solution for each $f \in \mathcal{H}$ or else there is a non-trivial finite dimensional space of solutions to

$$u - Au = 0 \tag{3.134}$$

and then (3.133) *has a solution if and only if f is orthogonal to all these solutions.*

PROOF. This is just saying that the null space of $\mathrm{Id} - A$ is a complement to the range – which is closed. So, either $\mathrm{Id} - A$ is invertible or if not then the range is precisely the orthocomplement of $\mathrm{Nul}(\mathrm{Id} - A)$. You might say there is not much alternative from this point of view, since it just says the range is *always* the orthocomplement of the null space. $\square$

Let me separate off the heart of the argument from the bookkeeping.

LEMMA 34. *If $A \in \mathcal{K}(\mathcal{H})$ is a self-adjoint compact operator on a separable (possibly finite-dimensional) Hilbert space then*

$$F(u) = (Au, u),\ \ F : \{u \in \mathcal{H}; \|u\| = 1\} \longrightarrow \mathbb{R} \tag{3.135}$$

is a continuous function on the unit sphere which attains its supremum and infimum where

$$\sup_{\|u\|=1} |F(u)| = \|A\|. \tag{3.136}$$

Furthermore, if the maximum or minimum of $F(u)$ is non-zero it is attained at an eivenvector of A with this extremal value as eigenvalue.

PROOF. Since $|F(u)|$ is the function considered in (3.126), (3.136) is a direct consequence of Lemma 33. Moreover, continuity of F follows from continuity of A and of the inner product so

$$|F(u)-F(u')| \le |(Au,u)-(Au,u')|+|(Au,u')-(Au',u')| \le 2\|A\|\|u-u'\| \tag{3.137}$$

since both u and u' have norm one.

If we were in finite dimensions this almost finishes the proof, since the sphere is then compact and a continuous function on a compact set attains its sup and inf. In the general case we need to use the compactness of A. Certainly F is bounded,

$$|F(u)| \le \sup_{\|u\|=1} |(Au,u)| \le \|A\|. \tag{3.138}$$

Thus, there is a sequence u_n^+ such that $F(u_n^+) \to \sup F$ and another u_n^- such that $F(u_n^-) \to \inf F$. The *weak* compactness of the unit sphere means that we can pass to a weakly convergent subsequence in each case, and so assume that $u_n^\pm \rightharpoonup u^\pm$ converges weakly. Then, by the compactness of A, $Au_n^\pm \to Au^\pm$ converges strongly, i.e. in norm. But then we can write

$$\begin{aligned} |F(u_n^\pm) - F(u^\pm)| &\le |(A(u_n^\pm - u^\pm), u_n^\pm)| + |(Au^\pm, u_n^\pm - u^\pm)| \\ &= |(A(u_n^\pm - u^\pm), u_n^\pm)| + |(u^\pm, A(u_n^\pm - u^\pm))| \le 2\|Au_n^\pm - Au^\pm\| \end{aligned} \tag{3.139}$$

to deduce that $F(u^\pm) = \lim F(u_n^\pm)$ are respectively the sup and inf of F. Thus indeed, as in the finite dimensional case, the sup and inf are attained, and hence are the max and min. Note that this is NOT typically true if A is not compact as well as self-adjoint.

Now, suppose that $\Lambda^+ = \sup F > 0$. Then for any $v \in \mathcal{H}$ with $v \perp u^+$ and $\|v\| = 1$, the curve

$$L_v : (-\pi, \pi) \ni \theta \longmapsto \cos\theta u^+ + \sin\theta v \tag{3.140}$$

lies in the unit sphere. Expanding out

$$\begin{aligned} &F(L_v(\theta)) = \\ &\quad (AL_v(\theta), L_v(\theta)) = \cos^2\theta F(u^+) + 2\sin(2\theta)\operatorname{Re}(Au^+, v) + \sin^2(\theta)F(v) \end{aligned} \tag{3.141}$$

we know that this function must take its maximum at $\theta = 0$. The derivative there (it is certainly continuously differentiable on $(-\pi,\pi)$) is $\operatorname{Re}(Au^+, v)$ which must therefore vanish. The same is true for iv in place of v so in fact

$$(Au^+, v) = 0 \ \forall\ v \perp u^+,\ \|v\| = 1. \tag{3.142}$$

Taking the span of these v's it follows that $(Au^+, v) = 0$ for all $v \perp u^+$ so A^+u must be a multiple of u^+ itself. Inserting this into the definition of F it follows that $Au^+ = \Lambda^+ u^+$ is an eigenvector with eigenvalue $\Lambda^+ = \sup F$.

The same argument applies to $\inf F$ if it is negative, for instance by replacing A by $-A$. This completes the proof of the Lemma. □

PROOF OF THEOREM 15. First consider the Hilbert space $\mathcal{H}_0 = \operatorname{Nul}(A)^\perp \subset \mathcal{H}$. Then, as noted above, A maps $\mathcal{H}_0$ into itself, since

$$(Au, v) = (u, Av) = 0\ \forall\ u \in \mathcal{H}_0,\ v \in \operatorname{Nul}(A) \Longrightarrow Au \in \mathcal{H}_0. \tag{3.143}$$

Moreover, A_0, which is A restricted to $\mathcal{H}_0$, is again a compact self-adjoint operator – where the compactness follows from the fact that $A(B(0,1))$ for $B(0,1) \subset \mathcal{H}_0$ is smaller than (actually of course equal to) the whole image of the unit ball.

Thus we can apply the Lemma above to A_0, with quadratic form F_0, and find an eigenvector. Let's agree to take the one associated to $\sup F_0$ unless $\sup F_0 < -\inf F_0$ in which case we take one associated to the inf . Now, what can go wrong here? Nothing except if $F_0 \equiv 0$. However in that case we know from Lemma 33 that $\|A\| = 0$ so $A = 0$.

So, we now know that we can find an eigenvector with non-zero eigenvalue unless $A \equiv 0$ which would implies $\text{Nul}(A) = \mathcal{H}$. Now we proceed by induction. Suppose we have found N mutually orthogonal eigenvectors e_j for A all with norm 1 and eigenvectors λ_j – an orthonormal set of eigenvectors and all in $\mathcal{H}_0$. Then we consider

$$\mathcal{H}_N = \{u \in \mathcal{H}_0 = \text{Nul}(A)^\perp; (u, e_j) = 0,\ j = 1, \dots, N\}. \tag{3.144}$$

From the argument above, A maps $\mathcal{H}_N$ into itself, since

$$(Au, e_j) = (u, Ae_j) = \lambda_j(u, e_j) = 0 \text{ if } u \in \mathcal{H}_N \Longrightarrow Au \in \mathcal{H}_N. \tag{3.145}$$

Moreover this restricted operator is self-adjoint and compact on $\mathcal{H}_N$ as before so we can again find an eigenvector, with eigenvalue either the max of min of the new F for $\mathcal{H}_N$. This process will not stop uness $F \equiv 0$ at some stage, but then $A \equiv 0$ on $\mathcal{H}_N$ and since $\mathcal{H}_N \perp \text{Nul}(A)$ which implies $\mathcal{H}_N = \{0\}$ so $\mathcal{H}_0$ must have been finite dimensional.

Thus, either $\mathcal{H}_0$ is finite dimensional or we can grind out an infinite orthonormal sequence e_i of eigenvectors of A in $\mathcal{H}_0$ with the corresponding sequence of eigenvalues such that $|\lambda_i|$ is non-increasing – since the successive F_N's are restrictions of the previous ones the max and min are getting closer to (or at least no further from) 0.

So we need to rule out the possibility that there is an infinite orthonormal sequence of eigenfunctions e_j with corresponding eigenvalues λ_j where $\inf_j |\lambda_j| = a > 0$. Such a sequence cannot exist since $e_j \rightharpoonup 0$ so by the compactness of A, $Ae_j \to 0$ (in norm) but $|Ae_j| \ge a$ which is a contradiction. Thus if $\text{null}(A)^\perp$ is not finite dimensional then the sequence of eigenvalues constructed above must converge to 0.

Finally then, we need to check that this orthonormal sequence of eigenvectors constitutes an orthonormal basis of $\mathcal{H}_0$. If not, then we can form the closure of the span of the e_i we have constructed, $\mathcal{H}'$, and its orthocomplement in $\mathcal{H}_0$ – which would have to be non-trivial. However, as before F restricts to this space to be F' for the restriction of A' to it, which is again a compact self-adjoint operator. So, if F' is not identically zero we can again construct an eigenfunction, with non-zero eigenvalue, which contracdicts the fact the we are always choosing a largest eigenvalue, in absolute value at least. Thus in fact $F' \equiv 0$ so $A' \equiv 0$ and the eigenvectors form and orthonormal basis of $\text{Nul}(A)^\perp$. This completes the proof of the theorem. □

19. Functional Calculus

So the non-zero eigenvalues of a compact self-adjoint operator form the image of a sequence in $[-\|A\|, \|A\|]$ either converging to zero or finite. If $f \in \mathcal{C}^0([-\|A\|, \|A\|)$ then one can define an operator

$$f(A) \in \mathcal{B}(H),\ f(A)u = \sum_i f(\lambda_u)(u, e_i)e_i \tag{3.146}$$

where $\{e_i\}$ is a complete orthonormal basis of eigenfunctions. Provided $f(0) = 0$ this is compact and if f is real it is self-adjoint. This formula actually defines a linear map

$$\mathcal{C}^0([-\|A\|, \|A\|]) \longrightarrow \mathcal{B}(H) \text{ with } f(A)g(A) = (fg)(A). \tag{3.147}$$

Such a map exists for any bounded self-adjoint operator. Even though it may not have eigenfunctions – or not a complete orthonormal basis of them anyway, it is still possible to define $f(A)$ for a continous function defined on $[-\|A\|, \|A\|]$ (in fact it only has to be defined on $\operatorname{Spec}(A) \subset [-\|A\|, \|A\|]$ which might be quite a lot smaller). This is an effective replacement for the spectral theorem in the compact case.

How does one define $f(A)$? Well, it is easy enough in case f is a polynomial, since then we can factorize it and set

$$f(z) = c(z - z_1)(z - z_2) \dots (z - z_N) \Longrightarrow f(A) = c(A - z_1)(A - z_2) \dots (A - z_N). \tag{3.148}$$

Notice that the result does not depend on the order of the factors or anything like that. To pass to the case of a general continuous function on $[-\|A\|, \|A\|]$ one can use the norm estimate in the polynomial case, that

$$\|f(A)\| \le \sup_{z \in [-\|A\|, \|A\|} |f(z)|. \tag{3.149}$$

This allows one to pass f in the uniform closure of the polynomials, which by the Stone-Weierstrass theorem is the whole of $\mathcal{C}^0([-\|A\|, \|A\|])$. The proof of (3.149) is outlined in Problem 5.33 below.

20. Compact perturbations of the identity

I have generally not had a chance to discuss most of the material in this section, or the next, in the lectures.

Compact operators are, as we know, 'small' in the sense that the are norm limits of finite rank operators. If you accept this, then you will want to say that an operator such as

$$\operatorname{Id} - K,\ K \in \mathcal{K}(\mathcal{H}) \tag{3.150}$$

is 'big'. We are quite interested in this operator because of spectral theory. To say that $\lambda \in \mathbb{C}$ is an eigenvalue of K is to say that there is a non-trivial solution of

$$Ku - \lambda u = 0 \tag{3.151}$$

where non-trivial means other than than the solution $u = 0$ which always exists. If λ is an eigenvalue of K then certainly $\lambda \in \operatorname{Spec}(K)$, since $\lambda - K$ cannot be invertible. For general operators the converse is not correct, but for compact operators it is.

LEMMA 35. *If $K \in \mathcal{B}(H)$ is a compact operator then $\lambda \in \mathbb{C}\backslash\{0\}$ is an eigenvalue of K if and only if $\lambda \in \operatorname{Spec}(K)$.*

PROOF. Since we can divide by λ we may replace K by $\lambda^{-1}K$ and consider the special case $\lambda = 1$. Now, if K is actually finite rank the result is straightforward. By Lemma 26 we can choose a basis so that (3.76) holds. Let the span of the e_i be W – since it is finite dimensional it is closed. Then $\operatorname{Id} - K$ acts rather simply – decomposing $H = W \oplus W^\perp$, $u = w + w'$

$$(\operatorname{Id} - K)(w + w') = w + (\operatorname{Id}_W - K')w',\ K' : W \longrightarrow W \tag{3.152}$$

being a matrix with respect to the basis. Now, 1 is an eigenvalue of K if and only if 1 is an eigenvalue of K' as an operator on the finite-dimensional space W. Now, a matrix, such as $\mathrm{Id}_W - K'$, is invertible if and only if it is injective, or equivalently surjective. So, the same is true for $\mathrm{Id} - K$.

In the general case we use the approximability of K by finite rank operators. Thus, we can choose a finite rank operator F such that $\|K - F\| < 1/2$. Thus, $(\mathrm{Id} - K + F)^{-1} = \mathrm{Id} - B$ is invertible. Then we can write

$$\mathrm{Id} - K = \mathrm{Id} - (K - F) - F = (\mathrm{Id} - (K - F))(\mathrm{Id} - L),\ L = (\mathrm{Id} - B)F. \tag{3.153}$$

Thus, $\mathrm{Id} - K$ is invertible if and only if $\mathrm{Id} - L$ is invertible. Thus, if $\mathrm{Id} - K$ is *not* invertible then $\mathrm{Id} - L$ is not invertible and hence has null space and from (3.153) it follows that $\mathrm{Id} - K$ has non-trivial null space, i.e. K has 1 as an eigenvalue. □

A little more generally:-

PROPOSITION 35. *If $K \in \mathcal{K}(\mathcal{H})$ is a compact operator on a separable Hilbert space then*

$$\begin{gathered} \mathrm{null}(\mathrm{Id} - K) = \{u \in \mathcal{H}; (\mathrm{Id}_K)u = 0\} \text{ is finite dimensional} \\ \mathrm{Ran}(\mathrm{Id} - K) = \{v \in \mathcal{H}; \exists u \in \mathcal{H},\ v = (\mathrm{Id} - K)u\} \text{ is closed and} \\ \mathrm{Ran}(\mathrm{Id} - K)^{\perp} = \{w \in \mathcal{H}; (w, Ku) = 0\ \forall\ u \in \mathcal{H}\} \text{ is finite dimensional} \end{gathered} \tag{3.154}$$

and moreover

$$\dim\left(\mathrm{null}(\mathrm{Id} - K)\right) = \dim\left(\mathrm{Ran}(\mathrm{Id} - K)^{\perp}\right). \tag{3.155}$$

PROOF OF PROPOSITION 35. First let's check this in the case of a finite rank operator $K = T$. Then

$$\mathrm{Nul}(\mathrm{Id} - T) = \{u \in \mathcal{H}; u = Tu\} \subset \mathrm{Ran}(T). \tag{3.156}$$

A subspace of a finite dimensional space is certainly finite dimensional, so this proves the first condition in the finite rank case.

Similarly, still assuming that T is finite rank consider the range

$$\mathrm{Ran}(\mathrm{Id} - T) = \{v \in \mathcal{H}; v = (\mathrm{Id} - T)u \ \text{ for some } u \in \mathcal{H}\}. \tag{3.157}$$

Consider the subspace $\{u \in \mathcal{H}; Tu = 0\}$. We know that this this is closed, since T is certainly continuous. On the other hand from (3.157),

$$\mathrm{Ran}(\mathrm{Id} - T) \supset \mathrm{Nul}(T). \tag{3.158}$$

Remember that a finite rank operator can be written out as a finite sum

$$Tu = \sum_{i=1}^{N} (u, e_i) f_i \tag{3.159}$$

where we can take the f_i to be a basis of the range of T. We also know in this case that the e_i must be linearly independent – if they weren't then we could write one of them, say the last since we can renumber, out as a sum, $e_N = \sum\limits_{j<N} c_i e_j$, of multiples of the others and then find

$$Tu = \sum_{i=1}^{N-1} (u, e_i)(f_i + \overline{c_j} f_N) \tag{3.160}$$

showing that the range of T has dimension at most $N - 1$, contradicting the fact that the f_i span it.

So, going back to (3.159) we know that $\operatorname{Nul}(T)$ has finite *codimension* – every element of $\mathcal{H}$ is of the form

$$u = u' + \sum_{i=1}^{N} d_i e_i,\ u' \in \operatorname{Nul}(T). \tag{3.161}$$

So, going back to (3.158), if $\operatorname{Ran}(\operatorname{Id} - T) \neq \operatorname{Nul}(T)$, and it need not be equal, we can choose – using the fact that $\operatorname{Nul}(T)$ *is* closed – an element $g \in \operatorname{Ran}(\operatorname{Id} - T) \setminus \operatorname{Nul}(T)$ which is orthogonal to $\operatorname{Nul}(T)$. To do this, start with any a vector g' in $\operatorname{Ran}(\operatorname{Id} - T)$ which is not in $\operatorname{Nul}(T)$. It can be split as $g' = u'' + g$ where $g \perp \operatorname{Nul}(T)$ (being a closed subspace) and $u'' \in \operatorname{Nul}(T)$, then $g \neq 0$ is in $\operatorname{Ran}(\operatorname{Id} - T)$ and orthongonal to $\operatorname{Nul}(T)$. Now, the new space $\operatorname{Nul}(T) \oplus \mathbb{C}g$ is again closed and contained in $\operatorname{Ran}(\operatorname{Id} - T)$. But we can continue this process replacing $\operatorname{Nul}(T)$ by this larger closed subspace. After a a finite number of steps we conclude that $\operatorname{Ran}(\operatorname{Id} - T)$ itself is closed.

What we have just proved is:

Lemma 36. *If $V \subset \mathcal{H}$ is a subspace of a Hilbert space which contains a closed subspace of finite codimension in $\mathcal{H}$ – meaning $V \supset W$ where W is closed and there are finitely many elements $e_i \in \mathcal{H}$, $i = 1, \dots, N$ such that every element $u \in \mathcal{H}$ is of the form*

$$u = u' + \sum_{i=1}^{N} c_i e_i,\ c_i \in \mathbb{C}, \tag{3.162}$$

then V itself is closed.

So, this takes care of the case that $K = T$ has finite rank! What about the general case where K is compact? Here we just use a consequence of the approximation of compact operators by finite rank operators proved last time. Namely, if K is compact then there exists $B \in \mathcal{B}(\mathcal{H})$ and T of finite rank such that

$$K = B + T,\ \|B\| < \frac{1}{2}. \tag{3.163}$$

Now, consider the null space of $\operatorname{Id} - K$ and use (3.163) to write

$$\operatorname{Id} - K = (\operatorname{Id} - B) - T = (\operatorname{Id} - B)(\operatorname{Id} - T'),\ T' = (\operatorname{Id} - B)^{-1}T. \tag{3.164}$$

Here we have used the convergence of the Neumann series, so $(\operatorname{Id} - B)^{-1}$ does exist. Now, T' is of finite rank, by the ideal property, so

$$\operatorname{Nul}(\operatorname{Id} - K) = \operatorname{Nul}(\operatorname{Id} - T') \text{ is finite dimensional.} \tag{3.165}$$

Here of course we use the fact that $(\operatorname{Id} - K)u = 0$ is equivalent to $(\operatorname{Id} - T')u = 0$ since $\operatorname{Id} - B$ is invertible. So, this is the first condition in (3.154).

Similarly, to examine the second we do the same thing but the other way around and write

$$\operatorname{Id} - K = (\operatorname{Id} - B) - T = (\operatorname{Id} - T'')(\operatorname{Id} - B),\ T'' = T(\operatorname{Id} - B)^{-1}. \tag{3.166}$$

Now, T'' is again of finite rank and

$$\operatorname{Ran}(\operatorname{Id} - K) = \operatorname{Ran}(\operatorname{Id} - T'') \text{ is closed} \tag{3.167}$$

again using the fact that $\operatorname{Id} - B$ is invertible – so every element of the form $(\operatorname{Id} - K)u$ is of the form $(\operatorname{Id} - T'')u'$ where $u' = (\operatorname{Id} - B)u$ and conversely.

So, now we have proved all of (3.154) – the third part following from the first as discussed before.

What about (3.155)? This time let's first check that it is enough to consider the finite rank case. For a compact operator we have written

$$(\mathrm{Id} - K) = G(\mathrm{Id} - T) \tag{3.168}$$

where $G = \mathrm{Id} - B$ with $\|B\| < \frac{1}{2}$ is invertible and T is of finite rank. So what we want to see is that

$$\dim \mathrm{Nul}(\mathrm{Id} - K) = \dim \mathrm{Nul}(\mathrm{Id} - T) = \dim \mathrm{Nul}(\mathrm{Id} - K^*). \tag{3.169}$$

However, $\mathrm{Id} - K^* = (\mathrm{Id} - T^*)G^*$ and G^* is also invertible, so

$$\dim \mathrm{Nul}(\mathrm{Id} - K^*) = \dim \mathrm{Nul}(\mathrm{Id} - T^*) \tag{3.170}$$

and hence it is enough to check that $\dim \mathrm{Nul}(\mathrm{Id} - T) = \dim \mathrm{Nul}(\mathrm{Id} - T^*)$ – which is to say the same thing for finite rank operators.

Now, for a finite rank operator, written out as (3.159), we can look at the vector space W spanned by all the f_i's and all the e_i's together – note that there is nothing to stop there being dependence relations among the combination although separately they are independent. Now, $T : W \longrightarrow W$ as is immediately clear and

$$T^* v = \sum_{i=1}^{N} (v, f_i) e_i \tag{3.171}$$

so $T : W \longrightarrow W$ too. In fact $Tw' = 0$ and $T^* w' = 0$ if $w' \in W^\perp$ since then $(w', e_i) = 0$ and $(w', f_i) = 0$ for all i. It follows that if we write $R : W \longleftrightarrow W$ for the linear map on this finite dimensional space which is equal to $\mathrm{Id} - T$ acting on it, then R^* is given by $\mathrm{Id} - T^*$ acting on W and we use the Hilbert space structure on W induced as a subspace of $\mathcal{H}$. So, what we have just shown is that

$$(\mathrm{Id} - T)u = 0 \Longleftrightarrow u \in W \text{ and } Ru = 0,\ (\mathrm{Id} - T^*)u = 0 \Longleftrightarrow u \in W \text{ and } R^* u = 0. \tag{3.172}$$

Thus we really are reduced to the finite-dimensional theorem

$$\dim \mathrm{Nul}(R) = \dim \mathrm{Nul}(R^*) \text{ on } W. \tag{3.173}$$

You no doubt know this result. It follows by observing that in this case, everything now on W, $\mathrm{Ran}(W) = \mathrm{Nul}(R^*)^\perp$ and finite dimensions

$$\dim \mathrm{Nul}(R) + \dim \mathrm{Ran}(R) = \dim W = \dim \mathrm{Ran}(W) + \dim \mathrm{Nul}(R^*). \tag{3.174}$$

$\square$

21. Fredholm operators

Definition 21. A bounded operator $F \in \mathcal{B}(\mathcal{H})$ on a Hilbert space is said to be *Fredholm*, written $F \in \mathcal{F}(H)$, if it has the three properties in (3.154) – its null space is finite dimensional, its range is closed and the orthocomplement of its range is finite dimensional.

For general Fredholm operators the row-rank=colum-rank result (3.155) does not hold. Indeed the difference of these two integers, called the index of the operator,

$$\mathrm{ind}(F) = \dim\left(\mathrm{null}(\mathrm{Id} - K)\right) - \dim\left(\mathrm{Ran}(\mathrm{Id} - K)^\perp\right) \tag{3.175}$$

is a very important number with lots of interesting properties and uses.

Notice that the last two conditions in (3.154) are really independent since the orthocomplement of a subspace is the same as the orthocomplement of its closure. There is for instance a bounded operator on a separable Hilbert space with trivial null space and dense range which is not closed. How could this be? Think for instance of the operator on $L^2(0,1)$ which is multiplication by the function x. This is assuredly bounded and an element of the null space would have to satisfy $xu(x) = 0$ almost everywhere, and hence vanish almost everywhere. Moreover the density of the L^2 functions vanishing in $x < \epsilon$ for some (non-fixed) $\epsilon > 0$ shows that the range is dense. However it is clearly not invertible.

Before proving this result let's check that, in the case of operators of the form $\mathrm{Id} - K$, with K compact the third conclusion in (3.154) really follows from the first. This is a general fact which I mentioned, at least, earlier but let me pause to prove it.

PROPOSITION 36. *If $B \in \mathcal{B}(\mathcal{H})$ is a bounded operator on a Hilbert space and B^* is its adjoint then*

$$\mathrm{Ran}(B)^\perp = (\overline{\mathrm{Ran}}(B))^\perp = \{v \in \mathcal{H}; (v,w) = 0 \ \forall \ w \in \mathrm{Ran}(B)\} = \mathrm{Nul}(B^*). \tag{3.176}$$

PROOF. The definition of the orthocomplement of $\mathrm{Ran}(B)$ shows immediately that

$$\begin{aligned} v \in (\mathrm{Ran}(B))^\perp \Longleftrightarrow (v,w) = 0 \ \forall \ w \in \mathrm{Ran}(B) \longleftrightarrow (v, Bu) = 0 \ \forall \ u \in \mathcal{H} \\ \Longleftrightarrow (B^*v, u) = 0 \ \forall \ u \in \mathcal{H} \Longleftrightarrow B^*v = 0 \Longleftrightarrow v \in \mathrm{Nul}(B^*). \end{aligned} \tag{3.177}$$

On the other hand we have already observed that $V^\perp = (\overline{V})^\perp$ for any subspace – since the right side is certainly contained in the left and $(u,v) = 0$ for all $v \in V$ implies that $(u,w) = 0$ for all $w \in \overline{V}$ by using the continuity of the inner product to pass to the limit of a sequence $v_n \to w$. □

Thus as a corrollary we see that if $\mathrm{Nul}(\mathrm{Id} - K)$ is always finite dimensional for K compact (i. e. we check it for all compact operators) then $\mathrm{Nul}(\mathrm{Id} - K^*)$ is finite dimensional and hence so is $\mathrm{Ran}(\mathrm{Id} - K)^\perp$.

There is a more 'analytic' way of characterizing Fredholm operators, rather than Definition 21.

LEMMA 37. *An operator $F \in \mathcal{B}(H)$ is Fredholm, $F \in \mathcal{F}(H)$, if and only if it has a generalized inverse P satisfying*

$$\begin{aligned} PF &= \mathrm{Id} - \Pi_{(F)} \\ FP &= \mathrm{Id} - \Pi_{(F)^\perp} \end{aligned} \tag{3.178}$$

with the two projections of finite rank.

PROOF. If (3.178) holds then F must be Fredholm, since its null space is finite dimensional, from the second identity the range of F must contain the range of $\mathrm{Id} - Pi_{(F)^\perp}$ and hence it must be closed and of finite codimension (and in fact be equal to this closed subspace.

Conversely, suppose that $F \in \mathcal{F}(H)$. We can divide H into two pieces in two ways as $H = (F) \oplus (F)^\perp$ and $H = \mathrm{Ran}(F)^\perp \oplus \mathrm{Ran}(F)$ where in each case the first summand is finite-dimensional. Then F defines four maps, from each of the two first summands to each of the two second ones but all but one of these is zero and so F corresponds to a bounded linear map $\tilde{F} : (F)^\perp \longrightarrow \mathrm{Ran}(F)$. These are two Hilbert spaces with bounded linear bijection between them, so the inverse map, $\tilde{P} : \mathrm{Ran}(F) \longrightarrow (F)^\perp$ is bounded by the Open Mapping Theorem and we can define

$$P = \tilde{P} \circ \Pi(F)^\perp v). \tag{3.179}$$

Then (3.178) follows directly. □

What we want to show is that the Fredholm operators form an open set in $\mathcal{B}(H)$ and that the index is locally constant. To do this we show that a weaker version of (3.178) also implies that F is Fredholm.

Lemma 38. *An operator $F \in \mathcal{F}(H)$ is Fredholm if and only if it has a* parametrix $Q \in \mathcal{B}(H)$ *in the sense that*

$$\begin{aligned} QF &= \mathrm{Id} - E_R \\ FQ &= \mathrm{Id} - E_L \end{aligned} \tag{3.180}$$

with E_R and E_L of finite rank. Moreover any two such parametrices differ by a finite rank operator.

Proof. If F is Fredholm then $Q = P$ certainly is a parameterix in this sense. Conversely suppose that Q as in (3.180) exists. Then $(\mathrm{Id} - E_R)$ is finite dimensional – from (3.154) for instance. However, from the first identity $(F) \subset (QF) = (\mathrm{Id} - E_R)$ so (F) is finite dimensional too. Similarly, the second identity shows that $\mathrm{Ran}(F) \supset \mathrm{Ran}(FQ) = \mathrm{Ran}(\mathrm{Id} - E_L)$ and the last space is closed and of finite codimension, hence so is the first.

Now if Q and Q' both satisfy (3.180) with finite ranke error terms E'_R and E'_L for Q' then

$$(Q' - Q)F = E_R - E'_R \tag{3.181}$$

is of finite rank. Applying the generalized inverse, P of F on the right shows that the difference

$$(Q' - Q) = (E_R - E'_R)P + (Q' - Q)\Pi_{(F)} \tag{3.182}$$

is indeed of finite rank. □

Now recall (in 2014 from Problems7) that finite-rank operators are of trace class, that the trace is well-defined and that the trace of a commutator where one factor is bounded and the other trace class vanishes. Using this we show

Lemma 39. *If Q and F satisfy* (3.180) *then*

$$\mathrm{ind}(F) = \mathrm{Tr}(E_L) - \mathrm{Tr}(E_R). \tag{3.183}$$

Proof. We certainly know that (3.183) holds in the special case that $Q = P$ is the generalized inverse of F, since then $E_L = \Pi_{(F)}$ and $E_R = \Pi_{\mathrm{Ran}(F)^\perp}$ and the traces are the dimensions of these spaces.

Now, if Q is a parameterix as in (3.180) consider the straight line of operators $Q_t = (1-t)P + tQ$. Using the two sets of identities for the generalized inverse and paramaterix

$$\begin{aligned} Q_tF &= (1-t)PF + tQF = \mathrm{Id} - (1-t)\Pi_{(F)} - tE_L, \\ FQ_t &= (1-t)FP + tFQ = \mathrm{Id} - (1-t)\Pi_{\mathrm{Ran}(F)^\perp} - tE_R. \end{aligned} \tag{3.184}$$

Thus Q_t is a curve of parameterices and what we need to show is that

$$J(t) = \mathrm{Tr}((1-t)\Pi_{(F)} + tE_L) - \mathrm{Tr}((1-t)\Pi_{\mathrm{Ran}(F)^\perp} + tE_R) \tag{3.185}$$

is constant. This is a linear function of t as is Q_t. We can differentiate (3.184) with respect to t and see that

$$\frac{d}{dt}((1-t)\Pi_{(F)}+tE_L)-\frac{d}{dt}((1-t)\Pi_{\mathrm{Ran}(F)^\perp}+tE_R)=[Q-P,F] \\ \Longrightarrow J'(t)=0 \tag{3.186}$$

since it is the trace of the commutator of a bounded and a finite rank operator (using the last part of Lemma 38. □

Proposition 37. *The Fredholm operators form an open set in $\mathcal{B}(H)$ on which the index is locally constant.*

Proof. We need to show that if F is Fredholm then there exists $\epsilon>0$ such that $F+B$ is Fredholm if $\|B\|<\epsilon$. Set $B'=\Pi_{\mathrm{Ran}(F)}B\Pi_{(F)^\perp}$ then $\|B'\|\le\|B\|$ and $B-B'$ is finite rank. If $\tilde F$ is the operator constructed in the proof of Lemma 37 then $\tilde F+B'$ is invertible as an operator from $(F)^\perp$ to $\mathrm{Ran}(F)$ if $\epsilon>0$ is small. The inverse, P'_B, extended as 0 to (F) as P is defined in that proof, satisfies

$$\begin{gathered}P'_B(F+B)=\mathrm{Id}-\Pi_{(F)}+P'_B(B-B'),\\ (F+B)P'_B=\mathrm{Id}-\Pi)\mathrm{Ran}(F)^\perp+(B-B])P'_B\end{gathered}\tag{3.187}$$

and so is a parametrix for $F+B$. Thus the set of Fredholm operators is open.

The index of $F+B$ is given by the difference of the trace of the finite rank error terms in the second and first lines here. It depends continuously on B in $\|B\|<\epsilon$ so, being integer valued, is constant. □

This shows in particular that there is an open subset of $\mathcal{B}(H)$ which contains no invertible operators, in strong contrast to the finite dimensional case. Still even the Fredholm operators do no form a dense subset of $\mathcal{B}(H)$. One such open subset consists of the *sem-Fredholm* operators, those with closed range and with *either* null space of complement of range finite-dimensional.

22. Kuiper's theorem

For finite dimensional spaces, such as $\mathbb{C}^N$, the group of invertible operators, denoted typically $\mathrm{GL}(N)$, is a particularly important example of a Lie group. One reason it is important is that it carries a good deal of 'topological' structure. In particular – I'm assuming you have done a little topology – its fundamental group is not trivial, in fact it is isomorphic to $\mathbb{Z}$. This corresponds to the fact that a continuous closed curve $c:\mathbb{S}\longrightarrow\mathrm{GL}(N)$ is *contractible* if and only if its winding number is zero – the effective number of times that the determinant goes around the origin in $\mathbb{C}$. There is a lot more topology than this and it is actually quite complicated.

Perhaps surprisingly, the corresponding group of the bounded operators on a separable (complex) infinite-dimensional Hilbert space which have bounded inverses (or equivalently those which are bijections in view of the open mapping theorem) is contractible. This is Kuiper's theorem, and means that this group, $\mathrm{GL}(H)$, has no 'topology' at all, no holes in any dimension and for topological purposes it is like a big open ball. The proof is not really hard, but it is not exactly obvious either. It depends on an earlier idea, 'Eilenberg's swindle', which shows how the infinite-dimensionality is exploited. As you can guess, this is sort of amusing (if you have the right attitude . . .).

Let's denote by $\mathrm{GL}(H)$ this group, as remarked above in view of the open mapping theorem we know that

$$\mathrm{GL}(H) = \{A \in \mathcal{B}(H); A \text{ is injective and surjective.}\}. \tag{3.188}$$

Contractibility is the topological notion of 'topologically trivial'. It means precisely that there is a continuous map

$$\begin{gathered} \gamma : [0,1] \times \mathrm{GL}(H) \longrightarrow \mathrm{GL}(H) \text{ s.t.} \\ \gamma(0, A) = A,\ \gamma(1, A) = \mathrm{Id},\ \forall\ A \in \mathrm{GL}(H). \end{gathered} \tag{3.189}$$

Continuity here means for the metric space $[0,1] \times \mathrm{GL}(H)$ where the metric comes from the norms on $\mathbb{R}$ and $\mathcal{B}(H)$.

As a warm-up exercise, let us show that the group $\mathrm{GL}(H)$ is contractible to the unitary subgroup

$$\mathrm{U}(H) = \{U \in \mathrm{GL}(H); U^{-1} = U^*\}. \tag{3.190}$$

These are the isometric isomorphisms.

Proposition 38. *There is a continuous map*

$$\Gamma : [0,1] \times \mathrm{GL}(H) \longrightarrow \mathrm{GL}(H) \textit{ s.t. } \Gamma(0, A) = A,\ \Gamma(1, A) \in \mathrm{U}(H)\ \forall\ A \in \mathrm{GL}(H). \tag{3.191}$$

Proof. This is a consequence of the functional calculus, giving the 'polar decomposition' of invertible (and more generally bounded) operators. Namely, if $A \,\mathrm{GL}(H)$ then $AA^* \in \mathrm{GL}(H)$ is self-adjoint. Its spectrum is then contained in an interval $[a, b]$, where $0 < a \le b = \|A\|^2$. It follows from what we showed earlier that $R = (AA^*)^{\frac{1}{2}}$ is a well-defined bounded self-adjoint operator and $R^2 = AA^*$. Moreover, R is invertible and the operator $U_A = R^{-1}A \in \mathrm{U}(H)$. Certainly it is bounded and $U_A^* = A^*R^{-1}$ so $U_A^*U_A = A^*R^{-2}A = \mathrm{Id}$ since $R^{-2} = (AA^*)^{-1} = (A^*)^{-1}A^{-1}$. Thus U_A^* is a right inverse of U_A, and (since U_A is a bijection) is the unique inverse so $U_A \in \mathrm{U}(H)$. So we have shown $A = RU_A$ (this is the polar decomposition) and then

$$\Gamma(s, A) = (s \,\mathrm{Id} + (1 - s)R)U_A,\ \ s \in [0, 1] \tag{3.192}$$

satisfies (3.191). $\square$

Initially we will consider only the notion of 'weak contractibility'. This has nothing to do with weak convergence, rather just means that we only look for an homotopy over compact sets. So, for any compact subset $X \subset \mathrm{GL}(H)$ we seek a continuous map

$$\begin{gathered} \gamma : [0,1] \times X \longrightarrow \mathrm{GL}(H) \text{ s.t.} \\ \gamma(0, A) = A,\ \gamma(1, A) = \mathrm{Id},\ \forall\ A \in X, \end{gathered} \tag{3.193}$$

note that this is not contractibility *of* X, but *of* X in $\mathrm{GL}(H)$.

In fact, to carry out the construction without having to worry about too many things at one, just consider (path) connectedness of $\mathrm{GL}(H)$ meaning that there is a continuous map as in (3.193) where $X = \{A\}$ just consists of one point – so the map is just $\gamma : [0, 1] \longrightarrow \mathrm{GL}(H)$ such that $\gamma(0) = A$, $\gamma(1) = \mathrm{Id}$.

The construction of γ is in three stages

(1) Creating a gap
(2) Rotating to a trivial factor
(3) Eilenberg's swindle.

This approach follows ideas of B. Mityagin, [**2**].

LEMMA 40 (Creating a gap). *If $A \in \mathcal{B}(H)$ and $\epsilon > 0$ is given there is a decomposition $H = H_K \oplus H_L \oplus H_O$ into three closed mutually orthogonal infinite-dimensional subspaces such that if Q_I is the orthogonal projections onto H_I for $I = K,\ L,\ O$ then*

$$\|Q_L B Q_K\| < \epsilon. \tag{3.194}$$

PROOF. Choose an orthonormal basis e_j, $j \in \mathbb{N}$, of H. The subspaces H_i will be determined by a corresponding decomposition

$$\mathbb{N} = K \cup L \cup O,\ \ K \cap L = K \cap O = L \cap O = \emptyset. \tag{3.195}$$

Thus H_I has orthonormal basis e_k, $k \in I$, $I = K,\ L,\ O$. To ensure (3.194) we choose the decomposition (3.195) so that all three sets are infinite and so that

$$|(e_l, Be_k)| < 2^{-l-1}\epsilon \ \forall\ l \in L,\ \ k \in K. \tag{3.196}$$

Once we have this, then for $u \in H$, $Q_K u \in H_K$ can be expanded to $\sum\limits_{k\in K}(Q_k u, e_k)e_k$ and expanding in H_L similalry,

$$\begin{aligned} Q_L B Q_K u &= \sum_{l\in L}(BQ_K u, e_l)e_l = \sum_{k\in L}\sum_{k\in K}(Be_k, e_l)(Q_K u, e_k)e_l \\ \Longrightarrow \|Q_L B Q_K u\|^2 &\le \sum_{k\in K}\left(|(Q_k u, e_k)|^2 \sum_{l\in L}|(Be_k, e_l)|^2\right) \\ &\le \frac{1}{2}\epsilon^2\sum_{k\in K}|(Q_k u, e_k)|^2 \le \frac{1}{2}\epsilon^2\|u\|^2 \end{aligned} \tag{3.197}$$

giving (3.194). The absolute convergence of the series following from (3.196).

Thus, it remains to find a decomposition (3.195) for which (3.196) holds. This follows from Bessel's inequality. First choose $1 \in K$ then $(Be_1, e_l) \to 0$ as $l \to \infty$ so $|(Be_1, e_{l_1})| < \epsilon/4$ for l_1 large enough and we will take $l_1 > 2k_1$. Then we use induction on N, choosing $K(N)$, $L(N)$ and $O(N)$ with

$$\begin{gathered} K(N) = \{k_1 = 1 < k_2 < \dots, k_N\}, \\ L(N) = \{l_1 < l_2 < \cdots < l_N\},\ l_r > 2k_r,\ k_r > l_{r-1} \text{ for } 1 < r \le N \text{ and} \\ O(N) = \{1, \dots, l_N\} \setminus (K(N) \cup L(N)). \end{gathered}$$

Now, choose $k_{N+1} > l_N$ by such that $|(e_l, Be_{k_{N+1}})| < 2^{-l-N}\epsilon$, for all $l \in L(N)$, and then $l_{N+1} > 2k_{N+1}$ such that $|(e_{l_{N+1}}, B_k)| < e^{-N-1-k}\epsilon$ for $k \in K(N+1) = K(N) \cup \{k_{N+1}\}$ and the inductive hypothesis follows with $L(N+1) = N(N) \cup \{l_{N+1}\}$. $\square$

Given a fixed operator $A \in \mathrm{GL}(H)$ Lemma 40 can be applied with $\epsilon = \|A^{-1}\|^{-1}$. It then follows, from the convergence of the Neumann series, that the curve

$$A(s) = A - sQ_L A Q_K,\ \ s \in [0, 1] \tag{3.198}$$

lies in $\mathrm{GL}(H)$ and has endpoint satisfying

$$Q_L B Q_K = 0,\ \ B = A(1),\ \ Q_L Q_K = 0 = Q_K Q_L,\ \ Q_K = Q_K^2,\ \ Q_L = Q_L^2 \tag{3.199}$$

where all three projections, Q_L, Q_K and $\mathrm{Id} - Q_K - Q_L$ have infinite rank.

These three projections given an identification of $H = H \oplus H \oplus H$ and so replace the bounded operators by 3×3 matrices with entries which are bounded operators on H. The condition (3.199) means that

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix},\ Q_K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Q_L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.200}$$

So, now we have a 'little hole'. Under the conditions (3.199) consider

$$P = BQ_KB^{-1}(\mathrm{Id} - Q_L). \tag{3.201}$$

The condition $Q_LBQ_K = 0$ and the definition show that $Q_LP = 0 = PQ_L$. Moreover,

$$P^2 = BQ_KB^{-1}(\mathrm{Id} - Q_L)BQ_KB^{-1}(\mathrm{Id} - Q_L) = BQ_KB^{-1}BQ_KB^{-1}(\mathrm{Id} - Q_L) = P.$$

So, P is a projection which acts on the range of $\mathrm{Id} - Q_L$; from its definition, the range of P is contained in the range of BQ_K. Since

$$PBQ_K = BQ_KB^{-1}(\mathrm{Id} - Q_L)BQ_K = BQ_K$$

it follows that P is a projection *onto* the range of BQ_K.

The next part of the proof can be thought of as a result on 3×3 matrices but applied to a decomposition of Hilbert space. First, observe a little result on rotations.

LEMMA 41. *If P and Q are projections on a Hilbert space with $PQ = QP = 0$ and $M = MP = QM$ restricts to an isomorphism from the range of P to the range of Q with 'inverse' $M' = M'Q = PM'$ (so $M'M = P$ and $MM' = Q$)*

$$[-\pi/2, \pi/2] \ni \theta \longmapsto R(\theta) = \cos\theta P + \sin\theta M - \sin\theta M' + \cos\theta Q + (\mathrm{Id} - P - Q) \tag{3.202}$$

is a path in the space of invertible operators such that

$$R(0)P = P,\ R(\pi/2)P = M'P. \tag{3.203}$$

PROOF. Computing directly, $R(\theta)R(-\theta) = \mathrm{Id}$ from which the invertibility follows as does (3.203). □

We have shown above that the projection P has range equal to the range of BQ_K; apply Lemma 41 with $M = S(BQ_K)^{-1}P$ where S is a fixed isomorphism of the range of Q_K to the range of Q_L. Then

$$L_1(\theta) = R(\theta)B \text{ has } L_1(0) = B,\ L(\pi/2) = B' \text{ with } B'Q_K = Q_LSQ_K \tag{3.204}$$

an isomorphism onto the range of Q.

Next apply Lemma 41 again but for the projections Q_K and Q_L with the isomorphism S, giving

$$R'(\theta) = \cos\theta Q_K + \sin\theta S - \sin\theta S' + \cos\theta Q_L + Q_O. \tag{3.205}$$

Then the curve of invertibles

$$L_2(\theta) = R'(\theta - \theta')B' \text{ has } L(0) = B',\ L(\pi/2) = B'',\ B''Q_K = Q_K.$$

So, we have succeed by succesive homotopies through invertible elements in arriving at an operator

$$B'' = \begin{pmatrix} \mathrm{Id} & E \\ 0 & F \end{pmatrix} \tag{3.206}$$

where we are looking at the decomposition of $H = H \oplus H$ according to the projections Q_K and $\operatorname{Id} - Q_K$. The invertibility of this is equivalent to the invertibility of F and the homotopy

$$B''(s) = \begin{pmatrix} \operatorname{Id} & (1-s)E \\ 0 & F \end{pmatrix} \tag{3.207}$$

connects it to

$$L = \begin{pmatrix} \operatorname{Id} & 0 \\ 0 & F \end{pmatrix},\ (B''(s))^{-1} = \begin{pmatrix} \operatorname{Id} & -(1-s)EF^{-1} \\ 0 & F^{-1} \end{pmatrix} \tag{3.208}$$

through invertibles.

The final step is 'Eilenberg's swindle'. Start from the form of L in (3.208), choose an isomorphism $\operatorname{Ran}(Q_K) = l^2(H) \oplus l^2(H)$ and then consider the successive rotations in terms of this 2×2 decomposition

$$\begin{gathered} L(\theta) = \begin{pmatrix} \cos\theta & \sin\theta F^{-1} \\ -\sin\theta F & \cos\theta \end{pmatrix},\ \theta \in [0, \pi/2], \\ L(\theta) = \begin{pmatrix} \cos\theta F^{-1} & \sin\theta F^{-1} \\ -\sin\theta F & \cos\theta F \end{pmatrix},\ \theta \in [\pi/2, \pi] \end{gathered} \tag{3.209}$$

extended to be the constant isomorphism F on the extra factor. Then take the isomorphism

$$l^2(H) \oplus l^2(H) \oplus H \longrightarrow L^2(H) \oplus l^2(H),\ (\{u_i\}, \{w_i\}, v) \longmapsto (\{u_i\}, \{v, w_i\}) \tag{3.210}$$

in which the last element of H is place at the beginning of the second sequence. Now the rotations in (3.209) act on this space and $L(\pi - \theta)$ gives a homotopy connecting $\tilde{B}$ to the identity.

Theorem 16. *[Kuiper] For any compact subset $X \subset \operatorname{GL}(H)$ there is a retraction γ as in* (3.193).

Proof. It is only necessary to go through the construction above, for the family parameterized by X to check continuity in the variable $B \in X$. Compactness of X is used in the proof of the extension of Lemma 40; to arrange (3.196) uniformly for the whole family we need to use the compactness of the images of various finite sets under the action of all the elements of X – namely that the Fourier-Bessel series converges uniformly for such sets. After that it is only necessary to check that the choices made are either fixed for the family, or depend continuously on it (as is the case for the operators P and M for instance). $\square$

CHAPTER 4

Differential equations

The last part of the course includes some applications of Hilbert space and the spectral theorem – the completeness of the Fourier basis, some spectral theory for second-order differential operators on an interval or the circle and enough of a treatment of the eigenfunctions for the harmonic oscillator to show that the Fourier transform is an isomorphism on $L^2(\mathbb{R})$. Once one has all this, one can do a lot more, but there is no time left. Such is life.

1. Fourier series and $L^2(0, 2\pi)$.

Let us now try applying our knowledge of Hilbert space to a concrete Hilbert space such as $L^2(a, b)$ for a finite interval $(a, b) \subset \mathbb{R}$. You showed that this is indeed a Hilbert space. One of the reasons for developing Hilbert space techniques originally was precisely the following result.

THEOREM 17. *If $u \in L^2(0, 2\pi)$ then the Fourier series of u,*

$$\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} c_k e^{ikx},\ c_k = \int_{(0,2\pi)} u(x)e^{-ikx}dx \tag{4.1}$$

converges in $L^2(0, 2\pi)$ to u.

Notice that this does not say the series converges pointwise, or pointwise almost everywhere. In fact it is true that the Fourier series of a function in $L^2(0, 2\pi)$ converges almost everywhere to u, but it is hard to prove! In fact it is an important result of L. Carleson. Here we are just claiming that

$$\lim_{n\to\infty}\int |u(x) - \frac{1}{2\pi}\sum_{|k|\le n} c_k e^{ikx}|^2 = 0 \tag{4.2}$$

for any $u \in L^2(0, 2\pi)$.

Our abstract Hilbert space theory has put us quite close to proving this. First observe that if $e'_k(x) = \exp(ikx)$ then these elements of $L^2(0, 2\pi)$ satisfy

$$\int e'_k \overline{e'_j} = \int_0^{2\pi} \exp(i(k-j)x) = \begin{cases} 0 & \text{if } k \ne j \\ 2\pi & \text{if } k = j. \end{cases} \tag{4.3}$$

Thus the functions

$$e_k = \frac{e'_k}{\|e'_k\|} = \frac{1}{\sqrt{2\pi}}e^{ikx} \tag{4.4}$$

form an orthonormal set in $L^2(0, 2\pi)$. It follows that (4.1) is just the Fourier-Bessel series for u with respect to this orthonormal set:-

$$c_k = \sqrt{2\pi}(u, e_k) \Longrightarrow \frac{1}{2\pi}c_k e^{ikx} = (u, e_k)e_k. \tag{4.5}$$

So, we already know that this series converges in $L^2(0,2\pi)$ thanks to Bessel's inequality. So 'all' we need to show is

PROPOSITION 39. *The* e_k, $k \in \mathbb{Z}$, *form an orthonormal basis of* $L^2(0,2\pi)$, *i.e. are complete:*

$$\int ue^{ikx} = 0 \ \forall \ k \Longrightarrow u = 0 \ \text{in} \ L^2(0,2\pi). \tag{4.6}$$

This however, is not so trivial to prove. An equivalent statement is that the finite linear span of the e_k is *dense* in $L^2(0,2\pi)$. I will prove this using Fejér's method. In this approach, we check that any continuous function on $[0,2\pi]$ satisfying the additional condition that $u(0) = u(2\pi)$ is the uniform limit on $[0,2\pi]$ of a sequence in the finite span of the e_k. Since uniform convergence of continuous functions certainly implies convergence in $L^2(0,2\pi)$ and we already know that the continuous functions which vanish near 0 and 2π are dense in $L^2(0,2\pi)$ this is enough to prove Proposition 39. However the proof is a serious piece of analysis, at least it seems so to me! There are other approaches, for instance we could use the Stone-Weierstrass Theorem. On the other hand Fejér's approach is clever and generalizes in various ways as we will see.

So, the problem is to find the sequence in the span of the e_k which converges to a given continuous function and the trick is to use the Fourier expansion that we want to check. The idea of Cesàro is close to one we have seen before, namely to make this Fourier expansion 'converge faster', or maybe better. For the moment we can work with a general function $u \in L^2(0,2\pi)$ – or think of it as continuous if you prefer. The truncated Fourier series of u is a finite linear combination of the e_k :

$$U_n(x) = \frac{1}{2\pi} \sum_{|k| \le n} (\int_{(0,2\pi)} u(t)e^{-ikt}dt)e^{ikx} \tag{4.7}$$

where I have just inserted the definition of the c_k's into the sum. Since this is a finite sum we can treat x as a parameter and use the linearity of the integral to write it as

$$U_n(x) = \int_{(0,2\pi)} D_n(x-t)u(t), \ D_n(s) = \frac{1}{2\pi} \sum_{|k| \le n} e^{iks}. \tag{4.8}$$

Now this sum can be written as an explicit quotient, since, by telescoping,

$$2\pi D_n(s)(e^{is/2} - e^{-is/2}) = e^{i(n+\frac{1}{2})s} - e^{-i(n+\frac{1}{2})s}. \tag{4.9}$$

So in fact, at least where $s \ne 0$,

$$D_n(s) = \frac{e^{i(n+\frac{1}{2})s} - e^{-i(n+\frac{1}{2})s}}{2\pi(e^{is/2} - e^{-is/2})} \tag{4.10}$$

and the limit as $s \to 0$ exists just fine.

As I said, Cesàro's idea is to speed up the convergence by replacing U_n by its average

$$V_n(x) = \frac{1}{n+1} \sum_{l=0}^{n} U_l. \tag{4.11}$$

Again plugging in the definitions of the U_l's and using the linearity of the integral we see that

$$V_n(x) = \int_{(0,2\pi)} S_n(x-t)u(t),\ S_n(s) = \frac{1}{n+1}\sum_{l=0}^{n} D_l(s). \tag{4.12}$$

So again we want to compute a more useful form for $S_n(s)$ – which is the Fejér kernel. Since the denominators in (4.10) are all the same,

$$2\pi(n+1)(e^{is/2} - e^{-is/2})S_n(s) = \sum_{l=0}^{n} e^{i(l+\frac{1}{2})s} - \sum_{l=0}^{n} e^{-i(l+\frac{1}{2})s}. \tag{4.13}$$

Using the same trick again,

$$(e^{is/2} - e^{-is/2})\sum_{l=0}^{n} e^{i(l+\frac{1}{2})s} = e^{i(n+1)s} - 1 \tag{4.14}$$

so

$$\begin{gathered} 2\pi(n+1)(e^{is/2} - e^{-is/2})^2 S_n(s) = e^{i(n+1)s} + e^{-i(n+1)s} - 2 \\ \Longrightarrow S_n(s) = \frac{1}{n+1}\frac{\sin^2(\frac{(n+1)}{2}s)}{2\pi\sin^2(\frac{s}{2})}. \end{gathered} \tag{4.15}$$

Now, what can we say about this function? One thing we know immediately is that if we plug $u = 1$ into the disucssion above, we get $U_n = 1$ for $n \ge 0$ and hence $V_n = 1$ as well. Thus in fact

$$\int_{(0,2\pi)} S_n(x - \cdot) = 1,\ \forall\ x \in (0, 2\pi). \tag{4.16}$$

Looking directly at (4.15) the first thing to notice is that $S_n(s) \ge 0$. Also, we can see that the denominator only vanishes when $s = 0$ or $s = 2\pi$ in $[0, 2\pi]$. Thus if we stay away from there, say $s \in (\delta, 2\pi - \delta)$ for some $\delta > 0$ then – since $\sin(t)$ is a bounded function

$$|S_n(s)| \le (n+1)^{-1}C_\delta \text{ on } (\delta, 2\pi - \delta). \tag{4.17}$$

We are interested in how close $V_n(x)$ is to the given $u(x)$ in supremum norm, where now we will take u to be continuous. Because of (4.16) we can write

$$u(x) = \int_{(0,2\pi)} S_n(x-t)u(x) \tag{4.18}$$

where t denotes the variable of integration (and x is fixed in $[0, 2\pi]$). This 'trick' means that the difference is

$$V_n(x) - u(x) = \int_{(0,2\pi)} S_n(x-t)(u(t) - u(x)). \tag{4.19}$$

For each x we split this integral into two parts, the set $\Gamma(x)$ where $x - t \in [0, \delta]$ or $x - t \in [2\pi - \delta, 2\pi]$ and the remainder. So

$$|V_n(x) - u(x)| \le \int_{\Gamma(x)} S_n(x-t)|u(t) - u(x)| + \int_{(0,2\pi)\setminus\Gamma(x)} S_n(x-t)|u(t) - u(x)|. \tag{4.20}$$

Now on $\Gamma(x)$ either $|t - x| \le \delta$ – the points are close together – or t is close to 0 and x to 2π so $2\pi - x + t \le \delta$ or conversely, x is close to 0 and t to 2π so $2\pi - t + x \le \delta$.

In any case, by assuming that $u(0) = u(2\pi)$ and using the uniform continuity of a continuous function on $[0, 2\pi]$, given $\epsilon > 0$ we can choose δ so small that

$$|u(x) - u(t)| \le \epsilon/2 \text{ on } \Gamma(x). \tag{4.21}$$

On the complement of $\Gamma(x)$ we have (4.17) and since u is bounded we get the estimate

$$|V_n(x) - u(x)| \le \epsilon/2 \int_{\Gamma(x)} S_n(x-t) + (n+1)^{-1}C'(\delta) \le \epsilon/2 + (n+1)^{-1}C'(\delta). \tag{4.22}$$

Here the fact that S_n is non-negative and has integral one has been used again to estimate the integral of $S_n(x-t)$ over $\Gamma(x)$ by 1. Having chosen δ to make the first term small, we can choose n large to make the second term small and it follows that

$$V_n(x) \to u(x) \text{ uniformly on } [0, 2\pi] \text{ as } n \to \infty \tag{4.23}$$

under the assumption that $u \in \mathcal{C}([0, 2\pi])$ satisfies $u(0) = u(2\pi)$.

So this proves Proposition 39 subject to the density in $L^2(0, 2\pi)$ of the continuous functions which vanish near (but not of course in a fixed neighbourhood of) the ends. In fact we know that the L^2 functions which vanish near the ends are dense since we can chop off and use the fact that

$$\lim_{\delta \to 0} \left(\int_{(0,\delta)} |f|^2 + \int_{(2\pi-\delta, 2\pi)} |f|^2 \right) = 0. \tag{4.24}$$

This proves Theorem 17.

2. Dirichlet problem on an interval

I want to do a couple more 'serious' applications of what we have done so far. There are many to choose from, and I will mention some more, but let me first consider the Diriclet problem on an interval. I will choose the interval $[0, 2\pi]$ because we looked at it before but of course we could work on a general bounded interval instead. So, we are supposed to be trying to solve

$$-\frac{d^2u(x)}{dx^2} + V(x)u(x) = f(x) \text{ on } (0, 2\pi), \ u(0) = u(2\pi) = 0 \tag{4.25}$$

where the last part are the Dirichlet boundary conditions. I will assume that the 'potential'

$$V : [0, 2\pi] \longrightarrow \mathbb{R} \text{ is continuous and real-valued.} \tag{4.26}$$

Now, it certainly makes sense to try to solve the equation (4.25) for say a given $f \in \mathcal{C}^0([0, 2\pi])$, looking for a solution which is twice continuously differentiable on the interval. It may not exist, depending on V but one thing we can shoot for, which has the virtue of being explicit, is the following:

PROPOSITION 40. *If* $V \ge 0$ *as in* (4.26) *then for each* $f \in \mathcal{C}^0([0, 2\pi])$ *there exists a unique twice continuously differentiable solution,* u, *to* (4.25).

You will see that it is a bit hard to approach this directly – especially if you have some ODE theory at your fingertips. There are in fact various approaches to this but we want to go through L^2 theory – not surprisingly of course. How to start?

Well, we do know how to solve (4.25) if $V \equiv 0$ since we can use (Riemann) integration. Thus, ignoring the boundary conditions for the moment, we can find a solution to $-d^2v/dx^2 = f$ on the interval by integrationg twice:

$$v(x) = -\int_0^x \int_0^y f(t)dtdy \text{ satifies } -d^2v/dx^2 = f \text{ on } (0, 2\pi). \tag{4.27}$$

Moroever v really is twice continuously differentiable if f is continuous. So, what has this got to do with operators? Well, we can change the order of integration in (4.27) to write v as

$$v(x) = -\int_0^x \int_t^x f(t)dydt = \int_0^{2\pi} a(x,t)f(t)dt,\ a(x,t) = (t-x)H(x-t) \tag{4.28}$$

where the Heaviside function $H(y)$ is 1 when $y \geq 0$ and 0 when $y < 0$. Thus $a(x,t)$ is actually continuous on $[0, 2\pi] \times [0, 2\pi]$ since the $t - x$ factor vanishes at the jump in $H(t - x)$. So (4.28) shows that v is given by applying an integral operator, with continuous kernel on the square, to f.

Before thinking more seriously about this, recall that there is also the matter of the boundary conditions. Clearly, $v(0) = 0$ since we integrated from there. On the other hand, there is no particular reason why

$$v(2\pi) = \int_0^{2\pi} (t - 2\pi)f(t)dt \tag{4.29}$$

should vanish. However, we can always add to v any linear function and still satify the differential equation. Since we do not want to spoil the vanishing at $x = 0$ we can only afford to add cx but if we choose the constant c correctly this will work. Namely consider

$$c = \frac{1}{2\pi} \int_0^{2\pi} (2\pi - t)f(t)dt, \text{ then } (v + cx)(2\pi) = 0. \tag{4.30}$$

So, finally the solution we want is

$$w(x) = \int_0^{2\pi} b(x,t)f(t)dt,\ b(x,t) = \min(t,x) - \frac{tx}{2\pi} \in \mathcal{C}([0, 2\pi]^2) \tag{4.31}$$

with the formula for b following by simple manipulation from

$$b(x,t) = a(x,t) + x - \frac{tx}{2\pi} \tag{4.32}$$

Thus there is a unique, twice continuously differentiable, solution of $-d^2w/dx^2 = f$ in $(0, 2\pi)$ which vanishes at both end points and it is given by the *integral operator* (4.31).

Lemma 42. *The integral operator* (4.31) *extends by continuity from* $\mathcal{C}^0([0, 2\pi])$ *to a compact, self-adjoint operator on* $L^2(0, 2\pi)$.

Proof. Since w is given by an integral operator with a continuous real-valued kernel which is even in the sense that (check it)

$$b(t,x) = b(x,t) \tag{4.33}$$

we might as well give a more general result. □

PROPOSITION 41. *If* $b \in \mathcal{C}^0([0,2\pi]^2)$ *then*

$$Bf(x) = \int_0^{2\pi} b(x,t)f(t)dt \tag{4.34}$$

defines a compact operator on $L^2(0,2\pi)$ *if in addition* b *satisfies*

$$\overline{b(t,x)} = b(x,t) \tag{4.35}$$

then B *is self-adjoint.*

PROOF. If $f \in L^2((0,2\pi))$ and $v \in \mathcal{C}([0,2\pi])$ then the product $vf \in L^2((0,2\pi))$ and $\|vf\|_{L^2} \le \|v\|_\infty \|f\|_{L^2}$. This can be seen for instance by taking an absolutely summable approcimation to f, which gives a sequence of continuous functions converging a.e. to f and bounded by a fixed L^2 function and observing that $vf_n \to vf$ a.e. with bound a constant multiple, $\sup |v|$, of that function. It follows that for $b \in \mathcal{C}([0,2\pi]^2)$ the product

$$b(x,y)f(y) \in L^2(0,2\pi) \tag{4.36}$$

for each $x \in [0,2\pi]$. Thus $Bf(x)$ is well-defined by (4.35) since $L^2((0,2\pi) \subset L^1((0,2\pi))$.

Not only that, but $Bf \in \mathcal{C}([0,2\pi])$ as can be seen from the Cauchy-Schwarz inequality,

$$|Bf(x') - Bf(x)| = |\int (b(x',y) - b(x,y))f(y)| \le \sup_y |b(x',y - b(x,y)|(2\pi)^{\frac{1}{2}} \|f\|_{L^2}. \tag{4.37}$$

Essentially the same estimate shows that

$$\sup_x \|Bf(x)\| \le (2\pi)^{\frac{1}{2}} \sup_{(x,y)} |b| \|f\|_{L^2} \tag{4.38}$$

so indeed, $B : L^2(0,2\pi) \longrightarrow \mathcal{C}([0,2\pi])$ is a bounded linear operator.

When b satisfies (4.35) and f and g are continuous

$$\int Bf(x)\overline{g(x)} = \int f(x)\overline{Bg(x)} \tag{4.39}$$

and the general case follows by approximation in L^2 by continuous functions.

So, we need to see the compactness. If we fix x then $b(x,y) \in \mathcal{C}([0,2\pi])$ and then if we let x vary,

$$[0,2\pi] \ni x \longmapsto b(x,\cdot) \in \mathcal{C}([0,2\pi]) \tag{4.40}$$

is continuous as a map into this Banach space. Again this is the uniform continuity of a continuous function on a compact set, which shows that

$$\sup_y |b(x',y) - b(x,y)| \to 0 \text{ as } x' \to x. \tag{4.41}$$

Since the inclusion map $\mathcal{C}([0,2\pi]) \longrightarrow L^2((0,2\pi))$ is bounded, i.e continuous, it follows that the map (I have reversed the variables)

$$[0,2\pi] \ni y \longmapsto b(\cdot,y) \in L^2((0,2\pi)) \tag{4.42}$$

is continuous and so has a compact range.

Take the Fourier basis e_k for $[0,2\pi]$ and expand b in the first variable. Given $\epsilon > 0$ the compactness of the image of (4.42) implies that for some N

$$\sum_{|k|>N} |(b(x,y),e_k(x))|^2 < \epsilon \ \forall\ y \in [0,2\pi]. \tag{4.43}$$

The finite part of the Fourier series is continuous as a function of both arguments

$$b_N(x,y) = \sum_{|k|\le N} e_k(x)c_k(y),\ \ c_k(y) = (b(x,y),e_k(x)) \tag{4.44}$$

and so defines another bounded linear operator B_N as before. This operator can be written out as

$$B_N f(x) = \sum_{|k|\le N} e_k(x) \int c_k(y)f(y)dy \tag{4.45}$$

and so is of finite rank – it always takes values in the span of the first $2N+1$ trigonometric functions. On the other hand the remainder is given by a similar operator with corresponding to $q_N = b - b_N$ and this satisfies

$$\sup_y \|q_N(\cdot,y)\|_{L^2((0,2\pi))} \to 0 \text{ as } N \to \infty. \tag{4.46}$$

Thus, q_N has small norm as a bounded operator on $L^2((0,2\pi))$ so B is compact – it is the norm limit of finite rank operators. □

Now, recall from Problem# that $u_k = c\sin(kx/2)$, $k \in \mathbb{N}$, is also an orthonormal basis for $L^2(0,2\pi)$ (it is not the Fourier basis!) Moreover, differentiating we find straight away that

$$-\frac{d^2u_k}{dx^2} = \frac{k^2}{4}u_k. \tag{4.47}$$

Since of course $u_k(0) = 0 = u_k(2\pi)$ as well, from the uniqueness above we conclude that

$$Bu_k = \frac{4}{k^2}u_k \ \forall\ k. \tag{4.48}$$

Thus, in this case we know the orthonormal basis of eigenfunctions for B. They are the u_k, each eigenspace is 1 dimensional and the eigenvalues are $4k^{-2}$. So, this happenstance allows us to decompose B as the square of another operator defined directly on the othornormal basis. Namely

$$Au_k = \frac{2}{k}u_k \Longrightarrow B = A^2. \tag{4.49}$$

Here again it is immediate that A is a compact self-adjoint operator on $L^2(0,2\pi)$ since its eigenvalues tend to 0. In fact we can see quite a lot more than this.

Lemma 43. *The operator A maps $L^2(0,2\pi)$ into $\mathcal{C}^0([0,2\pi])$ and $Af(0) = Af(2\pi) = 0$ for all $f \in L^2(0,2\pi)$.*

Proof. If $f \in L^2(0,2\pi)$ we may expand it in Fourier-Bessel series in terms of the u_k and find

$$f = \sum_k c_k u_k,\ \ \{c_k\} \in l^2. \tag{4.50}$$

Then of course, by definition,

$$Af = \sum_k \frac{2c_k}{k} u_k. \tag{4.51}$$

Here each u_k is a bounded continuous function, with the bound on u_k being independent of k. So in fact (4.51) converges uniformly and absolutely since it is uniformly Cauchy, for any $q > p$,

$$|\sum_{k=p}^{q} \frac{2c_k}{k} u_k| \le 2|c| \sum_{k=p}^{q} |c_k| k^{-1} \le 2|c| \left(\sum_{k=p}^{q} k^{-2} \right)^{\frac{1}{2}} \|f\|_{L^2} \tag{4.52}$$

where Cauchy-Schwarz has been used. This proves that

$$A : L^2(0, 2\pi) \longrightarrow \mathcal{C}^0([0, 2\pi])$$

is bounded and by the uniform convergence $u_k(0) = u_k(2\pi) = 0$ for all k implies that $Af(0) = Af(2\pi) = 0$. □

So, going back to our original problem we try to solve (4.25) by moving the Vu term to the right side of the equation (don't worry about regularity yet) and hope to use the observation that

$$u = -A^2(Vu) + A^2 f \tag{4.53}$$

should satisfy the equation and boundary conditions. In fact, let's anticipate that $u = Av$, which has to be true if (4.53) holds with $v = -AVu + Af$, and look instead for

$$v = -AVAv + Af \Longrightarrow (\mathrm{Id} + AVA)v = Af. \tag{4.54}$$

So, we know that multiplication by V, which is real and continuous, is a bounded self-adjoint operator on $L^2(0, 2\pi)$. Thus AVA is a self-adjoint compact operator so we can apply our spectral theory to it and so examine the invertibility of $\mathrm{Id} + AVA$. Working in terms of a complete orthonormal basis of eigenfunctions e_i of AVA we see that $\mathrm{Id} + AVA$ is invertible if and only if it has trivial null space, i.e. if -1 is *not* an eigenvalue of AVA. Indeed, an element of the null space would have to satisfy $u = -AVAu$. On the other hand we know that AVA is *positive* since

$$(AVAw, w) = (VAv, Av) = \int_{(0,2\pi)} V(x)|Av|^2 \ge 0 \Longrightarrow \int_{(0,2\pi)} |u|^2 = 0, \tag{4.55}$$

using the non-negativity of V. So, there can be no null space – all the eigenvalues of AVA are at least non-negative and the inverse is the bounded operator given by its action on the basis

$$(\mathrm{Id} + AVA)^{-1} e_i = (1 + \tau_i)^{-1}, \ AVAe_i = \tau_i e_i. \tag{4.56}$$

Thus $\mathrm{Id} + AVA$ is invertible on $L^2(0, 2\pi)$ with inverse of the form $\mathrm{Id} + Q$, Q again compact and self-adjoint since $(1 + \tau_i)^1 - 1 \to 0$. Now, to solve (4.54) we just need to take

$$v = (\mathrm{Id} + Q)Af \Longleftrightarrow v + AVAv = Af \text{ in } L^2(0, 2\pi). \tag{4.57}$$

Then indeed

$$u = Av \text{ satisfies } u + A^2Vu = A^2 f. \tag{4.58}$$

In fact since $v \in L^2(0, 2\pi)$ from (4.57) we already know that $u \in \mathcal{C}^0([0, 2\pi])$ vanishes at the end points.

Moreover if $f \in \mathcal{C}^0([0, 2\pi])$ we know that $Bf = A^2 f$ is twice continuously differentiable, since it is given by two integrations – that is where B came from. Now, we know that u in L^2 satisfies $u = -A^2(Vu) + A^2 f$. Since $Vu \in L^2((0, 2\pi)$ so is $A(Vu)$ and then, as seen above, $A(A(Vu)$ is continuous. So combining this with the result about $A^2 f$ we see that u itself is continuous and hence so is Vu. But then, going through the routine again

$$u = -A^2(Vu) + A^2 f \tag{4.59}$$

is the sum of two twice continuously differentiable functions. Thus it is so itself. In fact from the properties of $B = A^2$ it satisifes

$$-\frac{d^2 u}{dx^2} = -Vu + f \tag{4.60}$$

which is what the result claims. So, we have proved the existence part of Proposition 40.

The uniqueness follows pretty much the same way. If there were two twice continuously differentiable solutions then the difference w would satisfy

$$-\frac{d^2 w}{dx^2} + Vw = 0,\ w(0) = w(2\pi) = 0 \Longrightarrow w = -Bw = -A^2 Vw. \tag{4.61}$$

Thus $w = A\phi$, $\phi = -AVw \in L^2(0, 2\pi)$. Thus ϕ in turn satisfies $\phi = AVA\phi$ and hence is a solution of $(\operatorname{Id} + AVA)\phi = 0$ which we know has none (assuming $V \ge 0$). Since $\phi = 0$, $w = 0$.

This completes the proof of Proposition 40. To summarize, what we have shown is that $\operatorname{Id} + AVA$ is an invertible bounded operator on $L^2(0, 2\pi)$ (if $V \ge 0$) and then the solution to (4.25) is precisely

$$u = A(\operatorname{Id} + AVA)^{-1} Af \tag{4.62}$$

which is twice continuously differentiable and satisfies the Dirichlet conditions for each $f \in \mathcal{C}^0([0, 2\pi])$.

Now, even if we do not assume that $V \ge 0$ we pretty much know what is happening.

Proposition 42. *For any $V \in \mathcal{C}^0([0, 2\pi])$ real-valued, there is an orthonormal basis w_k of $L^2(0, 2\pi)$ consisting of twice-continuously differentiable functions on $[0, 2\pi]$, vanishing at the end-points and satisfying $-\frac{d^2 w_k}{dx^2} + Vw_k = T_k w_k$ where $T_k \to \infty$ as $k \to \infty$. The equation (4.25) has a (twice continuously differentiable) solution for given $f \in \mathcal{C}^0([0, 2\pi])$ if and only if*

$$T_k = 0 \Longrightarrow \int_{(0,2\pi)} f w_k = 0, \tag{4.63}$$

i.e. f is orthogonal to the null space of $\operatorname{Id} + A^2 V$, which is the image under A of the null space of $\operatorname{Id} + AVA$, in $L^2(0, 2\pi)$.

Proof. Notice the form of the solution in case $V \ge 0$ in (4.62). In general, we can choose a constant c such that $V + c \ge 0$. Then the equation can be rewritten

$$-\frac{d^2 w}{dx^2} + Vw = Tw_k \Longleftrightarrow -\frac{d^2 w}{dx^2} + (V + c)w = (T + c)w. \tag{4.64}$$

Thus, if w satisfies this eigen-equation then it also satisfies

$$w = (T+c)A(\mathrm{Id} + A(V+c)A)^{-1}Aw \Longleftrightarrow \\ Sw = (T+c)^{-1}w,\ S = A(\mathrm{Id} + A(V+c)A)^{-1}A. \tag{4.65}$$

Now, we have shown that S is a compact self-adjoint operator on $L^2(0,2\pi)$ so we know that it has a complete set of eigenfunctions, e_k, with eigenvalues $\tau_k \neq 0$. From the discussion above we then know that each e_k is actually continuous – since it is Aw' with $w' \in L^2(0,2\pi)$ and hence also twice continuously differentiable. So indeed, these e_k satisfy the eigenvalue problem (with Dirichlet boundary conditions) with eigenvalues

$$T_k = \tau_k^{-1} + c \to \infty \text{ as } k \to \infty. \tag{4.66}$$

The solvability part also follows in much the same way. □

3. Friedrichs' extension

Next I will discuss an abstract Hilbert space set-up which covers the treatment of the Dirichlet problem above and several other applications to differential equations and indeed to other problems. I am attributing this method to Friedrichs and he certainly had a hand in it.

Instead of just one Hilbert space we will consider two at the same time. First is a 'background' space, H, a separable infinite-dimensional Hilbert space which you can think of as being something like $L^2(I)$ for some interval I. The inner product on this I will denote $(\cdot,\cdot)_H$ or maybe sometimes leave off the 'H' since this is the basic space. Let me denote a second, separable infinite-dimensional, Hilbert space as D, which maybe stands for 'domain' of some operator. So D comes with its own inner product $(\cdot,\cdot)_D$ where I will try to remember not to leave off the subscript. The relationship between these two Hilbert spaces is given by a linear map

$$i : D \longrightarrow H. \tag{4.67}$$

This is denoted 'i' because it is supposed to be an 'inclusion'. In particular I will always require that

$$i \text{ is injective.} \tag{4.68}$$

Since we will not want to have parts of H which are inaccessible, I will also assume that

$$i \text{ has dense range } i(D) \subset H. \tag{4.69}$$

In fact because of these two conditions it is quite safe to identify D with $i(D)$ and think of each element of D as really being an element of H. The subspace '$i(D) = D$' will not be closed, which is what we are used to thinking about (since it is dense) but rather has its own inner product $(\cdot,\cdot)_D$. Naturally we will also suppose that i is continuous and to avoid too many constants showing up I will suppose that i has norm at most 1 so that

$$\|i(u)\|_H \leq \|u\|_D. \tag{4.70}$$

If you are comfortable identifying $i(D)$ with D this just means that the 'D-norm' on D is *bigger* than the H norm restricted to D. A bit later I will assume one more thing about i.

What can we do with this setup? Well, consider an arbitrary element $f \in H$. Then consider the linear map

$$T_f : D \ni u \longrightarrow (i(u), f)_H \in \mathbb{C}. \tag{4.71}$$

where I have put in the identification i but will leave it out from now on, so just write $T_f(u) = (u, f)_H$. This is in fact a continuous linear functional on D since by Cauchy-Schwarz and then (4.70),

$$|T_f(u)| = |(u, f)_H| \le \|u\|_H \|f\|_H \le \|f\|_H \|u\|_D. \tag{4.72}$$

So, by the Riesz' representation – so using the assumed completeness of D (with respect to the D-norm of course) there exists a unique element $v \in D$ such that

$$(u, f)_H = (u, v)_D \ \forall\ u \in D. \tag{4.73}$$

Thus, v only depends on f and always exists, so this defines a map

$$B : H \longrightarrow D,\ Bf = v \text{ iff } (f, u)_H = (v, u)_D \ \forall\ u \in D \tag{4.74}$$

where I have taken complex conjugates of both sides of (4.73).

LEMMA 44. *The map B is a continuous linear map $H \longrightarrow D$ and restricted to D is self-adjoint:*

$$(Bw, u)_D = (w, Bu)_D \ \forall\ u, w \in D. \tag{4.75}$$

The assumption that $D \subset H$ is dense implies that $B : H \longrightarrow D$ is injective.

PROOF. The linearity follows from the uniqueness and the definition. Thus if $f_i \in H$ and $c_i \in \mathbb{C}$ for $i = 1, 2$ then

$$\begin{aligned} (c_1 f_1 + c_2 f_2, u)_H &= c_1 (f_1, u)_H + c_2 (f_2, u)_H \\ &= c_1 (Bf_1, u)_D + c_2 (Bf_2, u)_D = (c_1 Bf_1 + c_2 Bf_2, u) \ \forall\ u \in D \end{aligned} \tag{4.76}$$

shows that $B(c_1 f_1 + c_2 f_2) = c_1 Bf_1 + c_2 Bf_2$. Moreover from the estimate (4.72),

$$|(Bf, u)_D| \le \|f\|_H \|u\|_D \tag{4.77}$$

and setting $u = Bf$ it follows that $\|Bf\|_D \le \|f\|_H$ which is the desired continuity.

To see the self-adjointness suppose that u, $w \in D$, and hence of course since we are erasing i, u, $w \in H$. Then, from the definitions

$$(Bu, w)_D = (u, w)_H = \overline{(w, u)_H} = \overline{(Bw, u)_D} = (u, Bw)_D \tag{4.78}$$

so B is self-adjoint.

Finally observe that $Bf = 0$ implies that $(Bf, u)_D = 0$ for all $u \in D$ and hence that $(f, u)_H = 0$, but since D is dense, this implies $f = 0$ so B is injective. $\square$

To go a little further we will assume that the inclusion i is *compact*. Explicitly this means

$$u_n \rightharpoonup_D u \Longrightarrow u_n (= i(u_n)) \to_H u \tag{4.79}$$

where the subscript denotes which space the convergence is in. Thus compactness means that a weakly convergent sequence in D is, or is mapped to, a strongly convergent sequence in H.

LEMMA 45. *Under the assumptions* (4.67), (4.68), (4.69), (4.70) *and* (4.79) *on the inclusion of one Hilbert space into another, the operator B in* (4.74) *is compact as a self-adjoint operator on D and has only positive eigenvalues.*

PROOF. Suppose $u_n \rightharpoonup u$ is weakly convergent in D. Then, by assumption it is strongly convergent in H. But B is continuous as a map from H to D so $Bu_n \to Bu$ in D and it follows that B is compact as an operator on D.

So, we know that D has an orthonormal basis of eigenvectors of B. None of the eigenvalues λ_j can be zero since B is injective. Moreover, from the definition if $Bu_j = \lambda_j u_j$ then

$$\|u_j\|_H^2 = (u_j, u_j)_H = (Bu_j, u_j)_D = \lambda_j \|u_j\|_D^2 \tag{4.80}$$

showing that $\lambda_j > 0$. □

Now, in view of this we can define another compact operator on D by

$$Au_j = \lambda_j^{\frac{1}{2}} u_j \tag{4.81}$$

taking the positive square-roots. So of course $A^2 = B$. In fact $A : H \longrightarrow D$ is also a bounded operator.

LEMMA 46. *If u_j is an orthonormal basis of D of eigenvectors of B then $f_j = \lambda^{-\frac{1}{2}} u_j$ is an orthonormal basis of H and $A : D \longrightarrow D$ extends by continuity to an isometric isomorphism $A : H \longrightarrow D$.*

PROOF. The identity (4.80) extends to pairs of eigenvectors

$$(u_j, u_k)_H = (Bu_j, u_k)_D = \lambda_j \delta_{jk} \tag{4.82}$$

which shows that the f_j form an orthonormal sequence in H. The span is dense in D (in the H norm) and hence is dense in H so this set is complete. Thus A maps an orthonormal basis of H to an orthonormal basis of D, so it is an isometric isomorphism. □

If you think about this a bit you will see that this is an abstract version of the treatment of the 'trivial' Dirichlet problem above, except that I did not describe the Hilbert space D concretely in that case.

There are various ways this can be extended. One thing to note is that the failure of injectivity, i.e. the loss of (4.68) is not so crucial. If i is not injective, then its null space is a closed subspace and we can take its orthocomplement in place of D. The result is the same except that the operator D is only defined on this orthocomplement.

An additional thing to observe is that the completeness of D, although used crucially above in the application of Riesz' Representation theorem, is not really such a big issue either

PROPOSITION 43. *Suppose that $\tilde{D}$ is a* pre-Hilbert *space with inner product $(\cdot, \cdot)_D$ and $i : \tilde{A} \longrightarrow H$ is a linear map into a Hilbert space. If this map is injective, has dense range and satisfies* (4.70) *in the sense that*

$$\|i(u)\|_H \le \|u\|_D \ \forall \ u \in \tilde{D} \tag{4.83}$$

then it extends by continuity to a map of the completion, D, of $\tilde{D}$, satisfying (4.68), (4.69) *and* (4.70) *and if bounded sets in $\tilde{D}$ are mapped by i into precompact sets in H then* (4.79) *also holds.*

PROOF. We know that a completion exists, $\tilde{D} \subset D$, with inner product restricting to the given one and every element of D is then the limit of a Cauchy sequence in $\tilde{D}$. So we denote without ambiguity the inner product on D again as

$(\cdot,\cdot)_D$. Since i is continuous with respect to the norm on D (and on H of course) it extends by continuity to the closure of $\tilde{D}$, namely D as $i(u) = \lim_n i(u_n)$ if u_n is Cauchy in $\tilde{D}$ and hence converges in D; this uses the completeness of H since $i(u_n)$ is Cauchy in H. The value of $i(u)$ does not depend on the choice of approximating sequence, since if $v_n \to 0$, $i(v_n) \to 0$ by continuity. So, it follows that $i : D \longrightarrow H$ exists, is linear and continuous and its norm is no larger than before so (4.67) holds. $\square$

The map extended map may not be injective, i.e. it might happen that $i(u_n) \to 0$ even though $u_n \to u \neq 0$.

The general discussion of the set up of Lemmas 45 and 46 can be continued further. Namely, having defined the operators B and A we can define a new positive-definite Hermitian form on H by

$$(u, v)_E = (Au, Av)_H,\ u,\ v \in H \tag{4.84}$$

with the same relationship as between $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_D$. Now, it follows directly that

$$\|u\|_H \le \|u\|_E \tag{4.85}$$

so if we let E be the completion of H with respect to this new norm, then $i : H \longrightarrow E$ is an injection with dense range and A extends to an isometric isomorphism $A : E \longrightarrow H$. Then if u_j is an orthonormal basis of H of eigenfunctions of A with eigenvalues $\tau_j > 0$ it follows that $u_j \in D$ and that the $\tau_j^{-1} u_j$ form an orthonormal basis for D while the $\tau_j u_j$ form an orthonormal basis for E.

Lemma 47. *With E defined as above as the completion of H with respect to the inner product* (4.84)*, B extends by continuity to an isomoetric isomorphism $B : E \longrightarrow D$.*

Proof. Since $B = A^2$ on H this follows from the properties of the eigenbases above. $\square$

The typical way that Friedrichs' extension arises is that we are actually given an explicit 'operator', a linear map $P : \tilde{D} \longrightarrow H$ such that $(u, v)_D = (u, Pv)_H$ satisfies the conditions of Proposition 43. Then P extends by continuity to an isomorphism $P : D \longrightarrow E$ which is precisely the inverse of B as in Lemma 47. We shall see examples of this below.

4. Dirichlet problem revisited

So, does the setup of the preceding section work for the Dirichlet problem? We take $H = L^2((0, 2\pi))$. Then, and this really is Friedrichs' extension, we take as a subspace $\tilde{D} \subset H$ the space of functions which are once continuously differentiable and vanish outside a compact subset of $(0, 2\pi)$. This just means that there is some smaller interval, depending on the function, $[\delta, 2\pi - \delta]$, $\delta > 0$, on which we have a continuously differentiable function f with $f(\delta) = f'(\delta) = f(2\pi - \delta) = f'(2\pi - \delta) = 0$ and then we take it to be zero on $(0, \delta)$ and $(2\pi - \delta, 2\pi)$. There are lots of these, let's call the space $\tilde{D}$ as above

$$\begin{aligned} \tilde{D} = \{u \in \mathcal{C}^0[0, 2\pi];\ & u \text{ continuously differentiable on } [0, 2\pi], \\ & u(x) = 0 \text{ in } [0, \delta] \cup [2\pi - \delta, 2\pi] \text{ for some } \delta > 0\}. \end{aligned} \tag{4.86}$$

Then our first claim is that

$$\tilde{D} \text{ is dense in } L^2(0, 2\pi) \tag{4.87}$$

with respect to the norm on L^2 of course.

What inner product should we take on $\tilde{D}$? Well, we can just integrate formally by parts and set

$$(u, v)_D = \frac{1}{2\pi} \int_{[0,2\pi]} \frac{du}{dx} \overline{\frac{dv}{dx}} dx. \tag{4.88}$$

This is a pre-Hilbert inner product. To check all this note first that $(u, u)_D = 0$ implies $du/dx = 0$ by Riemann integration (since $|du/dx|^2$ is continuous) and since $u(x) = 0$ in $x < \delta$ for some $\delta > 0$ it follows that $u = 0$. Thus $(u, v)_D$ makes $\tilde{D}$ into a pre-Hilbert space, since it is a positive definite sesquilinear form. So, what about the completion? Observe that, the elements of $\tilde{D}$ being continuosly differentiable, we can always integrate from $x = 0$ and see that

$$u(x) = \int_0^x \frac{du}{dx} dx \tag{4.89}$$

as $u(0) = 0$. Now, to say that $u_n \in \tilde{D}$ is Cauchy is to say that the continuous functions $v_n = du_n/dx$ are Cauchy in $L^2(0, 2\pi)$. Thus, from the completeness of L^2 we know that $v_n \to v \in L^2(0, 2\pi)$. On the other hand (4.89) applies to each u_n so

$$|u_n(x) - u_m(x)| = |\int_0^x (v_n(s) - v_m(s))ds| \le \sqrt{2\pi}\|v_n - v_m\|_{L^2} \tag{4.90}$$

by applying Cauchy-Schwarz. Thus in fact the sequence u_n is uniformly Cauchy in $C([0, 2\pi])$ if u_n is Cauchy in $\tilde{D}$. From the completeness of the Banach space of continuous functions it follows that $u_n \to u$ in $C([0, 2\pi])$ so each element of the completion, $\tilde{D}$, 'defines' (read 'is') a continuous function:

$$u_n \to u \in D \Longrightarrow u \in \mathcal{C}([0, 2\pi]),\ u(0) = u(2\pi) = 0 \tag{4.91}$$

where the Dirichlet condition follows by continuity from (4.90).

Thus we do indeed get an injection

$$D \ni u \longrightarrow u \in L^2(0, 2\pi) \tag{4.92}$$

where the injectivity follows from (4.89) that if $v = \lim du_n/dx$ vanishes in L^2 then $u = 0$.

Now, you can go ahead and check that with these definitions, B and A are the same operators as we constructed in the discussion of the Dirichlet problem.

5. Harmonic oscillator

As a second 'serious' application of our Hilbert space theory I want to discuss the harmonic oscillator, the corresponding Hermite basis for $L^2(\mathbb{R})$. Note that so far we have not found an explicit orthonormal basis on the whole real line, even though we know $L^2(\mathbb{R})$ to be separable, so we certainly know that such a basis exists. How to construct one explicitly and with some handy properties? One way is to simply orthonormalize – using Gram-Schmidt – some countable set with dense span. For instance consider the basic Gaussian function

$$\exp(-\frac{x^2}{2}) \in L^2(\mathbb{R}). \tag{4.93}$$

This is so rapidly decreasing at infinity that the product with any polynomial is also square integrable:

$$x^k \exp(-\frac{x^2}{2}) \in L^2(\mathbb{R}) \ \forall \ k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}. \tag{4.94}$$

Orthonormalizing this sequence gives an orthonormal basis, where completeness can be shown by an appropriate approximation technique but as usual is not so simple. This is in fact the Hermite basis as we will eventually show.

Rather than proceed directly we will work up to this by discussing the eigenfunctions of the harmonic oscillator

$$P = -\frac{d^2}{dx^2} + x^2 \tag{4.95}$$

which we want to think of as an operator – although for the moment I will leave vague the question of what it operates *on*.

As you probably already know, and we will show later, it is straightforward to show that P has a lot of eigenvectors using the 'creation' and 'annihilation operators. We want to know a bit more than this and in particular I want to apply the abstract discussion above to this case but first let me go through the 'formal' theory. There is nothing wrong (I hope) here, just that we cannot easily conclude the completeness of the eigenfunctions.

The first thing to observe is that the Gaussian is an eigenfunction of H

$$\begin{aligned} Pe^{-x^2/2} = -\frac{d}{dx}(-xe^{-x^2/2} + x^2 e^{-x^2/2} \\ - (x^2-1)e^{-x^2/2} + x^2 e^{-x^2/2} = e^{-x^2/2} \end{aligned} \tag{4.96}$$

with eigenvalue 1. It is an eigenfunctions but not, for the moment, of a bounded operator on any Hilbert space – in this sense it is only a formal eigenfunctions.

In this special case there is an essentially algebraic way to generate a whole sequence of eigenfunctions from the Gaussian. To do this, write

$$\begin{aligned} Pu = (-\frac{d}{dx} + x)(\frac{d}{dx} + x)u + u = (CA+1)u, \\ \mathrm{Cr} = (-\frac{d}{dx} + x), \ \mathrm{An} = (\frac{d}{dx} + x) \end{aligned} \tag{4.97}$$

again formally as operators. Then note that

$$\mathrm{An}\, e^{-x^2/2} = 0 \tag{4.98}$$

which again proves (4.96). The two operators in (4.97) are the 'creation' operator and the 'annihilation' operator. They almost commute in the sense that

$$(\mathrm{An}\,\mathrm{Cr} - \mathrm{Cr}\,\mathrm{An})u = 2u \tag{4.99}$$

for say any twice continuously differentiable function u.

Now, set $u_0 = e^{-x^2/2}$ which is the 'ground state' and consider $u_1 = \mathrm{Cr}\, u_0$. From (4.99), (4.98) and (4.97),

$$Pu_1 = (\mathrm{Cr}\,\mathrm{An}\,\mathrm{Cr} + \mathrm{Cr})u_0 = \mathrm{Cr}^2\,\mathrm{An}\, u_0 + 3\,\mathrm{Cr}\, u_0 = 3u_1. \tag{4.100}$$

Thus, u_1 is an eigenfunction with eigenvalue 3.

LEMMA 48. *For* $j \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ *the function* $u_j = \mathrm{Cr}^j\, u_0$ *satisfies* $Pu_j = (2j+1)u_j$.

PROOF. This follows by induction on j, where we know the result for $j = 0$ and $j = 1$. Then

$$P \operatorname{Cr} u_j = (\operatorname{Cr} \operatorname{An} + 1) \operatorname{Cr} u_j = \operatorname{Cr}(P - 1)u_j + 3 \operatorname{Cr} u_j = (2j + 3)u_j. \tag{4.101}$$

□

Again by induction we can check that $u_j = (2^j x^j + q_j(x))e^{-x^2/2}$ where q_j is a polynomial of degree at most $j - 2$. Indeed this is true for $j = 0$ and $j = 1$ (where $q_0 = q_1 \equiv 0$) and then

$$\operatorname{Cr} u_j = (2^{j+1}x^{j+1} + \operatorname{Cr} q_j)e^{-x^2/2}. \tag{4.102}$$

and $q_{j+1} = \operatorname{Cr} q_j$ is a polynomial of degree at most $j - 1$ – one degree higher than q_j.

From this it follows in fact that the finite span of the u_j consists of all the products $p(x)e^{-x^2/2}$ where $p(x)$ is any polynomial.

Now, all these functions are in $L^2(\mathbb{R})$ and we want to compute their norms. First, a standard integral computation[1] shows that

$$\int_{\mathbb{R}} (e^{-x^2/2})^2 = \int_{\mathbb{R}} e^{-x^2} = \sqrt{\pi} \tag{4.103}$$

For $j > 0$, integration by parts (easily justified by taking the integral over $[-R, R]$ and then letting $R \to \infty$) gives

$$\int_{\mathbb{R}} (\operatorname{Cr}^j u_0)^2 = \int_{\mathbb{R}} \operatorname{Cr}^j u_0(x) \operatorname{Cr}^j u_0(x) dx = \int_{\mathbb{R}} u_0 \operatorname{An}^j \operatorname{Cr}^j u_0. \tag{4.104}$$

Now, from (4.99), we can move one factor of An through the j factors of Cr until it emerges and 'kills' u_0

$$\begin{aligned} \operatorname{An} \operatorname{Cr}^j u_0 &= 2 \operatorname{Cr}^{j-1} u_0 + \operatorname{Cr} \operatorname{An} \operatorname{Cr}^{j-1} u_0 \\ &= 2 \operatorname{Cr}^{j-1} u_0 + \operatorname{Cr}^2 \operatorname{An} \operatorname{Cr}^{j-2} u_0 = 2j \operatorname{Cr}^{j-1} u_0. \end{aligned} \tag{4.105}$$

So in fact,

$$\int_{\mathbb{R}} (\operatorname{Cr}^j u_0)^2 = 2j \int_{\mathbb{R}} (\operatorname{Cr}^{j-1} u_0)^2 = 2^j j! \sqrt{\pi}. \tag{4.106}$$

A similar argument shows that

$$\int_{\mathbb{R}} u_k u_j = \int_{\mathbb{R}} u_0 \operatorname{An}^k \operatorname{Cr}^j u_0 = 0 \text{ if } k \neq j. \tag{4.107}$$

Thus the functions

$$e_j = 2^{-j/2}(j!)^{-\frac{1}{2}}\pi^{-\frac{1}{4}} C^j e^{-x^2/2} \tag{4.108}$$

form an orthonormal sequence in $L^2(\mathbb{R})$.

---

[1] To compute the Gaussian integral, square it and write as a double integral then introduce polar coordinates

$$(\int_{\mathbb{R}} e^{-x^2} dx)^2 = \int_{\mathbb{R}^2} e^{-x^2-y^2} dxdy = \int_0^\infty \int_0^{2\pi} e^{-r^2} r dr d\theta = \pi\big[-e^{-r^2}\big]_0^\infty = \pi.$$

We would like to show this orthonormal sequence is complete. Rather than argue through approximation, we can guess that in some sense the operator

$$\mathrm{An}\,\mathrm{Cr} = (\frac{d}{dx} + x)(-\frac{d}{dx} + x) = -\frac{d^2}{dx^2} + x^2 + 1 \tag{4.109}$$

should be invertible, so one approach is to use the ideas above of Friedrichs' extension to construct its 'inverse' and show this really exists as a compact, self-adjoint operator on $L^2(\mathbb{R})$ and that its only eigenfunctions are the e_i in (4.108). Another, more indirect approach is described below.

6. Isotropic space

There are some functions which should be in the domain of P, namely the twice continuously differentiable functions on $\mathbb{R}$ with compact support, those which vanish outside a finite interval. Recall that there are actually a lot of these, they are dense in $L^2(\mathbb{R})$. Following what we did above for the Dirichlet problem set

$$\begin{aligned}\tilde{D} = \{u : \mathbb{R} \longmapsto \mathbb{C}; \exists\ R \text{ s.t. } u = 0 \text{ in } |x| > R,\\ u \text{ is twice continuously differentiable on } \mathbb{R}\}.\end{aligned} \tag{4.110}$$

Now for such functions integration by parts on a large enough interval (depending on the functions) produces no boundary terms so

$$(Pu, v)_{L^2} = \int_{\mathbb{R}} (Pu)\overline{v} = \int_{\mathbb{R}} \left(\frac{du}{dx}\overline{\frac{dv}{dx}} + x^2 u\overline{v}\right) = (u, v)_{\mathrm{iso}} \tag{4.111}$$

is a positive definite hermitian form on $\tilde{D}$. Indeed the vanishing of $\|u\|_S$ implies that $\|xu\|_{L^2} = 0$ and so $u = 0$ since $u \in \tilde{D}$ is continuous. The suffix 'iso' here stands for 'isotropic' and refers to the fact that xu and du/dx are essentially on the same footing here. Thus

$$(u, v)_{\mathrm{iso}} = (\frac{du}{dx}, \frac{dv}{dx})_{L^2} + (xu, xv)_{L^2}. \tag{4.112}$$

This may become a bit clearer later when we get to the Fourier transform.

Definition 22. Let $H^1_{\mathrm{iso}}(\mathbb{R})$ be the completion of $\tilde{D}$ in (4.110) with respect to the inner product $(\cdot, \cdot)_{\mathrm{iso}}$.

Proposition 44. *The inclusion map $i : \tilde{D} \longrightarrow L^2(\mathbb{R})$ extends by continuity to $i : H^1_{\mathrm{iso}} \longrightarrow L^2(\mathbb{R})$ which satisfies* (4.67)*,* (4.68)*,* (4.69)*,* (4.70) *and* (4.79) *with $D = H^1_{\mathrm{iso}}$ and $H = L^2(\mathbb{R})$ and the derivative and multiplication maps define an injection*

$$H^1_{\mathrm{iso}} \longrightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R}). \tag{4.113}$$

Proof. Let us start with the last part, (4.113). The map here is supposed to be the continuous extension of the map

$$\tilde{D} \ni u \longmapsto (\frac{du}{dx}, xu) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \tag{4.114}$$

where du/dx and xu are both compactly supported continuous functions in this case. By definition of the inner product $(\cdot, \cdot)_{\mathrm{iso}}$ the norm is precisely

$$\|u\|^2_{\mathrm{iso}} = \|\frac{du}{dx}\|^2_{L^2} + \|xu\|^2_{L^2} \tag{4.115}$$

so if u_n is Cauchy in $\tilde{D}$ with respect to $\|\cdot\|_{\text{iso}}$ then the sequences du_n/dx and xu_n are Cauchy in $L^2(\mathbb{R})$. By the completeness of L^2 they converge defining an element in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ as in (4.113). Moreover the elements so defined only depend on the element of the completion that the Cauchy sequence defines. The resulting map (4.113) is clearly continuous.

Now, we need to show that the inclusion i extends to H^1_{iso} from $\tilde{D}$. This follows from another integration identity. Namely, for $u \in \tilde{D}$ the Fundamental theorem of calculus applied to

$$\frac{d}{dx}(ux\overline{u}) = |u|^2 + \frac{du}{dx}x\overline{u} + ux\overline{\frac{du}{dx}}$$

gives

$$\|u\|^2_{L^2} \le \int_{\mathbb{R}} |\frac{du}{dx}x\overline{u}| + \int |ux\overline{\frac{du}{dx}}| \le \|u\|^2_{\text{iso}}. \tag{4.116}$$

Thus the inequality (4.70) holds for $u \in \tilde{D}$.

It follows that the inclusion map $i : \tilde{D} \longrightarrow L^2(\mathbb{R})$ extends by continuity to H^1_{iso} since if $u_n \in \tilde{D}$ is Cauchy with respect in H^1_{iso} it is Cauchy in $L^2(\mathbb{R})$. It remains to check that i is injective and compact, since the range is already dense on $\tilde{D}$.

If $u \in H^1_{\text{iso}}$ then to say $i(u) = 0$ (in $L^2(\mathbb{R})$) is to say that for any $u_n \to u$ in H^1_{iso}, with $u_n \in \tilde{D}$, $u_n \to 0$ in $L^2(\mathbb{R})$ and we need to show that this means $u_n \to 0$ in H^1_{iso} to conclude that $u = 0$. To do so we use the map (4.113). If $u_n\tilde{D}$ converges in H^1_{iso} then it follows that the sequence $(\frac{du}{dx}, xu)$ converges in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. If v is a continuous function of compact support then $(xu_n, v)_{L^2} = (u_n, xv) \to (u, xv)_{L^2}$, for if $u = 0$ it follows that $xu_n \to 0$ as well. Similarly, using integration by parts the limit U of $\frac{du_n}{dx}$ in $L^2(\mathbb{R})$ satisfies

$$(U, v)_{L^2} = \lim_n \int \frac{du_n}{dx}\overline{v} = -\lim_n \int u_n \overline{\frac{dv}{dx}} = -(u, \frac{dv}{dx})_{L^2} = 0 \tag{4.117}$$

if $u = 0$. It therefore follows that $U = 0$ so in fact $u_n \to 0$ in H^1_{iso} and the injectivity of i follows. □

We can see a little more about the metric on H^1_{iso}.

LEMMA 49. *Elements of H^1_{iso} are continuous functions and convergence with respect to $\|\cdot\|_{\text{iso}}$ implies uniform convergence on bounded intervals.*

PROOF. For elements of the dense subspace $\tilde{D}$, (twice) continuously differentiable and vanishing outside a bounded interval the Fundamental Theorem of Calculus shows that

$$\begin{gathered} u(x) = e^{x^2/2}\int_{-\infty}^x (\frac{d}{dt}(e^{-t^2/2}u) = e^{x^2/2}\int_{-\infty}^x (e^{-t^2/2}(-tu + \frac{du}{dt})) \Longrightarrow \\ |u(x)| \le e^{x^2/2}(\int_{-\infty}^x e^{-t^2})^{\frac{1}{2}}\|u\|_{\text{iso}} \end{gathered} \tag{4.118}$$

where the estimate comes from the Cauchy-Schwarz applied to the integral. It follows that if $u_n \to u$ with respect to the isotropic norm then the sequence converges uniformly on bounded intervals with

$$\sup_{[-R,R]} |u(x)| \le C(R)\|u\|_{\text{iso}}. \tag{4.119}$$

□

Now, to proceed further we either need to apply some 'regularity theory' or do a computation. I choose to do the latter here, although the former method (outlined below) is much more general. The idea is to show that

LEMMA 50. *The linear map* $(P+1) : \mathcal{C}^2_c(\mathbb{R}) \longrightarrow \mathcal{C}_c(\mathbb{R})$ *is injective with range dense in* $L^2(\mathbb{R})$ *and if* $f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ *there is a sequence* $u_n \in \mathcal{C}^2_c(\mathbb{R})$ *such that* $u_n \to u$ *in* H^1_{iso}, $u_n \to u$ *locally uniformly with its first two derivatives and* $(P+1)u_n \to f$ *in* $L^2(\mathbb{R})$ *and locally uniformly.*

PROOF. Why $P+1$ and not P? The result is actually true for P but not so easy to show directly. The advantage of $P+1$ is that it factorizes

$$(P+1) = \operatorname{An}\operatorname{Cr} \text{ on } \mathcal{C}^2_{\mathrm{c}}(\mathbb{R}).$$

so we proceed to solve the equation $(P+1)u = f$ in two steps.

First, if $f \in \mathrm{c}(\mathbb{R})$ then using the natural integrating factor

$$v(x) = e^{x^2/2}\int_{-\infty}^{x} e^{t^2/2} f(t)dt + ae^{-x^2/2} \text{ satisfies } \operatorname{An} v = f. \tag{4.120}$$

The integral here is not in general finite if f does not vanish in $x < -R$, which by assumption it does. Note that $\operatorname{An} e^{-x^2/2} = 0$. This solution is of the form

$$v \in \mathcal{C}^1(\mathbb{R}),\ v(x) = a_\pm e^{-x^2/2} \text{ in } \pm x > R \tag{4.121}$$

where R depends on f and the constants can be different.

In the second step we need to solve away such terms – in general one cannot. However, we can always choose a in (4.120) so that

$$\int_{\mathbb{R}} e^{-x^2/2} v(x) = 0. \tag{4.122}$$

Now consider

$$u(x) = e^{x^2/2}\int_{-\infty}^{x} e^{-t^2/2} v(t)dt. \tag{4.123}$$

Here the integral does make sense because of the decay in v from (4.121) and $u \in \mathcal{C}^2(\mathbb{R})$. We need to understand how it behaves as $x \to \pm\infty$. From the second part of (4.121),

$$u(x) = a_- \operatorname{erf}_-(x),\ x < -R,\ \operatorname{erf}_-(x) = \int_{(-\infty,x]} e^{x^2/2 - t^2} \tag{4.124}$$

is an incomplete error function. It's derivative is e^{-x^2} but it actually satisfies

$$|x \operatorname{erf}_-(x)| \le Ce^{x^2},\ x < -R. \tag{4.125}$$

In any case it is easy to get an estimate such as Ce^{-bx^2} as $x \to -\infty$ for any $0 < b < 1$ by Cauchy-Schwarz.

As $x \to \infty$ we would generally expect the solution to be rapidly increasing, but precisely because of (4.122). Indeed the vanishing of this integral means we can rewrite (4.123) as an integral from $+\infty$:

$$u(x) = -e^{x^2/2}\int_{[x,\infty)} e^{-t^2/2} v(t)dt \tag{4.126}$$

and then the same estimates analysis yields

$$u(x) = -a_+ \operatorname{erf}_+(x),\ x < -R,\ \operatorname{erf}_+(x) = \int_{[x,\infty)} e^{x^2/2-t^2} \tag{4.127}$$

So for any $f \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ we have found a solution of $(P+1)u = f$ with u satisfying the rapid decay conditions (4.124) and (4.127). These are such that if $\chi \in \mathcal{C}^2_{\mathrm{c}}(\mathbb{R})$ has $\chi(t) = 1$ in $|t| < 1$ then the sequence

$$u_n = \chi(\frac{x}{n})u(x) \to u,\ u_n' \to u',\ u_n'' \to u'' \tag{4.128}$$

in all cases with convergence in $L^2(\mathbb{R})$ and uniformly and even such that $x^2u_n \to xu$ uniformly and in $L^2(\mathbb{R})$.

This yields the first part of the Lemma, since if $f \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ and u is the solution just constructed to $(P+1)u = f$ then $(P+1)u_n \to f$ in L^2. So the closure $L^2(\mathbb{R})$ in range of $(P+1)$ on $\mathcal{C}^2_{\mathrm{c}}(\mathbb{R})$ includes $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$ so is certainly dense in $L^2(\mathbb{R})$.

The second part also follows from this construction. If $f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ then the sequence

$$f_n = \chi(\frac{x}{n})f(x) \in \mathcal{C}_{\mathrm{c}}(\mathbb{R}) \tag{4.129}$$

converges to f both in $L^2(\mathbb{R})$ and locally uniformly. Consider the solution, u_n to $(P+1)u_n = f_n$ constructed above. We want to show that $u_n \to u$ in L^2 and locally uniformly with its first two derivatives. The decay in u_n is enough to allow integration by parts to see that

$$\int_{\mathbb{R}} (P+1)u_n\overline{u_n} = \|u_n\|^2_{\mathrm{iso}} + \|u\|^2_{L^2} = |(f_n, u_n)| \le \|f_n\|_{l^2}\|u_n\|_{L^2}. \tag{4.130}$$

This shows that the sequence is bounded in H^1_{iso} and applying the same estimate to $u_n - u_m$ that it is Cauchy and hence convergent in H^1_{iso}. This implies $u_n \to u$ in H^1_{iso} and so both in $L^2(\mathbb{R})$ and locally uniformly. The differential equation can be written

$$(u_n)'' = x^2u_n - u_n - f_n \tag{4.131}$$

where the right side converges locally uniformly. It follows from a standard result on uniform convergence of sequences of derivatives that in fact the uniform limit u is twice continuously differentiable and that $(u_n)'' \to u''$ locally uniformly. So in fact $(P+1)u = f$ and the last part of the Lemma is also proved. □

7. Fourier transform

The Fourier transform for functions on $\mathbb{R}$ is in a certain sense the limit of the definition of the coefficients of the Fourier series on an expanding interval, although that is not generally a good way to approach it. We know that if $u \in L^1(\mathbb{R})$ and $v \in \mathcal{C}_\infty(\mathbb{R})$ is a bounded continuous function then $vu \in L^1(\mathbb{R})$ – this follows from our original definition by approximation. So if $u \in L^1(\mathbb{R})$ the integral

$$\hat{u}(\xi) = \int e^{-ix\xi}u(x)dx,\ \xi \in \mathbb{R} \tag{4.132}$$

exists for each $\xi \in \mathbb{R}$ as a Legesgue integral. Note that there are many different normalizations of the Fourier transform in use. This is the standard 'analysts' normalization.

PROPOSITION 45. *The Fourier tranform,* (4.132)*, defines a bounded linear map*

$$\mathcal{F} : L^1(\mathbb{R}) \ni u \longmapsto \hat{u} \in \mathcal{C}_0(\mathbb{R}) \tag{4.133}$$

into the closed subspace $\mathcal{C}_0(\mathbb{R}) \subset \mathcal{C}_\infty(\mathbb{R})$ *of continuous functions which vanish at infinity (with respect to the supremum norm).*

PROOF. We know that the integral exists for each ξ and from the basic properties of the Lebesgue integal

$$|\hat{u}(\xi)| \le \|u\|_{L^1}, \text{ since } |e^{-ix\xi}u(x)| = |u(x)|. \tag{4.134}$$

To investigate its properties we restrict to $u \in \rfloor(\mathbb{R})$, a compactly-supported continuous function. Then the integral becomes a Riemann integral and the integrand is a continuous function of both variables. It follows that the result is uniformly continuous:-

$$|\hat{u}(\xi) - \frac{1}{2}u(\xi')| \le \int_{|x|\le R} |e^{-ix\xi} - e^{-ix\xi'}||u(x)|dx \le C(u) \sup_{|x|\le R} |e^{-ix\xi} - e^{-ix\xi'}| \tag{4.135}$$

with the right side small by the uniform continuity of continuous functions on compact sets. From (4.134), if $u_n \to u$ in $L^1(\mathbb{R})$ with $u_n \in \mathcal{C}_c(\mathbb{R})$ it follows that $\hat{u}_n \to \hat{u}$ uniformly on $\mathbb{R}$. Thus the Fourier transform is uniformly continuous on $\mathbb{R}$ for any $u \in L^1(\mathbb{R})$ (you can also see this from the continuity-in-the-mean of L^1 functions).

Now, we know that even the compactly-supported once continuously differentiable functions, forming $\mathcal{C}^1_c(\mathbb{R})$ are dense in $L^1(\mathbb{R})$ so we can also consider (4.132) where $u \in \mathcal{C}^1_c(\mathbb{R})$. Then the integration by parts as follows is justified

$$\xi\hat{u}(\xi) = i\int (\frac{de^{-ix\xi}}{dx})u(x)dx = -i\int e^{-ix\xi}\frac{du(x)}{dx}dx. \tag{4.136}$$

Now, $du/dx \in \mathcal{C}_c(\mathbb{R})$ (by assumption) so the estimate (4.134) now gives

$$\sup_{\xi\in\mathbb{R}} |\xi\hat{u}(\xi)| \le \sup_{x\in\mathbb{R}} |\frac{du}{dx}|. \tag{4.137}$$

This certainly implies the weaker statement that

$$\lim_{|\xi|\to\infty} |\hat{u}(\xi)| = 0 \tag{4.138}$$

which is 'vanishing at infinity'. Now we again use the density, this time of $\mathcal{C}^1_c(\mathbb{R})$, in $L^1(\mathbb{R})$ and the uniform estimate (4.134), plus the fact that is a sequence of continuous functions on $\mathbb{R}$ converges uniformly on $\mathbb{R}$ and each element vanishes at infinity then the limit vanishes at infinity to complete the proof of the Proposition. □

We will use the explicit eigenfunctions of the harmonic oscillator below to show that the Fourier tranform extends by continuity from $\mathcal{C}_c(\mathbb{R})$ to define an isomorphism

$$\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \tag{4.139}$$

with inverse given by the corresponding continuous extension of

$$\mathcal{G}v(x) = (2\pi)^{-1}\int e^{ix\xi}v(\xi). \tag{4.140}$$

8. Fourier inversion

This year, 2015, I decided to go directly to the proof of the Fourier inversion formula, via Schwartz space and an elegant argument due to Hörmander.

We have shown above that the Fourier transform is defined as an integral if $u \in L^1(\mathbb{R})$. Suppose that in addition we know that $xu \in L^1(\mathbb{R})$. We can summarize the combined information as (why?)

$$(1+|x|)u \in L^1(\mathbb{R}). \tag{4.141}$$

Lemma 51. *If u satisfies* (4.141) *then $\hat{u}$ is continuously differentiable and $d\hat{u}/d\xi = \mathcal{F}(-ixu)$ is bounded.*

Proof. Consider the difference quotient for the Fourier transform:

$$\frac{\hat{u}(\xi+s)-\hat{u}(\xi)}{s} = \int \frac{e^{-ixs}-1}{s} e^{-ix\xi} u(x). \tag{4.142}$$

We can use the standard proof of Taylor's formula to write the difference quotient inside the integral as

$$D(x,s) = -ix \int_0^1 e^{-itxs} dt \Longrightarrow |D(x,s)| \le |x|. \tag{4.143}$$

It follows that as $s \to 0$ (along a sequence if you prefer) $D(x,s)e^{-ix\xi} f(x)$ is bounded by the $L^1(\mathbb{R})$ function $|x||u(x)|$ and converges pointwise to $-ie^{-ix\xi} xu(x)$. Dominated convergence therefore shows that the integral converges showing that the derivative exists and that

$$\frac{d\hat{u}(\xi)}{d\xi} = \mathcal{F}(-ixu). \tag{4.144}$$

From the earlier results it follows that the derivative is continuous and bounded, proving the lemma. □

Now, we can iterate this result and so conclude:

$$\begin{gathered} (1+|x|)^k u \in L^1(\mathbb{R})\ \forall\ k \Longrightarrow \\ \hat{u} \text{ is infinitely differentiable with bounded derivatives and} \\ \frac{d^k \hat{u}}{d\xi^k} = \mathcal{F}((-ix)^k u). \end{gathered} \tag{4.145}$$

This result shows that from 'decay' of u we deduce smoothness of $\hat{u}$. We can go the other way too. Note one way to ensure the assumption in (4.145) is to make the stronger assumption that

$$x^k u \text{ is bounded and continuous } \forall\ k. \tag{4.146}$$

Indeed, Dominated Convergence shows that if u is continuous and satisfies the bound

$$|u(x)| \le (1+|x|)^{-r},\ r > 1$$

then $u \in L^1(\mathbb{R})$. So the integrability of $x^j u$ follows from the bounds in (4.146) for $k \le j+2$. This is throwing away information but simplifies things below.

In the opposite direction, suppose that u is continuously differentiable and satisfies the estimates (4.146) and

$$|\frac{u(x)}{dx}| \le (1+|x|)^{-r},\ r > 1.$$

Then consider

$$\xi\hat{u} = i\int \frac{de^{-ix\xi}}{dx}u(x) = \lim_{R\to\infty} i\int_{-R}^{R} \frac{de^{-ix\xi}}{dx}u(x). \tag{4.147}$$

We may integrate by parts in this integral to get

$$\xi\hat{u} = \lim_{R\to\infty}\left(i\big[e^{-ix\xi}u(x)\big]_{-R}^{R} - i\int_{-R}^{R} e^{-ix\xi}\frac{du}{dx}\right). \tag{4.148}$$

The decay of u shows that the first term vanishes in the limit so

$$\xi\hat{u} = \mathcal{F}(-i\frac{du}{dx}). \tag{4.149}$$

Iterating this in turn we see that if u has continuous derivatives of all orders and for all j

$$|\frac{d^j u}{dx^j}| \le C_j(1+|x|)^{-r},\ r>1 \text{ then } \xi^j\hat{u} = \mathcal{F}((-i)^j\frac{d^j u}{dx^j}) \tag{4.150}$$

are all bounded.

Laurent Schwartz defined a space which handily encapsulates these results.

Definition 23. Schwartz space, $\mathcal{S}(\mathbb{R})$, consists of all the infinitely differentiable functions $u : \mathbb{R} \longrightarrow \mathbb{C}$ such that

$$\|u\|_{j,k} = \sup|x^j\frac{d^k u}{dx^k}| < \infty\ \forall\ j,\ k \ge 0 \tag{4.151}$$

This is clearly a linear space. In fact it is a complete metric space in a natural way. All the $\|\cdot\|_{j,k}$ in (4.151) are norms on $\mathcal{S}(\mathbb{R})$, but none of them is stronger than the others. So there is no natural norm on $\mathcal{S}(\mathbb{R})$ with respect to which it is complete. In the problems below you can find some discussion of the fact that

$$d(u,v) = \sum_{j,k\ge 0} 2^{-j-k}\frac{\|u-v\|_{j,k}}{1+\|u-v\|_{j,k}} \tag{4.152}$$

is a complete metric. We will not use this here.

Notice that there is some prejudice on the order of multiplication by x and differentiation in (4.151). This is only apparent, since these estimates (taken together) are equivalent to

$$\sup|\frac{d^k(x^j u)}{dx^k}| < \infty\ \forall\ j,\ k \ge 0. \tag{4.153}$$

To see the equivalence we can use induction over N where the inductive statement is the equivalence of (4.151) and (4.153) for $j+k \le N$. Certainly this is true for $N=0$ and to carry out the inductive step just differentiate out the product to see that

$$\frac{d^k(x^j u)}{dx^k} = x^j\frac{d^k u}{dx^k} + \sum_{l+m<k+j} c_{l,m,k,j}x^m\frac{d^l u}{dx^l}$$

where one can be much more precise about the extra terms, but the important thing is that they all are lower order (in fact both degrees go down). If you want to be careful, you can of course prove this identity by induction too! The equivalence of (4.151) and (4.153) for $N+1$ now follows from that for N.

THEOREM 18. *The Fourier transform restricts to a bijection on $\mathcal{S}(\mathbb{R})$ with inverse*

$$\mathcal{G}(v)(x) = \frac{1}{2\pi}\int e^{ix\xi}v(\xi). \tag{4.154}$$

PROOF. The proof (due to Hörmander as I said above) will take a little while because we need to do some computation, but I hope you will see that it is quite clear and elementary.

First we need to check that $\mathcal{F} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$, but this is what I just did the preparation for. Namely the estimates (4.151) imply that (4.150) applies to all the $\frac{d^k(x^j u)}{dx^k}$ and so

$$\xi^k \frac{d^j\hat{u}}{d\xi^j} \text{ is continuous and bounded } \forall\ k,\ j \Longrightarrow \hat{u} \in \mathcal{S}(\mathbb{R}). \tag{4.155}$$

This indeed is why Schwartz introduced this space.

So, what we want to show is that with $\mathcal{G}$ defined by (4.154), $u = \mathcal{G}(\hat{u})$ for all $u \in \mathcal{S}(\mathbb{R})$. Notice that there is only a sign change and a constant factor to get from $\mathcal{F}$ to $\mathcal{G}$ so certainly $\mathcal{G} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$. We start off with what looks like a small part of this. Namely we want to show that

$$I(\hat{u}) = \int \hat{u} = 2\pi u(0). \tag{4.156}$$

Here, $I : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ is just integration, so it is certainly well-defined. To prove (4.156) we need to use a version of Taylor's formula and then do a little computation.

LEMMA 52. *If $u \in \mathcal{S}(\mathbb{R})$ then*

$$u(x) = u(0)\exp(-\frac{x^2}{2}) + xv(x),\ v \in \mathcal{S}(\mathbb{R}). \tag{4.157}$$

PROOF. Here I will leave it to you (look in the problems) to show that the Gaussian

$$\exp(-\frac{x^2}{2}) \in \mathcal{S}(\mathbb{R}). \tag{4.158}$$

Observe then that the difference

$$w(x) = u(x) - u(0)\exp(-\frac{x^2}{2}) \in \mathcal{S}(\mathbb{R}) \text{ and } w(0) = 0.$$

This is clearly a necessary condition to see that $w = xv$ with $v \in \mathcal{S}(\mathbb{R})$ and we can then see from the Fundamental Theorem of Calculus that

$$w(x) = \int_0^x w'(y)dy = x\int_0^1 w'(tx)dt \Longrightarrow v(x) = \int_0^1 w'(tx) = \frac{w(x)}{x}. \tag{4.159}$$

From the first formula for v it follows that it is infinitely differentiable and from the second formula the derivatives decay rapidly since each derivative can be written in the form of a finite sum of terms $p(x)\frac{d^l w}{dx^l}/x^N$ where the ps are polynomials. The rapid decay of the derivatives of w therefore implies the rapid decay of the derivatives of v. So indeed we have proved Lemma 52. $\square$

Let me set $\gamma(x) = \exp(-\frac{x^2}{2})$ to simplify the notation. Taking the Fourier transform of each of the terms in (4.157) gives

$$\hat{u} = u(0)\hat{\gamma} + i\frac{d\hat{v}}{d\xi}. \tag{4.160}$$

Since $\hat{v} \in \mathcal{S}(\mathbb{R})$,

$$\int \frac{d\hat{v}}{d\xi} = \lim_{R\to\infty} \int_{-R}^{R} \frac{d\hat{v}}{d\xi} = \lim_{R\to\infty} \big[\hat{v}(\xi)\big]_{-R}^{R} = 0. \tag{4.161}$$

So now we see that

$$\int \hat{u} = cu(0), \ c = \int \hat{\gamma}$$

being a constant that we still need to work out!

Lemma 53. *For the Gaussian,* $\gamma(x) = \exp(-\frac{x^2}{2})$,

$$\hat{\gamma}(\xi) = \sqrt{2\pi}\gamma(\xi). \tag{4.162}$$

Proof. Certainly, $\hat{\gamma} \in \mathcal{S}(\mathbb{R})$ and from the identities for derivatives above

$$\frac{d\hat{\gamma}}{d\xi} = -i\mathcal{F}(x\gamma), \ \xi\hat{\gamma} = \mathcal{F}(-i\frac{d\gamma}{dx}). \tag{4.163}$$

Thus, $\hat{\gamma}$ satisfies the same differential equation as γ :

$$\frac{d\hat{\gamma}}{d\xi} + \xi\hat{\gamma} = -i\mathcal{F}(\frac{d\gamma}{dx} + x\gamma) = 0.$$

This equation we can solve and so we conclude that $\hat{\gamma} = c'\gamma$ where c' is also a constant that we need to compute. To do this observe that

$$c' = \hat{\gamma}(0) = \int \gamma = \sqrt{2\pi} \tag{4.164}$$

which gives (4.162). The computation of the integral in (4.164) is a standard clever argument which you probably know. Namely take the square and work in polar coordinates in two variables:

$$\begin{aligned}(\int \gamma)^2 &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy \\ &= \int_0^{2\pi} \int_0^\infty e^{-r^2/2} r dr d\theta = 2\pi\big[-e^{-r^2/2}\big]_0^\infty = 2\pi.\end{aligned} \tag{4.165}$$

□

So, finally we need to get from (4.156) to the inversion formula. Changing variable in the Fourier transform we can see that for any $y \in \mathbb{R}$, setting $u_y(x) = u(x+y)$, which is in $\mathcal{S}(\mathbb{R})$ if $u \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{F}(u_y) = \int e^{-ix\xi} u_y(x) dx = \int e^{-i(s-y)\xi} u(s) ds = e^{iy\xi}\hat{u}. \tag{4.166}$$

Now, plugging u_y into (4.156) we see that

$$\int \hat{u}_y(0) = 2\pi u_y(0) = 2\pi u(y) = \int e^{iy\xi}\hat{u}(\xi) d\xi \Longrightarrow u(y) = \mathcal{G}u, \tag{4.167}$$

the Fourier inversion formula. So we have proved the Theorem. □

9. Convolution

There is a discussion of convolution later in the notes, I have inserted a new (but not very different) treatment here to cover the density of $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$ needed in the next section.

Consider two continuous functions of compact support $u,\ v \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$. Their convolution is

$$u * v(x) = \int u(x-y)v(y)dy = \int u(y)v(x-y)dy. \tag{4.168}$$

The first integral is the definition, clearly it is a well-defined Riemann integral since the integrand is continuous as a function of y and vanishes whenever $v(y)$ vanishes – so has compact support. In fact if both u and v vanish outside $[-R, R]$ then $u * v = 0$ outside $[-2R, 2R]$.

From standard properties of the Riemann integral (or Dominated convergence if you prefer!) it follows easily that $u*v$ is continuous. What we need to understand is what happens if (at least) one of u or v is smoother. In fact we will want to take a very smooth function, so I pause here to point out

Lemma 54. *There exists a ('bump') function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ which is infinitely differentiable, i.e. has continuous derivatives of all orders, vanishes outside $[-1, 1]$, is strictly positive on $(-1, 1)$ and has integral* 1.

Proof. We start with an explicit function,

$$\phi(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0. \end{cases} \tag{4.169}$$

The exponential function grows faster than any polynomial at $+\infty$, since

$$\exp(x) > \frac{x^k}{k!} \text{ in } x > 0\ \forall\ k. \tag{4.170}$$

This can be seen directly from the Taylor series which converges on the whole line (indeed in the whole complex plane)

$$\exp(x) = \sum_{k\ge 0} \frac{x^k}{k!}.$$

From (4.170) we deduce that

$$\lim_{x\downarrow 0} \frac{e^{-1/x}}{x^k} = \lim_{R\to\infty} \frac{R^k}{e^R} = 0\ \forall\ k \tag{4.171}$$

where we substitute $R = 1/x$ and use the properties of $\exp$. In particular ϕ in (4.169) is continuous across the origin, and so everywhere. We can compute the derivatives in $x > 0$ and these are of the form

$$\frac{d^l\phi}{dx^l} = \frac{p_l(x)}{x^{2l}} e^{-1/x},\ x > 0, p_l \text{ a polynomial.} \tag{4.172}$$

As usual, do this by induction since it is true for $l = 0$ and differetiation the formula for a given l one finds

$$\frac{d^{l+1}\phi}{dx^{l+1}} = \left(\frac{p_l(x)}{x^{2l+2}} - 2l\frac{p_l(x)}{x^{2l+1}} + \frac{p_l'(x)}{x^{2l}} \right) e^{-1/x} \tag{4.173}$$

where the coefficient function is of the desired form p_{l+1}/x^{2l+2}.

Once we know (4.172) then we see from (4.171) that all these functions are continuous down to 0 where they vanish. From this it follows that ϕ in (4.169) is infinitely differentiable. For ϕ itself we can use the Fundamental Theorem of Calculus to write

$$\phi(x) = \int_{\epsilon}^{x} U(t)dt + \phi(\epsilon),\ x > \epsilon > 0. \tag{4.174}$$

Here U is the derivative in $x > 0$. Taking the limit as $\epsilon \downarrow 0$ both sides converge, and then we see that

$$\phi(x) = \int_{0}^{x} U(t)dt.$$

From this it follows that ϕ is continuously differentiable across 0 and it derivative is U, the continuous extension of the derivative from $x > 0$. The same argument applies to successive derivatives, so indeed ϕ is infinitely differentiable.

From ϕ we can construct a function closer to the desired bump function. Namely

$$\Phi(x) = \phi(x+1)\phi(1-x).$$

The first factor vanishes when $x \le -1$ and is otherwise positive while the second vanishes when $x \ge 1$ but is otherwise positive, so the product is infinitely differentiable on $\mathbb{R}$ and positive on $(-1,1)$ but otherwise 0. Then we can normalize the integral to 1 by taking

$$\psi(x) = \Phi(x)/\int \Phi. \tag{4.175}$$

$\square$

In particular from Lemma 54 we conclude that the space $\mathcal{C}_c^\infty(\mathbb{R})$, of infinitely differentiable functions of compact support, is not empty. Going back to convolution in (4.168) suppose now that is smooth. Then

$$u \in \mathcal{C}_{\mathrm{c}}(\mathbb{R}),\ v \in \mathcal{C}_c^\infty(\mathbb{R}) \Longrightarrow u * v \in \mathcal{C}_c^\infty(\mathbb{R}). \tag{4.176}$$

As usual this follows from properties of the Riemann integral or by looking directly at the difference quotient

$$\frac{u * v(x+t) - u * v(x)}{t} = \int u(y)\frac{v(x+t-y) - v(x-y)}{t}dt.$$

As $t \to 0$, the differce quotient for v converges uniformly (in y) to the derivative and hence the integral converges and the derivative of the convolution exists,

$$\frac{d}{dx}u * v(x) = u * (\frac{dv}{dx}). \tag{4.177}$$

This result allows of immediate iteration, showing that the convolution is smooth and we know that it has compact support

Proposition 46. *For any $u \in \mathcal{C}_c(\mathbb{R})$ there exists $u_n \to u$ uniformly on $\mathbb{R}$ where $u_n \in \mathcal{C}_c^\infty(\mathbb{R})$ with supports in a fixed compact set.*

Proof. For each $\epsilon > 0$ consider the rescaled bump function

$$\psi_\epsilon = \epsilon^{-1}\psi(\frac{x}{\epsilon}) \in \mathcal{C}_c^\infty(\mathbb{R}). \tag{4.178}$$

In fact, ψ_ϵ vanishes outside the interval (ϵ, ϵ), is positive within this interval and has integral 1 – which is what the factor of ϵ^{-1} does. Now set

$$u_\epsilon = u * \psi_\epsilon \in \mathcal{C}_c^\infty(\mathbb{R}),\ \epsilon > 0, \tag{4.179}$$

from what we have just seen. From the supports of these functions, u_ϵ vanishes outside $[-R-\epsilon, R+\epsilon]$ if u vanishes outside $[-R, R]$. So only the convergence remains. To get this we use the fact that the integral of ψ_ϵ is equal to 1 to write

$$u_\epsilon(x) - u(x) = \int (u(x-y)\psi_\epsilon(y) - u(x)\psi_\epsilon(y))dy. \tag{4.180}$$

Estimating the integral using the positivity of the bump function

$$|u_\epsilon(x) - u(x)| = \int_{-\epsilon}^{\epsilon} |u(x-y) - u(x)|\psi_\epsilon(y)dy. \tag{4.181}$$

By the uniformity of a continuous function on a compact set, given $\delta > 0$ there exists $\epsilon > 0$ such that

$$\sup_{[-\epsilon,\epsilon]} |u(x-y) - y(x)| < \delta\ \forall\ x \in \mathbb{R}.$$

So the uniform convergence follows:-

$$\sup |u_\epsilon(x) - u(x)| \le \delta \int \phi_\epsilon = \delta \tag{4.182}$$

Pass to a sequence $\epsilon_n \to 0$ if you wish, □

Corollary 5. *The spaces $\mathcal{C}_c^\infty(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ are dense in $L^2(\mathbb{R})$.*

Uniform convegence of continuous functions with support in a fixed subset is stronger than L^2 convergence the result follows from the Proposition above for $\mathcal{C}_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$.

10. Plancherel and Parseval

But which is which?

We proceed to show that $\mathcal{F}$ and $\mathcal{G}$ both extend to isomorphisms of $L^2(\mathbb{R})$ which are inverses of each other. The main step is to show that

$$\int u(x)\hat{v}(x)dx = \int \hat{u}(\xi)v(\xi)d\xi,\ u,\ v \in \mathcal{S}(\mathbb{R}). \tag{4.183}$$

Since the integrals are rapidly convergent at infinity we may substitute the definite of the Fourier transform into (4.183), write the result out as a double integral and change the order of integration

$$\begin{aligned}\int u(x)\hat{v}(x)dx &= \int u(x) \int e^{-ix\xi} v(\xi)d\xi dx \\ &= \int v(\xi) \int e^{-ix\xi} u(x)dxd\xi = \int \hat{u}(\xi)v(\xi)d\xi.\end{aligned} \tag{4.184}$$

Now, if $w \in \mathcal{S}(\mathbb{R})$ we may replace $v(\xi)$ by $\overline{\hat{w}}(\xi)$, since it is another element of $\mathcal{S}(\mathbb{R})$. By the Fourier Inversion formula,

$$w(x) = (2\pi)^{-1} \int e^{-ix\xi} \hat{w}(\xi) \Longrightarrow \overline{w(x)} = (2\pi)^{-1} \int e^{ix\xi} \overline{\hat{w}(\xi)} = (2\pi)^{-1}\hat{v}. \tag{4.185}$$

Substituting these into (4.183) gives Parseval's formula

$$\int u\overline{w} = \frac{1}{2\pi}\int \hat{u}\overline{\hat{w}},\ u,\ w \in \mathcal{S}(\mathbb{R}). \tag{4.186}$$

PROPOSITION 47. *The Fourier transform $\mathcal{F}$ extends from $\mathcal{S}(\mathbb{R})$ to an isomorphism on $L^2(\mathbb{R})$ with $\frac{1}{\sqrt{2\pi}}$ an isometric isomorphism with adjoint, and inverse, $\sqrt{2\pi}\mathcal{G}$.*

PROOF. Setting $u = w$ in (4.186) shows that

$$\|\mathcal{F}(u)\|_{L^2} = \sqrt{2\pi}\|u\|_{L^2} \tag{4.187}$$

for all $u \in \mathcal{S}(\mathbb{R})$. The density of $\mathcal{S}(\mathbb{R})$, established above, then implies that $\mathcal{F}$ extends by continuity to the whole of $L^2(\mathbb{R})$ as indicated. $\square$

This isomorphism of $L^2(\mathbb{R})$ has many implications. For instance, we would like to define the Sobolev space $H^1(\mathbb{R})$ by the conditions that $u \in L^2(\mathbb{R})$ and $\frac{du}{dx} \in L^2(\mathbb{R})$ but to do this we would need to make sense of the derivative. However, we can 'guess' that if it exists, the Fourier transform of du/dx should be $\xi\hat{u}(\xi)$. For a function in L^2, such as $\hat{u}$ given that $u \in L^2$, we *do* know what it means to require $\xi\hat{u}(\xi) \in L^2(\mathbb{R})$. We can then define the Sobolev spaces of any positive, even non-integral, order by

$$H^r(\mathbb{R}) = \{u \in L^2(\mathbb{R}); |\xi|^r\hat{u} \in L^2(\mathbb{R})\}. \tag{4.188}$$

Of course it would take us some time to investigate the properties of these spaces!

11. Completeness of the Hermite functions

In 2015 I gave a different proof of the completeness of the eigenfunctions of the harmonic operator, reducing it to the spectral theorem, discussed in Section 5 above.

The starting point is to find a (generalized) inverse to the creation operator. Namely $e^{-x^2/2}$ is an integrating factor for it, so acting on once differentiable functions

$$\mathrm{Cr}\, u = -\frac{du}{dx} + xu = e^{x^2/2}\frac{d}{dx}(e^{-x^2/2}u). \tag{4.189}$$

For a function, say $f \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$, we therefore get a solution by integration

$$u(x) = -e^{x^2/2}\int_{-\infty}^{x} e^{-t^2/2}f(t)dt. \tag{4.190}$$

This function vanishes for $x << 0$ but as $x \to +\infty$, after passing the top of the support of f,

$$u(x) = ce^{x^2/2},\ c = -\int_{\mathbb{R}} e^{-t^2/2}f(t)dt. \tag{4.191}$$

So, to have Sf decay in both directions we need to assume that this integral vanishes.

PROPOSITION 48. *The creation operator gives a bijection*

$$\mathrm{Cr} : \mathcal{S}(\mathbb{R}) \longrightarrow \{f \in \mathcal{S}(\mathbb{R}); \int e^{t^2/2}f(t)dt = 0\} \tag{4.192}$$

with two-sided inverse in this sense

$$u = Sf,\ Sf(x) = -e^{x^2/2}\int_{-\infty}^{x} e^{-t^2/2}(\Pi_0 f)(t)dt, \qquad \Pi_0 f = f - (\frac{1}{\sqrt{\pi}}\int e^{-t^2/2}f(t)dt)e^{-x^2/2}. \tag{4.193}$$

Note that Π_0 is the orthogonal projection *off* the ground state of the harmonic oscillator and gives a map from $\mathcal{S}(\mathbb{R})$ to the right side of (4.192).

Proof. For any $f \in \mathcal{S}(\mathbb{R})$ consider the behaviour of u given by (4.190) as $x \to -\infty$. [This is what I messed up in lecture.] What we wish to show is that

$$|x^k u(x)| \text{ is bounded as } x \to -\infty \tag{4.194}$$

for all k. Now, it is not possible to find an explicit primitive for $e^{t^2/2}$ but we can make do with the identity

$$\frac{d}{dt}\frac{e^{-t^2/2}}{t} = -e^{-t^2/2} - \frac{e^{-t^2/2}}{t^2}. \tag{4.195}$$

Inserting this into the integral defining u and integrating by parts we find

$$u(x) = -f(x)/x - e^{x^2/2}\int_{-\infty}^{x} e^{-t^2/2}(\frac{f'(t)}{t} - \frac{f(t)}{t^2})dt. \tag{4.196}$$

The first term here obviously satisfies the estimate (4.194) and we can substitute in the integral and repeat the procedure. Proceeding inductively we find after N steps

$$u(x) = \sum_{1\le k\le 2N+1}\frac{h_j}{x^j} + e^{x^2/2}\int_{-\infty}^{x} e^{-t^2/2}\left(\sum_{N\le j\le 2N}\frac{g_{j,n}(t)}{t^j}\right)dt,\ h_j,\ g_{j,N} \in \mathcal{S}(\mathbb{R}). \tag{4.197}$$

The first terms certainly satisfy (4.194) for any k and the integral is bounded by $C|x|^{-N}e^{-x^2/2}$ so indeed (4.194) holds for all k.

For $g \in \mathcal{S}(\mathbb{R})$ such that $\int e^{-t^2/2}g(t) = 0$ we can replace (4.190) by

$$u(x) = -e^{x^2/2}\int_{x}^{\infty} e^{-t^2/2}f(t)dt \tag{4.198}$$

to which the same argument applies as $x \to +\infty$. The effect of Π_0 is to ensure this, so

$$\sup(1+|x|)^k|Sf| < \infty\ \forall\ k. \tag{4.199}$$

By construction, $\frac{d}{dx}Sf = xSf(x) - \Pi_0 f$ so this also shows rapid decrease of the first derivative. In fact we may differentiate this equation N times and deduce, inductively, that *all* derivatives of Sf are rapidly decaying at infinity.

So, we see that S defined by (4.193) maps $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ with null space containing the span of $e^{-x^2/2}$. Since it solves the differetial equation we find

$$\operatorname{Cr} S = \operatorname{Id} - \Pi_0,\ S\operatorname{Cr} = \operatorname{Id} \text{ on } \mathcal{S}(\mathbb{R}). \tag{4.200}$$

Indeed, the first identity is what we have just shown and this shows that Cr in (4.192) is surjective. We already know it is injective since $\operatorname{Cr} f\|_{L^2} \ge \|f\|_{L^2}$ for $f \in \mathcal{S}(\mathbb{R})$. So S Cr in (4.192) is a bijection and S is the bijection inverting it, so the second identity in (4.200) follows. $\square$

Notice that we can deduce from (4.200) that S extends by continuity to a bounded operator

$$S : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}). \tag{4.201}$$

Namely, it is zero on the span of $e^{-x^2/2}$ and

$$\|\frac{dSf}{dx}\|^2_{L^2} + \|xSf\|^2_{L^2} + \|Sf\|^2_{L^2} = \|\Pi_0 f\|^2_{L^2} \le \|f\|^2_{L^2}. \tag{4.202}$$

This actually shows that the bounded extension of S is compact.

THEOREM 19. *The composite SS^* is a compact injective self-adjoint operator on $L^2(\mathbb{R})$ with eigenvalues $(2j+2)^{-1}$ for $f \ge 0$ and associated one-dimesnional eigenspaces $E_j \subset \mathcal{S}(\mathbb{R})$ spanned by $\mathrm{Cr}^j\, e^{-x^2/2}$; in particular the Hermite functions form an orthonormal basis of $L^2(\mathbb{R})$.*

PROOF. We know that S has dense range (since this is already true when it acts on $\mathcal{S}(\mathbb{R})$) so S^* is injective and has range dense in the orthocomplement of $e^{-x^2/2}$. From this it follows that SS^* is injective. Compactness follows from the discussion of the isotropic space above, showing the compactness of S. By the spectral theorem SS^* has an orthonormal basis of eigenfunctions in $L^2(\mathbb{R})$, say v_j, with eigenvalues $s_j > 0$ which we may assume to be decreasing to 0.

□

12. Mehler's formula and completeness

Starting from the ground state for the harmonic oscillator

$$P = -\frac{d^2}{dx^2} + x^2,\ Hu_0 = u_0,\ u_0 = e^{-x^2/2} \tag{4.203}$$

and using the creation and annihilation operators

$$\mathrm{An} = \frac{d}{dx} + x,\ \mathrm{Cr} = -\frac{d}{dx} + x,\ \mathrm{An}\,\mathrm{Cr} - \mathrm{Cr}\,\mathrm{An} = 2\,\mathrm{Id},\ H = \mathrm{Cr}\,\mathrm{An} + \mathrm{Id} \tag{4.204}$$

we have constructed the higher eigenfunctions:

$$u_j = \mathrm{Cr}^j\, u_0 = p_j(x)u_0(c),\ p(x) = 2^j x^j + \text{l.o.ts},\ Hu_j = (2j+1)u_j \tag{4.205}$$

and shown that these are orthogonal, $u_j \perp u_k$, $j \ne k$, and so when normalized give an orthonormal system in $L^2(\mathbb{R})$:

$$e_j = \frac{u_j}{2^{j/2}(j!)^{\frac{1}{2}}\pi^{\frac{1}{4}}}. \tag{4.206}$$

Now, what we want to see, is that these e_j form an orthonormal basis of $L^2(\mathbb{R})$, meaning they are complete as an orthonormal sequence. There are various proofs of this, but the only 'simple' ones I know involve the Fourier inversion formula and I want to use the completeness to *prove* the Fourier inversion formula, so that will not do. Instead I want to use a version of Mehler's formula.

To show the completeness of the e_j's it is enough to find a compact self-adjoint operator with these as eigenfunctions and no null space. It is the last part which is tricky. The first part is easy. Remembering that all the e_j are real, we can find an operator with the e_j;s as eigenfunctions with corresponding eigenvalues $\lambda_j > 0$ (say) by just defining

$$Au(x) = \sum_{j=0}^{\infty} \lambda_j (u, e_j) e_j(x) = \sum_{j=0}^{\infty} \lambda_j e_j(x) \int e_j(y) u(y). \tag{4.207}$$

For this to be a compact operator we need $\lambda_j \to 0$ as $j \to \infty$, although for boundedness we just need the λ_j to be bounded. So, the problem with this is to show that A has no null space – which of course is just the completeness of the e_j' since (assuming all the λ_j are positive)

$$Au = 0 \Longleftrightarrow u \perp e_j \ \forall\ j. \tag{4.208}$$

Nevertheless, this is essentially what we will do. The idea is to write A as an *integral operator* and then work with that. I will take the $\lambda_j = w^j$ where $w \in (0, 1)$. The point is that we can find an explicit formula for

$$A_w(x, y) = \sum_{j=0}^{\infty} w^j e_j(x) e_j(y) = A(w, x, y). \tag{4.209}$$

To find $A(w, x, y)$ we will need to compute the Fourier transforms of the e_j. Recall that

$$\begin{gathered} \mathcal{F} : L^1(\mathbb{R}) \longrightarrow \mathcal{C}^0_\infty(\mathbb{R}),\ \mathcal{F}(u) = \hat{u}, \\ \hat{u}(\xi) = \int e^{-ix\xi} u(x),\ \sup |\hat{u}| \le \|u\|_{L^1}. \end{gathered} \tag{4.210}$$

Lemma 55. *The Fourier transform of u_0 is*

$$(\mathcal{F}u_0)(\xi) = \sqrt{2\pi} u_0(\xi). \tag{4.211}$$

Proof. Since u_0 is both continuous and Lebesgue integrable, the Fourier transform is the limit of a Riemann integral

$$\hat{u}_0(\xi) = \lim_{R\to\infty} \int_{-R}^{R} e^{i\xi x} u_0(x). \tag{4.212}$$

Now, for the Riemann integral we can differentiate under the integral sign with respect to the parameter ξ – since the integrand is continuously differentiable – and see that

$$\begin{aligned} \frac{d}{d\xi}\hat{u}_0(\xi) &= \lim_{R\to\infty} \int_{-R}^{R} ixe^{i\xi x} u_0(x) \\ &= \lim_{R\to\infty} i \int_{-R}^{R} e^{i\xi x} (-\frac{d}{dx} u_0(x) \\ &= \lim_{R\to\infty} -i \int_{-R}^{R} \frac{d}{dx} \left(e^{i\xi x} u_0(x)\right) - \xi \lim_{R\to\infty} \int_{-R}^{R} e^{i\xi x} u_0(x) \\ &= -\xi \hat{u}_0(\xi). \end{aligned} \tag{4.213}$$

Here I have used the fact that $\mathrm{An}\, u_0 = 0$ and the fact that the boundary terms in the integration by parts tend to zero rapidly with R. So this means that $\hat{u}_0$ is annihilated by An :

$$(\frac{d}{d\xi} + \xi)\hat{u}_0(\xi) = 0. \tag{4.214}$$

Thus, it follows that $\hat{u}_0(\xi) = c\exp(-\xi^2/2)$ since these are the only functions in annihilated by An. The constant is easy to compute, since

$$\hat{u}_0(0) = \int e^{-x^2/2}dx = \sqrt{2\pi} \tag{4.215}$$

proving (4.211). □

We can use this formula, of if you prefer the argument to prove it, to show that

$$v = e^{-x^2/4} \Longrightarrow \hat{v} = \sqrt{\pi}e^{-\xi^2}. \tag{4.216}$$

Changing the names of the variables this just says

$$e^{-x^2} = \frac{1}{2\sqrt{\pi}}\int_{\mathbb{R}} e^{ixs-s^2/4}ds. \tag{4.217}$$

The definition of the u_j's can be rewritten

$$u_j(x) = (-\frac{d}{dx}+x)^j e^{-x^2/2} = e^{x^2/2}(-\frac{d}{dx})^j e^{-x^2} \tag{4.218}$$

as is easy to see inductively – the point being that $e^{x^2/2}$ is an integrating factor for the creation operator. Plugging this into (4.217) and carrying out the derivatives – which is legitimate since the integral is so strongly convergent – gives

$$u_j(x) = \frac{e^{x^2/2}}{2\sqrt{\pi}}\int_{\mathbb{R}}(-is)^j e^{ixs-s^2/4}ds. \tag{4.219}$$

Now we can use this formula twice on the sum on the left in (4.209) and insert the normalizations in (4.206) to find that

$$\sum_{j=0}^{\infty} w^j e_j(x)e_j(y) = \sum_{j=0}^{\infty}\frac{e^{x^2/2+y^2/2}}{4\pi^{3/2}}\int_{\mathbb{R}^2}\frac{(-1)^j w^j s^j t^j}{2^j j!}e^{isx+ity-s^2/4-t^2/4}dsdt. \tag{4.220}$$

The crucial thing here is that we can sum the series to get an exponential, this allows us to finally conclude:

Lemma 56. *The identity* (4.209) *holds with*

$$A(w,x,y) = \frac{1}{\sqrt{\pi}\sqrt{1-w^2}}\exp\left(-\frac{1-w}{4(1+w)}(x+y)^2 - \frac{1+w}{4(1-w)}(x-y)^2\right) \tag{4.221}$$

Proof. Summing the series in (4.220) we find that

$$A(w,x,y) = \frac{e^{x^2/2+y^2/2}}{4\pi^{3/2}}\int_{\mathbb{R}^2}\exp(-\frac{1}{2}wst+isx+ity-\frac{1}{4}s^2-\frac{1}{4}t^2)dsdt. \tag{4.222}$$

Now, we can use the same formula as before for the Fourier transform of u_0 to evaluate these integrals explicitly. One way to do this is to make a change of variables by setting

$$s = (S+T)/\sqrt{2},\ t = (S-T)/\sqrt{2} \Longrightarrow dsdt = dSdT, \tag{4.223}$$
$$-\frac{1}{2}wst+isx+ity-\frac{1}{4}s^2-\frac{1}{4}t^2 = iS\frac{x+y}{\sqrt{2}}-\frac{1}{4}(1+w)S^2+iT\frac{x-y}{\sqrt{2}}-\frac{1}{4}(1-w)T^2.$$

Note that the integrals in (4.222) are 'improper' (but rapidly convergent) Riemann integrals, so there is no problem with the change of variable formula. The formula for the Fourier transform of $\exp(-x^2)$ can be used to conclude that

$$\begin{aligned}\int_{\mathbb{R}} \exp(iS\frac{x+y}{\sqrt{2}} - \frac{1}{4}(1+w)S^2)dS &= \frac{2\sqrt{\pi}}{\sqrt{(1+w)}}\exp(-\frac{(x+y)^2}{2(1+w)}) \\ \int_{\mathbb{R}} \exp(iT\frac{x-y}{\sqrt{2}} - \frac{1}{4}(1-w)T^2)dT &= \frac{2\sqrt{\pi}}{\sqrt{(1-w)}}\exp(-\frac{(x-y)^2}{2(1-w)}).\end{aligned} \tag{4.224}$$

Inserting these formulæ back into (4.222) gives

$$A(w,x,y) = \frac{1}{\sqrt{\pi}\sqrt{1-w^2}}\exp\left(-\frac{(x+y)^2}{2(1+w)} - \frac{(x-y)^2}{2(1-w)} + \frac{x^2}{2} + \frac{y^2}{2}\right) \tag{4.225}$$

which after a little adjustment gives (4.221). □

Now, this explicit representation of A_w as an integral operator allows us to show

Proposition 49. *For all real-valued $f \in L^2(\mathbb{R})$,*

$$\sum_{j=1}^{\infty} |(u,e_j)|^2 = \|f\|^2_{L^2}. \tag{4.226}$$

Proof. By definition of A_w

$$\sum_{j=1}^{\infty} |(u,e_j)|^2 = \lim_{w\uparrow 1}(f, A_w f) \tag{4.227}$$

so (4.226) reduces to

$$\lim_{w\uparrow 1}(f, A_w f) = \|f\|^2_{L^2}. \tag{4.228}$$

To prove (4.228) we will make our work on the integral operators rather simpler by assuming first that $f \in \mathcal{C}^0(\mathbb{R})$ is continuous and vanishes outside some bounded interval, $f(x) = 0$ in $|x| > R$. Then we can write out the L^2 inner product as a double integral, which is a genuine (iterated) Riemann integral:

$$(f, A_w f) = \int\int A(w,x,y)f(x)f(y)dydx. \tag{4.229}$$

Here I have used the fact that f and A are real-valued.

Look at the formula for A in (4.221). The first thing to notice is the factor $(1-w^2)^{-\frac{1}{2}}$ which blows up as $w \to 1$. On the other hand, the argument of the exponential has two terms, the first tends to 0 as $w \to 1$ and the becomes very large and negative, at least when $x - y \neq 0$. Given the signs, we see that

$$\begin{gathered}\text{if } \epsilon > 0,\ X = \{(x,y); |x| \le R, |y| \le R, |x-y| \ge \epsilon\} \text{ then} \\ \sup_X |A(w,x,y)| \to 0 \text{ as } w \to 1.\end{gathered} \tag{4.230}$$

So, the part of the integral in (4.229) over $|x-y| \ge \epsilon$ tends to zero as $w \to 1$.

So, look at the other part, where $|x-y| \le \epsilon$. By the (uniform) continuity of f, given $\delta > 0$ there exits $\epsilon > 0$ such that

$$|x-y| \le \epsilon \Longrightarrow |f(x) - f(y)| \le \delta. \tag{4.231}$$

Now we can divide (4.229) up into three pieces:-

$$(f, A_w f) = \int_{S\cap\{|x-y|\ge\epsilon\}} A(w,x,y)f(x)f(y)dydx \\ + \int_{S\cap\{|x-y|\le\epsilon\}} A(w,x,y)(f(x)-f(y))f(y)dydx \\ + \int_{S\cap\{|x-y|\le\epsilon\}} A(w,x,y)f(y)^2dydx \tag{4.232}$$

where $S = [-R,R]^2$.

Look now at the third integral in (4.232) since it is the important one. We can change variable of integration from x to $t = \sqrt{\frac{1+w}{1-w}}(x-y)$. Since $|x-y| \le \epsilon$, the new t variable runs over $|t| \le \epsilon\sqrt{\frac{1+w}{1-w}}$ and then the integral becomes

$$\int_{S\cap\{|t|\le\epsilon\sqrt{\frac{1+w}{1-w}}\}} A(w, y+t\sqrt{\frac{1-w}{1+w}}, y)f(y)^2dydt, \text{ where}$$

$$A(w, y+t\sqrt{\frac{1-w}{1+w}}, y) \\ = \frac{1}{\sqrt{\pi}(1+w)} \exp\left(-\frac{1-w}{4(1+w)}(2y+t\sqrt{1-w})^2\right)\exp\left(-\frac{t^2}{4}\right). \tag{4.233}$$

Here y is bounded; the first exponential factor tends to 1 and the t domain extends to $(-\infty,\infty)$ as $w \to 1$, so it follows that for any $\epsilon > 0$ the third term in (4.232) tends to

$$\|f\|^2_{L^2} \text{ as } w \to 1 \text{ since } \int e^{-t^2/4} = 2\sqrt{\pi}. \tag{4.234}$$

Noting that $A \ge 0$ the same argument shows that the second term is bounded by a constant multiple of δ. Now, we have already shown that the first term in (4.232) tends to zero as $\epsilon \to 0$, so this proves (4.228) – given some $\gamma > 0$ first choose $\epsilon > 0$ so small that the first two terms are each less than $\frac{1}{2}\gamma$ and then let $w \uparrow 0$ to see that the lim sup and lim inf as $w \uparrow 0$ must lie in the range $[\|f\|^2-\gamma, \|f\|^2+\gamma]$. Since this is true for all $\gamma > 0$ the limit exists and (4.226) follows under the assumption that f is continuous and vanishes outside some interval $[-R,R]$.

This actually suffices to prove the completeness of the Hermite basis. In any case, the general case follows by continuity since such continuous functions vanishing outside compact sets are dense in $L^2(\mathbb{R})$ and both sides of (4.226) are continuous in $f \in L^2(\mathbb{R})$. □

Now, (4.228) certainly implies that the e_j form an orthonormal basis, which is what we wanted to show – but hard work! It is done here in part to remind you of how we did the Fourier series computation of the same sort and to suggest that you might like to compare the two arguments.

13. Weak and strong derivatives

In approaching the issue of the completeness of the eigenbasis for harmonic oscillator more directly, rather than by the kernel method discussed above, we run into the issue of weak and strong solutions of differential equations. Suppose that

$u \in L^2(\mathbb{R})$, what does it *mean* to say that $\frac{du}{dx} \in L^2(\mathbb{R})$. For instance, we will want to understand what the 'possible solutions of'

$$\mathrm{An}\, u = f,\ u,\ f \in L^2(\mathbb{R}),\ \mathrm{An} = \frac{d}{dx} + x \tag{4.235}$$

are. Of course, if we assume that u is continuously differentiable then we know what this means, but we need to consider the possibilities of giving a meaning to (4.235) under more general conditions – without assuming too much regularity on u (or any at all).

Notice that there is a difference between the two terms in $\mathrm{An}\, u = \frac{du}{dx} + xu$. If $u \in L^2(\mathbb{R})$ we can assign a meaning to the second term, xu, since we know that $xu \in L^2_{\mathrm{loc}}(\mathbb{R})$. This is not a normed space, but it is a perfectly good vector space, in which $L^2(\mathbb{R})$ 'sits' – if you want to be pedantic it naturally injects into it. The point however, is that we *do know* what the statement $xu \in^2 (\mathbb{R})$ means, given that $u \in L^2(\mathbb{R})$, it means that there exists $v \in L^2(\mathbb{R})$ so that $xu = v$ in $L^2_{\mathrm{loc}}(\mathbb{R})$ (or $L^2_{\mathrm{loc}}(\mathbb{R})$). The derivative can actually be handled in a similar fashion using the Fourier transform but I will not do that here.

Rather, consider the following three 'L^2-based notions' of derivative.

Definition 24. (1) We say that $u \in L^2(\mathbb{R})$ has a *Sobolev derivative* if there exists a sequence $\phi_n \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ such that $\phi_n \to u$ in $L^2(\mathbb{R})$ and $\phi'_n \to v$ in $L^2(\mathbb{R})$, $\phi'_n = \frac{d\phi_n}{dx}$ in the usual sense of course.

(2) We say that $u \in L^2(\mathbb{R})$ has a *strong derivative* (in the L^2 sense) if the limit

$$\lim_{0 \neq s \to 0} \frac{u(x+s) - u(x)}{s} = \tilde{v} \text{ exists in } L^2(\mathbb{R}). \tag{4.236}$$

(3) Thirdly, we say that $u \in L^2(\mathbb{R})$ has a *weak derivative* in L^2 if there exists $w \in L^2(\mathbb{R})$ such that

$$(u, -\frac{df}{dx})_{L^2} = (w, f)_{L^2}\ \forall\ f \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R}). \tag{4.237}$$

In all cases, we will see that it is justified to write $v = \tilde{v} = w = \frac{du}{dx}$ because these defintions turn out to be equivalent. Of course *if* $u \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ then u is differentiable in each sense and the derivative is always du/dx – note that the integration by parts used to prove (4.237) is justified in that case. In fact we are most interested in the first and third of these definitions, the first two are both called 'strong derivatives.'

It is easy to see that the existence of a Sobolev derivative implies that this is also a weak derivative. Indeed, since ϕ_n, the approximating sequence whose existence is the definition of the Soboleve derivative, is in $\mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ so the integration by parts implicit in (4.237) is valid and so for all $f \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$,

$$(\phi_n, -\frac{df}{dx})_{L^2} = (\phi'_n, f)_{L^2}. \tag{4.238}$$

Since $\phi_n \to u$ in L^2 and $\phi'_n \to v$ in L^2 both sides of (4.238) converge to give the identity (4.237).

Before proceeding to the rest of the equivalence of these definitions we need to do some preparation. First let us investigate a little the consequence of the existence of a Sobolev derivative.

LEMMA 57. *If $u \in L^2(\mathbb{R})$ has a Sobolev derivative then $u \in \mathcal{C}(\mathbb{R})$ and there exists an unquely defined element $w \in L^2(\mathbb{R})$ such that*

$$u(x) - u(y) = \int_y^x w(s)ds \ \forall \ y \geq x \in \mathbb{R}. \tag{4.239}$$

PROOF. Suppose u has a Sobolev derivative, determined by some approximating sequence ϕ_n. Consider a general element $\psi \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$. Then $\tilde{\phi}_n = \psi\phi_n$ is a sequence in $\mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ and $\tilde{\phi}_n \to \psi u$ in L^2. Moreover, by the product rule for standard derivatives

$$\frac{d}{dx}\tilde{\phi}_n = \psi'\phi_n + \psi\phi_n' \to \psi' u + \psi w \text{ in } L^2(\mathbb{R}). \tag{4.240}$$

Thus in fact ψu also has a Sobolev derivative, namely $\phi' u + \psi w$ if w is the Sobolev derivative for u given by ϕ_n – which is to say that the product rule for derivatives holds under these conditions.

Now, the formula (4.239) comes from the Fundamental Theorem of Calculus which in this case really does apply to $\tilde{\phi}_n$ and shows that

$$\psi(x)\phi_n(x) - \psi(y)\phi_n(y) = \int_y^x (\frac{d\tilde{\phi}_n}{ds}(s))ds. \tag{4.241}$$

For any given $x = \bar{x}$ we can choose ψ so that $\psi(\bar{x}) = 1$ and then we can take y below the lower limit of the support of ψ so $\psi(y) = 0$. It follows that for this choice of ψ,

$$\phi_n(\bar{x}) = \int_y^{\bar{x}} (\psi'\phi_n(s) + \psi\phi_n'(s))ds. \tag{4.242}$$

Now, we can pass to the limit as $n \to \infty$ and the left side converges for each fixed $\bar{x}$ (with ψ fixed) since the integrand converges in L^2 and hence in L^1 on this compact interval. This actually shows that the limit $\phi_n(\bar{x})$ must exist for each fixed $\bar{x}$. In fact we can always choose ψ to be constant near a particular point and apply this argument to see that

$$\phi_n(x) \to u(x) \text{ locally uniformly on } \mathbb{R}. \tag{4.243}$$

That is, the limit exists locally uniformly, hence represents a continuous function but that continuous function must be equal to the original u almost everywhere (since $\psi\phi_n \to \psi u$ in L^2).

Thus in fact we conclude that '$u \in \mathcal{C}(\mathbb{R})$' (which really means that u *has a representative* which is continuous). Not only that but we get (4.239) from passing to the limit on both sides of

$$u(x) - u(y) = \lim_{n\to\infty}(\phi_n(x) - \phi_n(y)) = \lim_{n\to\infty}\int_y^s (\phi'(s))ds = \int_y^s w(s)ds. \tag{4.244}$$

□

One immediate consequence of this is

(4.245) The Sobolev derivative is unique if it exists.

Indeed, if w_1 and w_2 are both Sobolev derivatives then (4.239) holds for both of them, which means that $w_2 - w_1$ has vanishing integral on any finite interval and we know that this implies that $w_2 = w_1$ a.e.

So at least for Sobolev derivatives we are now justified in writing

$$w = \frac{du}{dx} \tag{4.246}$$

since w is unique and behaves like a derivative in the integral sense that (4.239) holds.

LEMMA 58. *If u has a Sobolev derivative then u has a stong derivative and if u has a strong derivative then this is also a weak derivative.*

PROOF. If u has a Sobolev derivative then (3.15) holds. We can use this to write the difference quotient as

$$\frac{u(x+s)-u(x)}{s} - w(x) = \frac{1}{s}\int_0^s (w(x+t)-w(x))dt \tag{4.247}$$

since the integral in the second term can be carried out. Using this formula twice the square of the L^2 norm, which is finite, is

$$\begin{aligned} &\|\frac{u(x+s)-u(x)}{s} - w(x)\|^2_{L^2} \\ &\qquad = \frac{1}{s^2}\int\int_0^s\int_0^s (w(x+t)-w(x)\overline{(w(x+t')-w(x))}dtdt'dx. \end{aligned} \tag{4.248}$$

There is a small issue of manupulating the integrals, but we can always 'back off a little' and replace u by the approximating sequence ϕ_n and then everything is fine – and we only have to check what happens at the end. Now, we can apply the Cauchy-Schwarz inequality as a triple integral. The two factors turn out to be the same so we find that

$$\|\frac{u(x+s)-u(x)}{s} - w(x)\|^2_{L^2} \le \frac{1}{s^2}\int\int_0^s\int_0^s |w(x+t)-w(x)|^2 dxdtdt'. \tag{4.249}$$

Now, something we checked long ago was that L^2 functions are 'continuous in the mean' in the sense that

$$\lim_{0\ne t\to 0}\int |w(x+t)-w(x)|^2 dx = 0. \tag{4.250}$$

Applying this to (4.249) and then estimating the t and t' integrals shows that

$$\frac{u(x+s)-u(x)}{s} - w(x) \to 0 \text{ in } L^2(\mathbb{R}) \text{ as } s \to 0. \tag{4.251}$$

By definition this means that u has w as a strong derivative. I leave it up to you to make sure that the manipulation of integrals is okay.

So, now suppose that u has a strong derivative, $\tilde{v}$. Obsever that if $f \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ then the limit defining the derivative

$$\lim_{0\ne s\to 0}\frac{f(x+s)-f(x)}{s} = f'(x) \tag{4.252}$$

is *uniform.* In fact this follows by writing down the Fundamental Theorem of Calculus, as in (4.239), again and using the properties of Riemann integrals. Now, consider

$$\begin{aligned} (u(x), \frac{f(x+s)-f(x)}{s})_{L^2} &= \frac{1}{s}\int u(x)\overline{f(x+s)}dx - \frac{1}{s}\int u(x)\overline{f(x)}dx \\ &= (\frac{u(x-s)-u(x)}{s}, f(x))_{L^2} \end{aligned} \tag{4.253}$$

where we just need to change the variable of integration in the first integral from x to $x+s$. However, letting $s \to 0$ the left side converges because of the uniform convergence of the difference quotient and the right side converges because of the assumed strong differentiability and as a result (noting that the parameter on the right is really $-s$)

$$(4.254) \qquad (u, \frac{df}{dx})_{L^2} = -(w, f)_{L^2} \ \forall \ f \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$$

which is weak differentiability with derivative $\tilde{v}$. □

So, at this point we know that Sobolev differentiabilty implies strong differentiability and either of the stong ones implies the weak. So it remains only to show that weak differentiability implies Sobolev differentiability and we can forget about the difference!

Before doing that, note again that a weak derivative, if it exists, is unique – since the difference of two would have to pair to zero in L^2 with all of $\mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ which is dense. Similarly, if u has a weak derivative then so does ψu for any $\psi \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ since we can just move ψ around in the integrals and see that

$$(4.255) \qquad \begin{aligned} (\psi u, -\frac{df}{dx}) &= (u, -\overline{\psi}\frac{df}{dx}) \\ &= (u, -\frac{d\overline{\psi}f}{dx}) + (u, \overline{\psi'}f) \\ = (w, \overline{\psi}f + (\psi' u, f) &= (\psi w + \psi' u, f) \end{aligned}$$

which also proves that the product formula holds for weak derivatives.

So, let us consider $u \in L^2_{\mathrm{c}}(\mathbb{R})$ which does have a weak derivative. To show that it has a Sobolev derivative we need to construct a sequence ϕ_n. We will do this by convolution.

Lemma 59. *If $\mu \in \mathcal{C}_c(\mathbb{R})$ then for any $u \in L^2_c(\mathbb{R})$,*

$$(4.256) \qquad \mu * u(x) = \int \mu(x-s)u(s)ds \in \mathcal{C}_c(\mathbb{R})$$

and if $\mu \in \mathcal{C}^1_c(\mathbb{R})$ then

$$(4.257) \qquad \mu * u(x) \in \mathcal{C}^1_c(\mathbb{R}), \ \frac{d\mu * u}{dx} = \mu' * u(x).$$

It folows that if μ has more continuous derivatives, then so does $\mu * u$.

Proof. Since u has compact support and is in L^2 it in L^1 so the integral in (4.256) exists for each $x \in \mathbb{R}$ and also vanishes if $|x|$ is large enough, since the integrand vanishes when the supports become separate – for some R, $\mu(x-s)$ is supported in $|s-x| \le R$ and $u(s)$ in $|s| < R$ which are disjoint for $|x| > 2R$. It is also clear that $\mu * u$ is continuous using the estimate (from uniform continuity of μ)

$$(4.258) \qquad |\mu * u(x') - \mu * u(x)| \le \sup |\mu(x-s) - \mu(x'-s)| \|u\|_{L^1}.$$

Similarly the difference quotient can be written

$$(4.259) \qquad \frac{\mu * u(x') - \mu * u(x)}{t} = \int \frac{\mu(x'-s) - \mu(x-s)}{s} u(s)ds$$

and the uniform convergence of the difference quotient shows that

$$\frac{d\mu * u}{dx} = \mu' * u. \tag{4.260}$$

□

One of the key properties of thes convolution integrals is that we can examine what happens when we 'concentrate' μ. Replace the one μ by the family

$$\mu_\epsilon(x) = \epsilon^{-1}\mu(\frac{x}{\epsilon}),\ \epsilon > 0. \tag{4.261}$$

The singular factor here is introduced so that $\int \mu_\epsilon$ is independent of $\epsilon > 0$,

$$\int \mu_\epsilon = \int \mu \ \forall\ \epsilon > 0. \tag{4.262}$$

Note that since μ has compact support, the support of μ_ϵ is concentrated in $|x| \le \epsilon R$ for some fixed R.

Lemma 60. *If $u \in L^2_c(\mathbb{R})$ and $0 \le \mu \in \mathcal{C}^1_c(\mathbb{R})$ then*

$$\lim_{0\ne\epsilon\to 0} \mu_\epsilon * u = (\int \mu) u \ \textit{in}\ L^2(\mathbb{R}). \tag{4.263}$$

In fact there is no need to assume that u has compact support for this to work.

Proof. First we can change the variable of integration in the definition of the convolution and write it intead as

$$\mu * u(x) = \int \mu(s)u(x-s)ds. \tag{4.264}$$

Now, the rest is similar to one of the arguments above. First write out the difference we want to examine as

$$\mu_\epsilon * u(x) - (\int \mu)(x) = \int_{|s|\le\epsilon R} \mu_\epsilon(s)(u(x-s) - u(x))ds. \tag{4.265}$$

Write out the square of the absolute value using the formula twice and we find that

$$\begin{aligned} &\int |\mu_\epsilon * u(x) - (\int \mu)(x)|^2 dx \\ &= \int \int_{|s|\le\epsilon R} \int_{|t|\le\epsilon R} \mu_\epsilon(s)\mu_\epsilon(t)(u(x-s) - u(x))\overline{(u(x-s) - u(x))} dsdtdx \end{aligned} \tag{4.266}$$

Now we can write the integrand as the product of two similar factors, one being

$$\mu_\epsilon(s)^{\frac12}\mu_\epsilon(t)^{\frac12}(u(x-s) - u(x)) \tag{4.267}$$

using the non-negativity of μ. Applying the Cauchy-Schwarz inequality to this we get two factors, which are again the same after relabelling variables, so

$$\int |\mu_\epsilon * u(x) - (\int \mu)(x)|^2 dx \le \int \int_{|s|\le\epsilon R} \int_{|t|\le\epsilon R} \mu_\epsilon(s)\mu_\epsilon(t)|u(x-s) - u(x)|^2. \tag{4.268}$$

The integral in x can be carried out first, then using continuity-in-the mean bounded by $J(s) \to 0$ as $\epsilon \to 0$ since $|s| < \epsilon R$. This leaves

$$\begin{aligned}(4.269) \qquad & \int |\mu_\epsilon * u(x) - (\int \mu) u(x)|^2 dx \\ & \le \sup_{|s|\le\epsilon R} J(s) \int_{|s|\le\epsilon R} \int_{|t|\le\epsilon R} \mu_\epsilon(s)\mu_\epsilon(t) = (\int \psi)^2 Y \sup_{|s|\le\epsilon R} \to 0.\end{aligned}$$

□

After all this preliminary work we are in a position to to prove the remaining part of 'weak=strong'.

Lemma 61. *If $u \in L^2(\mathbb{R})$ has w as a weak L^2-derivative then w is also the Sobolev derivative of u.*

Proof. Let's assume first that u has compact support, so we can use the discussion above. Then set $\phi_n = \mu_{1/n} * u$ where $\mu \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ is chosen to be non-negative and have integral $\int \mu = 0$; μ_ϵ is defined in (4.261). Now from Lemma 60 it follows that $\phi_n \to u$ in $L^2(\mathbb{R})$. Also, from Lemma 59, $\phi_n \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ has derivative given by (4.257). This formula can be written as a pairing in L^2 :

$$(4.270) \qquad (\mu_{1/n})' * u(x) = (u(s), -\frac{d\mu_{1/n}(x-s)}{ds})^2_L = (w(s), \frac{d\mu_{1/n}(x-s)}{ds})_{L^2}$$

using the definition of the weak derivative of u. It therefore follows from Lemma 60 applied again that

$$(4.271) \qquad \phi'_n = \mu_{/m1/n} * w \to w \text{ in } L^2(\mathbb{R}).$$

Thus indeed, ϕ_n is an approximating sequence showing that w is the Sobolev derivative of u.

In the general case that $u \in L^2(\mathbb{R})$ has a weak derivative but is not necessarily compactly supported, consider a function $\gamma \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$ with $\gamma(0) = 1$ and consider the sequence $v_m = \gamma(x)u(x)$ in $L^2(\mathbb{R})$ each element of which has compact support. Moreover, $\gamma(x/m) \to 1$ for each x so by Lebesgue dominated convergence, $v_m \to u$ in $L^2(\mathbb{R})$ as $m \to \infty$. As shown above, v_m has as weak derivative

$$\frac{d\gamma(x/m)}{dx} u + \gamma(x/m) w = \frac{1}{m}\gamma'(x/m)u + \gamma(x/m)w \to w$$

as $m \to \infty$ by the same argument applied to the second term and the fact that the first converges to 0 in $L^2(\mathbb{R})$. Now, use the approximating sequence $\mu_{1/n} * v_m$ discussed converges to v_m with its derivative converging to the weak derivative of v_m. Taking $n = N(m)$ sufficiently large for each m ensures that $\phi_m = \mu_{1/N(m)} * v_m$ converges to u and its sequence of derivatives converges to w in L^2. Thus the weak derivative is again a Sobolev derivative. □

Finally then we see that the three definitions are equivalent and we will freely denote the Sobolev/strong/weak derivative as du/dx or u'.

14. Fourier transform and L^2

Recall that one reason for proving the completeness of the Hermite basis was to apply it to prove some of the important facts about the Fourier transform, which we already know is a linear operator

$$L^1(\mathbb{R}) \longrightarrow \mathcal{C}^0_\infty(\mathbb{R}),\ \hat{u}(\xi) = \int e^{ix\xi}u(x)dx. \tag{4.272}$$

Namely we have already shown the effect of the Fourier transform on the 'ground state':

$$\mathcal{F}(u_0)(\xi) = \sqrt{2\pi}e_0(\xi). \tag{4.273}$$

By a similar argument we can check that

$$\mathcal{F}(u_j)(\xi) = \sqrt{2\pi}i^j u_j(\xi)\ \forall\ j \in \mathbb{N}. \tag{4.274}$$

As usual we can proceed by induction using the fact that $u_j = \operatorname{Cr} u_{j-1}$. The integrals involved here are very rapidly convergent at infinity, so there is no problem with the integration by parts in

$$\begin{aligned}
\mathcal{F}(\frac{d}{dx}u_{j-1}) &= \lim_{T\to\infty}\int_{-T}^{T} e^{-ix\xi}\frac{du_{j-1}}{dx}dx \\
&= \lim_{T\to\infty}\left(\int_{-T}^{T}(i\xi)e^{-ix\xi}u_{j-1}dx + \left[e^{-ix\xi}u_{j-1}(x)\right]_{-T}^{T}\right) = (i\xi)\mathcal{F}(u_{j-1}), \\
\mathcal{F}(xu_{j-1}) &= i\int\frac{de^{-ix\xi}}{d\xi}u_{j-1}dx = i\frac{d}{d\xi}\mathcal{F}(u_{j-1}).
\end{aligned} \tag{4.275}$$

Taken together these identities imply the validity of the inductive step:

$$\mathcal{F}(u_j) = \mathcal{F}((-\frac{d}{dx}+x)u_{j-1}) = (i(-\frac{d}{d\xi}+\xi)\mathcal{F}(u_{j-1}) = i\operatorname{Cr}(\sqrt{2\pi}i^{j-1}u_{j-1}) \tag{4.276}$$

so proving (4.274).

So, we have found an orthonormal basis for $L^2(\mathbb{R})$ with elements which are all in $L^1(\mathbb{R})$ and which are also eigenfunctions for $\mathcal{F}$.

Theorem 20. *The Fourier transform maps $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ and extends by continuity to an isomorphism of $L^2(\mathbb{R})$ such that $\frac{1}{\sqrt{2\pi}}\mathcal{F}$ is unitary with the inverse of $\mathcal{F}$ the continuous extension from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ of*

$$\mathcal{F}(f)(x) = \frac{1}{2\pi}\int e^{ix\xi}f(\xi). \tag{4.277}$$

Proof. This really is what we have already proved. The elements of the Hermite basis e_j are all in both $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ so if $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ its image under $\mathcal{F}$ is in $L^2(\mathbb{R})$ because we can compute the L^2 inner products and see that

$$(\mathcal{F}(u), e_j) = \int_{\mathbb{R}^2} e_j(\xi)e^{ix\xi}u(x)dxd\xi = \int\mathcal{F}(e_j)(x)u(x) = \sqrt{2\pi}i^j(u, e_j). \tag{4.278}$$

Now Bessel's inequality shows that $\mathcal{F}(u) \in L^2(\mathbb{R})$ (it is of course locally integrable since it is continuous).

Everything else now follows easily. □

Notice in particular that we have also proved Parseval's and Plancherel's identities for the Fourier transform:-

$$\|\mathcal{F}(u)\|_{L^2} = \sqrt{2\pi}\|u\|_{L^2},\ (\mathcal{F}(u), \mathcal{F}(v)) = 2\pi(u, v),\ \forall\ u, v \in L^2(\mathbb{R}). \tag{4.279}$$

Now there are lots of applications of the Fourier transform which we do not have the time to get into. However, let me just indicate the definitions of Sobolev spaces and Schwartz space and how they are related to the Fourier transform.

First Sobolev spaces. We now see that $\mathcal{F}$ maps $L^2(\mathbb{R})$ isomorphically onto $L^2(\mathbb{R})$ and we can see from (4.275) for instance that it 'turns differentiations by x into multiplication by ξ'. Of course we do not know how to differentiate L^2 functions so we have some problems making sense of this. One way, the usual mathematicians trick, is to turn what we want into a definition.

Definition 25. The Sobolev spaces of order s, for any $s \in (0, \infty)$, are defined as subspaces of $L^2(\mathbb{R})$:

$$H^s(\mathbb{R}) = \{u \in L^2(\mathbb{R}); (1 + |\xi|^2)^s\hat{u} \in L^2(\mathbb{R})\}. \tag{4.280}$$

It is natural to identify $H^0(\mathbb{R}) = L^2(\mathbb{R})$.

These Sobolev spaces, for each positive order s, are Hilbert spaces with the inner product and norm

$$(u, v)_{H^s} = \int (1 + |\xi|^2)^s\hat{u}(\xi)\overline{\hat{v}(\xi)},\ \|u\|_s = \|(1 + |\xi|^2)^{\frac{s}{2}}\hat{u}\|_{L^2}. \tag{4.281}$$

That they are pre-Hilbert spaces is clear enough. Completeness is also easy, given that we know the completeness of $L^2(\mathbb{R})$. Namely, if u_n is Cauchy in $H^s(\mathbb{R})$ then it follows from the fact that

$$\|v\|_{L^2} \le C\|v\|_s\ \forall\ v \in H^s(\mathbb{R}) \tag{4.282}$$

that u_n is Cauchy in L^2 and also that $(1 + |\xi|^2)^{\frac{s}{2}}\hat{u}_n(\xi)$ is Cauchy in L^2. Both therefore converge to a limit u in L^2 and the continuity of the Fourier transform shows that $u \in H^s(\mathbb{R})$ and that $u_n \to u$ in H^s.

These spaces are examples of what is discussed above where we have a dense inclusion of one Hilbert space in another, $H^s(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$. In this case the inclusion in *not* compact but it does give rise to a bounded self-adjoint operator on $L^2(\mathbb{R})$, $E_s : L^2(\mathbb{R}) \longrightarrow H^s(\mathbb{R}) \subset L^2(\mathbb{R})$ such that

$$(u, v)_{L^2} = (E_s u, E_s v)_{H^s}. \tag{4.283}$$

It is reasonable to denote this as $E_s = (1 + |D_x|^2)^{-\frac{s}{2}}$ since

$$u \in L^2(\mathbb{R}^n) \Longrightarrow \widehat{E_s u}(\xi) = (1 + |\xi|^2)^{-\frac{s}{2}}\hat{u}(\xi). \tag{4.284}$$

It is a form of 'fractional integration' which turns any $u \in L^2(\mathbb{R})$ into $E_s u \in H^s(\mathbb{R})$.

Having defined these spaces, which get smaller as s increases it can be shown for instance that if $n \ge s$ is an integer then the set of n times continuously differentiable functions on $\mathbb{R}$ which vanish outside a compact set are dense in H^s. This allows us to justify, by continuity, the following statement:-

Proposition 50. *The bounded linear map*

$$\frac{d}{dx} : H^s(\mathbb{R}) \longrightarrow H^{s-1}(\mathbb{R}),\ s \ge 1,\ v(x) = \frac{du}{dx} \Longleftrightarrow \hat{v}(\xi) = i\xi\hat{u}(\xi) \tag{4.285}$$

is consistent with differentiation on n times continuously differentiable functions of compact support, for any integer $n \ge s$.

In fact one can even get a 'strong form' of differentiation. The condition that $u \in H^1(\mathbb{R})$, that $u \in L^2$ 'has one derivative in L^2' is actually equivalent, for $u \in L^2(\mathbb{R})$ to the existence of the limit

$$\lim_{t\to 0} \frac{u(x+t)u(x)}{t} = v, \text{ in } L^2(\mathbb{R}) \tag{4.286}$$

and then $\hat{v} = i\xi\hat{u}$. Another way of looking at this is

$$\begin{gathered} u \in H^1(\mathbb{R}) \Longrightarrow u : \mathbb{R} \longrightarrow \mathbb{C} \text{ is continuous and} \\ u(x) - u(y) = \int_y^x v(t)dt, \ v \in L^2. \end{gathered} \tag{4.287}$$

If such a $v \in L^2(\mathbb{R})$ exists then it is unique – since the difference of two such functions would have to have integral zero over any finite interval and we know (from one of the exercises) that this implies that the function vansishes a.e.

One of the more important results about Sobolev spaces – of which there are many – is the relationship between these 'L^2 derivatives' and 'true derivatives'.

THEOREM 21 (Sobolev embedding). *If n is an integer and $s > n + \frac{1}{2}$ then*

$$H^s(\mathbb{R}) \subset \mathcal{C}^n_\infty(\mathbb{R}) \tag{4.288}$$

consists of n times continuosly differentiable functions with bounded derivatives to order n (which also vanish at infinity).

This is actually not so hard to prove, there are some hints in the exercises below.

These are not the only sort of spaces with 'more regularity' one can define and use. For instance one can try to treat x and ξ more symmetrically and define smaller spaces than the H^s above by setting

$$H^s_{\text{iso}}(\mathbb{R}) = \{u \in L^2(\mathbb{R}); (1+|\xi|^2)^{\frac{s}{2}}\hat{u} \in L^2(\mathbb{R}),\ (1+|x|^2)^{\frac{s}{2}}u \in L^2(\mathbb{R})\}. \tag{4.289}$$

The 'obvious' inner product with respect to which these 'isotropic' Sobolev spaces $H^s_{\text{iso}}(\mathbb{R})$ are indeed Hilbert spaces is

$$(u,v)_{s,\text{iso}} = \int_{\mathbb{R}} u\overline{v} + \int_{\mathbb{R}} |x|^{2s} u\overline{v} + \int_{\mathbb{R}} |\xi|^{2s}\hat{u}\overline{\hat{v}} \tag{4.290}$$

which makes them look rather symmetric between u and $\hat{u}$ and indeed

$$\mathcal{F} : H^s_{\text{iso}}(\mathbb{R}) \longrightarrow H^s_{\text{iso}}(\mathbb{R}) \text{ is an isomorphism } \forall\ s \geq 0. \tag{4.291}$$

At this point, by dint of a little, only moderately hard, work, it is possible to show that the harmonic oscillator extends by continuity to an isomorphism

$$H : H^{s+2}_{\text{iso}}(\mathbb{R}) \longrightarrow H^s_{\text{iso}}(\mathbb{R})\ \forall\ s \geq 2. \tag{4.292}$$

Finally in this general vein, I wanted to point out that Hilbert, and even Banach, spaces are not the end of the road! One very important space in relation to a direct treatment of the Fourier transform, is the Schwartz space. The definition is reasonably simple. Namely we denote Schwartz space by $\mathcal{S}(\mathbb{R})$ and say

$$\begin{gathered} u \in \mathcal{S}(\mathbb{R}) \Longleftrightarrow u : \mathbb{R} \longrightarrow \mathbb{C} \\ \text{is continuously differentiable of all orders and for every } n, \\ \|u\|_n = \sum_{k+p\leq n} \sup_{x\in\mathbb{R}} (1+|x|)^k |\frac{d^p u}{dx^p}| < \infty. \end{gathered} \tag{4.293}$$

All these inequalities just mean that all the derivatives of u are 'rapidly decreasing at ∞' in the sense that they stay bounded when multiplied by any polynomial.

So in fact we know already that $\mathcal{S}(\mathbb{R})$ is not empty since the elements of the Hermite basis, $e_j \in \mathcal{S}(\mathbb{R})$ for all j. In fact it follows immediately from this that

$$\mathcal{S}(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \text{ is dense.} \tag{4.294}$$

If you want to try your hand at something a little challenging, see if you can check that

$$\mathcal{S}(\mathbb{R}) = \bigcap_{s>0} H^s_{\text{iso}}(\mathbb{R}) \tag{4.295}$$

which uses the Sobolev embedding theorem above.

As you can see from the definition in (4.293), $\mathcal{S}(\mathbb{R})$ is not likely to be a Banach space. Each of the $\|\cdot\|_n$ is a norm. However, $\mathcal{S}(\mathbb{R})$ is pretty clearly not going to be complete with respect to any one of these. However it is complete with respect to all, countably many, norms. What does this mean? In fact $\mathcal{S}(\mathbb{R})$ is a *metric space* with the metric

$$d(u,v) = \sum_n 2^{-n} \frac{\|u-v\|_n}{1+\|u-v\|_n} \tag{4.296}$$

as you can check. So the claim is that $\mathcal{S}(\mathbb{R})$ *is* comlete as a metric space – such a thing is called a Fréchet space.

What has this got to do with the Fourier transform? The point is that

$$\mathcal{F} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}) \text{ is an isomorphism and } \mathcal{F}(\frac{du}{dx}) = i\xi\mathcal{F}(u),\ \mathcal{F}(xu) = -i\frac{d\mathcal{F}(u)}{d\xi} \tag{4.297}$$

where this now makes sense. The dual space of $\mathcal{S}(\mathbb{R})$ – the space of continuous linear functionals on it, is the space, denoted $\mathcal{S}'(\mathbb{R})$, of tempered distributions on $\mathbb{R}$.

15. Dirichlet problem

As a final application, which I do not have time to do in full detail in lectures, I want to consider the Dirichlet problem again, but now in higher dimensions. Of course this is a small issue, since I have not really gone through the treatment of the Lebesgue integral etc in higher dimensions – still I hope it is clear that with a little more application we could do it and for the moment I will just pretend that we have.

So, what is the issue? Consider Laplace's equation on an open set in $\mathbb{R}^n$. That is, we want to find a solution of

$$-(\frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2^2} + \cdots + \frac{\partial^2 u(x)}{\partial x_n^2}) = f(x) \text{ in } \Omega \subset \mathbb{R}^n. \tag{4.298}$$

Now, maybe some of you have not had a rigorous treatment of partical derivatives either. Just add that to the heap of unresolved issues. In any case, partial derivatives are just one-dimensional derivatives in the variable concerned with the other variables held fixed. So, we are looking for a function u which has *all* partial derivatives up to order 2 existing everywhere and continous. So, f will have to be continuous too. Unfortunately this is *not* enough to guarantee the existence of a twice continuously differentiable solution – later we will just suppose that f itself is once continuously differentiable.

Now, we want a solution of (4.298) which satisfies the Dirichlet condition. For this we need to have a reasonable domain, which has a decent boundary. To short cut the work involved, let's just suppose that $0 \in \Omega$ and that it is given by an inequality of the sort

$$\Omega = \{z \in \mathbb{R}^n; |z| < \rho(z/|z|) \tag{4.299}$$

where ρ is another once continuously differentiable, and strictly positive, function on $\mathbb{R}^n$ (although we only care about its values on the unit vectors). So, this is no worse than what we are already dealing with.

Now, the Dirichlet condition can be stated as

$$u \in \mathcal{C}^0(\overline{\Omega}),\ u\big|z| = \rho(z/|z|) = 0. \tag{4.300}$$

Here we need the first condition to make much sense of the second.

So, what I want to approach is the following result – which can be improved a lot and which I will not quite manage to prove anyway.

Theorem 22. *If* $0 < \rho \in \mathcal{C}^1(\mathbb{R}^n)$, *and* $f \in \mathcal{C}^1(\mathbb{R}^n)$ *then there exists a unique* $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ *satisfying* (4.298) *and* (4.300).

CHAPTER 5

Problems and solutions

1. Problems – Chapter 1

Missing or badly referenced:-

Outline of finite-dimensional theory.
Quotient space.
Norm from seminorm.
Norm on quotient and completeness.
Completness of the completion.
Subspace of functions vanishing at infinity.
Completeness of the space of k times differentiable functions.
Direct proof of open mapping.

Problem 5.1. Show from first principles that if V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) then for any set X the space

$$\mathcal{F}(X;V)=\{u:X\longrightarrow V\} \tag{5.1}$$

is a linear space over the same field, with 'pointwise operations'.

Problem 5.2. If V is a vector space and $S\subset V$ is a subset which is closed under addition and scalar multiplication:

$$v_1,v_2\in S,\ \lambda\in\mathbb{K}\Longrightarrow v_1+v_2\in S \text{ and } \lambda v_1\in S \tag{5.2}$$

then S is a vector space as well (called of course a subspace).

Problem 5.3. If $S\subset V$ be a linear subspace of a vector space show that the relation on V

$$v_1\sim v_2\Longleftrightarrow v_1-v_2\in S \tag{5.3}$$

is an equivalence relation and that the set of equivalence classes, denoted usually V/S, is a vector space in a natural way.

Problem 5.4. In case you do not know it, go through the basic theory of finite-dimensional vector spaces. Define a vector space V to be finite-dimensional if there is an integer N such that any N elements of V are linearly dependent – if $v_i\in V$ for $i=1,\dots N$, then there exist $a_i\in\mathbb{K}$, not *all* zero, such that

$$\sum_{i=1}^{N}a_iv_i=0 \text{ in } V. \tag{5.4}$$

Call the smallest such integer the dimension of V and show that a finite dimensional vector space always has a basis, $e_i\in V$, $i=1,\dots,\dim V$ such that any element of

V can be written uniquely as a linear combination

$$v = \sum_{i=1}^{\dim V} b_i e_i, \ b_i \in \mathbb{K}. \tag{5.5}$$

Problem 5.5. Recall the notion of a linear map between vector spaces (discussed above) and show that between two finite dimensional vector spaces V and W over the same field

(1) If $\dim V \le \dim W$ then there is an injective linear map $L : V \longrightarrow W$.
(2) If $\dim V \ge W$ then there is a surjective linear map $L : V \longrightarrow W$.
(3) if $\dim V = \dim W$ then there is a linear isomorphism $L : V \longrightarrow W$, i.e. an injective and surjective linear map.

Problem 5.6. Show that any two norms on a finite dimensional vector space are equivalent.

Problem 5.7. Show that if two norms on a vector space are equivalent then the topologies induced are the same – the sets open with respect to the distance from one are open with respect to the distance coming from the other. The converse is also true, you can use another result from this section to prove it.

Problem 5.8. Write out a proof (you can steal it from one of many places but at least write it out in your own hand) either for $p = 2$ or for each p with $1 \le p < \infty$ that

$$l^p = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_{j=1}^{\infty} |a_j|^p < \infty, \ a_j = a(j)\}$$

is a normed space with the norm

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}}.$$

This means writing out the proof that this is a linear space and that the three conditions required of a norm hold.

Problem 5.9. Prove directly that each l^p as defined in Problem 5.8 is complete, i.e. it is a Banach space.

Problem 5.10. The space l^∞ consists of the bounded sequences

$$l^\infty = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sup_n |a_n| < \infty\}, \ \|a\|_\infty = \sup_n |a_n|. \tag{5.6}$$

Show that it is a Banach space.

Problem 5.11. Another closely related space consists of the sequences converging to 0 :

$$c_0 = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \lim_{n\to\infty} a_n = 0\}, \ \|a\|_\infty = \sup_n |a_n|. \tag{5.7}$$

Check that this is a Banach space and that it is a closed subspace of l^∞ (perhaps in the opposite order).

Problem 5.12. Consider the 'unit sphere' in l^p. This is the set of vectors of length 1 :

$$S = \{a \in l^p; \|a\|_p = 1\}.$$

(1) Show that S is closed.
(2) Recall the sequential (so not the open covering definition) characterization of compactness of a set in a metric space (e.g. by checking in Rudin's book).
(3) Show that S is not compact by considering the sequence in l^p with kth element the sequence which is all zeros except for a 1 in the kth slot. Note that the main problem is not to get yourself confused about sequences of sequences!

Problem 5.13. Show that the norm on any normed space is continuous.

Problem 5.14. Finish the proof of the completeness of the space B constructed in the second proof of Theorem 1.

2. Hints for some problems

Hint 1 (Problem 5.9). You need to show that each Cauchy sequence converges. The problem here is to find the limit of a given Cauchy sequence. Show that for each N the sequence in $\mathbb{C}^N$ obtained by truncating each of the elements at point N is Cauchy with respect to the norm in Problem 5.2 on $\mathbb{C}^N$. Show that this is the same as being Cauchy in $\mathbb{C}^N$ in the usual sense (if you are doing $p = 2$ it is already the usual sense) and hence, this cut-off sequence converges. Use this to find a putative limit of the Cauchy sequence and then check that it works.

3. Solutions to problems

Solution 5.1 (5.1). If V is a vector space (over $\mathbb{K}$ which is $\mathbb{R}$ or $\mathbb{C}$) then for any set X consider

$$\mathcal{F}(X;V) = \{u : X \longrightarrow V\}. \tag{5.8}$$

Addition and scalar multiplication are defined 'pointwise':

$$(u+v)(x) = u(x) + v(x),\ (cu)(x) = cu(x),\ u, v \in \mathcal{F}(X;V),\ c \in \mathbb{K}. \tag{5.9}$$

These are well-defined functions since addition and multiplication are defined in $\mathbb{K}$.

So, one needs to check all the axioms of a vector space. Since an equality of functions is just equality at all points, these all follow from the corresponding identities for $\mathbb{K}$.

Solution 5.2 (5.2). If $S \subset V$ is a (non-empty) subset of a vector space and $S \subset V$ which is closed under addition and scalar multiplication:

$$v_1,\ v_2 \in S,\ \lambda \in \mathbb{K} \Longrightarrow v_1 + v_2 \in S \text{ and } \lambda v_1 \in S \tag{5.10}$$

then $0 \in S$, since $0 \in \mathbb{K}$ and for any $v \in S$, $0v = 0 \in S$. Similarly, if $v \in S$ then $-v = (-1)v \in S$. Then all the axioms of a vector space follow from the corresponding identities in V.

Solution 5.3. If $S \subset V$ be a linear subspace of a vector space consider the relation on V

$$v_1 \sim v_2 \Longleftrightarrow v_1 - v_2 \in S. \tag{5.11}$$

To say that this is an equivalence relation means that symmetry and transitivity hold. Since S is a subspace, $v \in S$ implies $-v \in S$ so

$$v_1 \sim v_2 \Longrightarrow v_1 - v_2 \in S \Longrightarrow v_2 - v_1 \in S \Longrightarrow v_2 \sim v_1.$$

Similarly, since it is also possible to add and remain in S

$$v_1 \sim v_2,\ v_2 \sim v_3 \Longrightarrow v_1 - v_2,\ v_2 - v_3 \in S \Longrightarrow v_1 - v_3 \in S \Longrightarrow v_1 \sim v_3.$$

So this is an equivalence relation and the quotient $V/\sim = V/S$ is well-defined – where the latter is notation. That is, and element of V/S is an equivalence class of elements of V which can be written $v + S$:

$$v + S = w + S \Longleftrightarrow v - w \in S. \tag{5.12}$$

Now, we can check the axioms of a linear space once we define addition and scalar multiplication. Notice that

$$(v + S) + (w + S) = (v + w) + S,\ \lambda(v + S) = \lambda v + S$$

are well-defined elements, independent of the choice of representatives, since adding an lement of S to v or w does not change the right sides.

Now, to the axioms. These amount to showing that S is a zero element for addition, $-v + S$ is the additive inverse of $v + S$ and that the other axioms follow directly from the fact that the hold as identities in V.

Solution 5.4 (5.4). In case you do not know it, go through the basic theory of finite-dimensional vector spaces. Define a vector space V to be *finite-dimensional* if there is an integer N such that any $N + 1$ elements of V are linearly dependent in the sense that the satisfy a non-trivial dependence relation – if $v_i \in V$ for $i = 1, \dots N + 1$, then there exist $a_i \in \mathbb{K}$, not *all* zero, such that

$$\sum_{i=1}^{N+1} a_i v_i = 0 \text{ in } V. \tag{5.13}$$

Call the smallest such integer the dimension of V – it is also the largest integer such that there are N linearly independent vectors – and show that a finite dimensional vector space always has a basis, $e_i \in V$, $i = 1, \dots, \dim V$ which are *not* linearly dependent and such that any element of V can be written as a linear combination

$$v = \sum_{i=1}^{\dim V} b_i e_i,\ b_i \in \mathbb{K}. \tag{5.14}$$

Solution 5.5 (5.6). Show that any two norms on a finite dimensional vector space are equivalent.

Solution 5.6 (5.7). Show that if two norms on a vector space are equivalent then the topologies induced are the same – the sets open with respect to the distance from one are open with respect to the distance coming from the other. The converse is also true, you can use another result from this section to prove it.

Solution 5.7 (5.8). Write out a proof (you can steal it from one of many places but at least write it out in your own hand) either for $p = 2$ or for each p with $1 \le p < \infty$ that

$$l^p = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_{j=1}^{\infty} |a_j|^p < \infty,\ a_j = a(j)\}$$

is a normed space with the norm

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{\frac{1}{p}}.$$

This means writing out the proof that this is a linear space and that the three conditions required of a norm hold.

SOLUTION 5.8 (). The 'tricky' part in Problem 5.1 is the triangle inequality. Suppose you knew – meaning I tell you – that for each N

$$\left(\sum_{j=1}^{N} |a_j|^p \right)^{\frac{1}{p}} \text{ is a norm on } \mathbb{C}^N$$

would that help?

SOLUTION 5.9 (5.9). Prove directly that each l^p as defined in Problem 5.1 is complete, i.e. it is a Banach space. At the risk of offending some, let me say that this means showing that each Cauchy sequence converges. The problem here is to find the limit of a given Cauchy sequence. Show that for each N the sequence in $\mathbb{C}^N$ obtained by truncating each of the elements at point N is Cauchy with respect to the norm in Problem 5.2 on $\mathbb{C}^N$. Show that this is the same as being Cauchy in $\mathbb{C}^N$ in the usual sense (if you are doing $p = 2$ it is already the usual sense) and hence, this cut-off sequence converges. Use this to find a putative limit of the Cauchy sequence and then check that it works.

SOLUTION 5.10 (5.10). The space l^∞ consists of the bounded sequences

$$l^\infty = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sup_n |a_n| < \infty\}, \ \|a\|_\infty = \sup_n |a_n|. \tag{5.15}$$

Show that it is a Banach space.

SOLUTION 5.11 (5.11). Another closely related space consists of the sequences converging to 0 :

$$c_0 = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \lim_{n\to\infty} a_n = 0\}, \ \|a\|_\infty = \sup_n |a_n|. \tag{5.16}$$

Check that this is a Banach space and that it is a closed subspace of l^∞ (perhaps in the opposite order).

SOLUTION 5.12 (5.12). Consider the 'unit sphere' in l^p. This is the set of vectors of length 1 :

$$S = \{a \in l^p; \|a\|_p = 1\}.$$

(1) Show that S is closed.
(2) Recall the sequential (so not the open covering definition) characterization of compactness of a set in a metric space (e .g . by checking in Rudin).
(3) Show that S is not compact by considering the sequence in l^p with kth element the sequence which is all zeros except for a 1 in the kth slot. Note that the main problem is not to get yourself confused about sequences of sequences!

SOLUTION 5.13 (5.13). Since the distance between two points is $\|x - y\|$ the continuity of the norm follows directly from the 'reverse triangle inequality'

$$|\|x\| - \|y\|| \le \|x - y\| \tag{5.17}$$

which in turn follows from the triangle inequality applied twice:-

$$\|x\| \le \|x - y\| + \|y\|, \ \|y\| \le \|x - y\| + \|x\|. \tag{5.18}$$

4. Problems – Chapter 2

Missing

PROBLEM 5.15. Let's consider an example of an absolutely summable sequence of step functions. For the interval $[0, 1)$ (remember there is a strong preference for left-closed but right-open intervals for the moment) consider a variant of the construction of the standard Cantor subset based on 3 proceeding in steps. Thus, remove the 'central interval $[1/3, 2/3)$. This leave $C_1 = [0, 1/3) \cup [2/3, 1)$. Then remove the central interval from each of the remaining two intervals to get $C_2 = [0, 1/9) \cup [2/9, 1/3) \cup [2/3, 7/9) \cup [8/9, 1)$. Carry on in this way to define successive sets $C_k \subset C_{k-1}$, each consisting of a finite union of semi-open intervals. Now, consider the *series* of step functions f_k where $f_k(x) = 1$ on C_k and 0 otherwise.

(1) Check that this is an absolutely summable series.
(2) For which $x \in [0, 1)$ does $\sum\limits_k |f_k(x)|$ converge?
(3) Describe a function on $[0, 1)$ which is shown to be Lebesgue integrable (as defined in Lecture 4) by the existence of this series and compute its Lebesgue integral.
(4) Is this function Riemann integrable (this is easy, not hard, if you check the definition of Riemann integrability)?
(5) Finally consider the function g which is equal to one on the union of all the intervals which are *removed* in the construction and zero elsewhere. Show that g is Lebesgue integrable and compute its integral.

PROBLEM 5.16. The covering lemma for $\mathbb{R}^2$. By a rectangle we will mean a set of the form $[a_1, b_1) \times [a_2, b_2)$ in $\mathbb{R}^2$. The area of a rectangle is $(b_1 - a_1) \times (b_2 - a_2)$.

(1) We may subdivide a rectangle by subdividing either of the intervals – replacing $[a_1, b_1)$ by $[a_1, c_1) \cup [c_1, b_1)$. Show that the sum of the areas of rectangles made by any repeated subdivision is always the same as that of the original.
(2) Suppose that a finite collection of disjoint rectangles has union a rectangle (always in this same half-open sense). Show, and I really mean prove, that the sum of the areas is the area of the whole rectange. Hint:- proceed by subdivision.
(3) Now show that for any countable collection of disjoint rectangles contained in a given rectange the sum of the areas is less than or equal to that of the containing rectangle.
(4) Show that if a finite collection of rectangles has union *containing* a given rectange then the sum of the areas of the rectangles is at least as large of that of the rectangle contained in the union.
(5) Prove the extension of the preceeding result to a countable collection of rectangles with union containing a given rectangle.

PROBLEM 5.17. (1) Show that any continuous function on $[0,1]$ is the *uniform limit* on $[0,1)$ of a sequence of step functions. Hint:- Reduce to the real case, divide the interval into 2^n equal pieces and define the step functions to take infimim of the continuous function on the corresponding interval. Then use uniform convergence.

(2) By using the 'telescoping trick' show that any continuous function on $[0,1)$ can be written as the sum

$$\sum_i f_j(x) \ \forall \ x \in [0,1) \tag{5.19}$$

where the f_j are step functions and $\sum\limits_j |f_j(x)| < \infty$ for all $x \in [0,1)$.

(3) Conclude that any continuous function on $[0,1]$, extended to be 0 outside this interval, is a Lebesgue integrable function on $\mathbb{R}$ and show that the Lebesgue integral is equal to the Riemann integral.

PROBLEM 5.18. If f and $g \in \mathcal{L}^1(\mathbb{R})$ are Lebesgue integrable functions on the line show that

(1) If $f(x) \geq 0$ a.e. then $\int f \geq 0$.
(2) If $f(x) \leq g(x)$ a.e. then $\int f \leq \int g$.
(3) If f is complex valued then its real part, $\operatorname{Re} f$, is Lebesgue integrable and $|\int \operatorname{Re} f| \leq \int |f|$.
(4) For a general complex-valued Lebesgue integrable function

$$|\int f| \leq \int |f|. \tag{5.20}$$

Hint: You can look up a proof of this easily enough, but the usual trick is to choose $\theta \in [0,2\pi)$ so that $e^{i\theta}\int f = \int (e^{i\theta} f) \geq 0$. Then apply the preceeding estimate to $g = e^{i\theta} f$.

(5) Show that the integral is a continuous linear functional

$$\int : L^1(\mathbb{R}) \longrightarrow \mathbb{C}. \tag{5.21}$$

PROBLEM 5.19. If $I \subset \mathbb{R}$ is an interval, including possibly $(-\infty, a)$ or (a, ∞), we *define* Lebesgue integrability of a function $f : I \longrightarrow \mathbb{C}$ to mean the Lebesgue integrability of

$$\tilde{f} : \mathbb{R} \longrightarrow \mathbb{C},\ \tilde{f}(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I. \end{cases} \tag{5.22}$$

The integral of f on I is then defined to be

$$\int_I f = \int \tilde{f}. \tag{5.23}$$

(1) Show that the space of such integrable functions on I is linear, denote it $\mathcal{L}^1(I)$.
(2) Show that is f is integrable on I then so is $|f|$.
(3) Show that if f is integrable on I and $\int_I |f| = 0$ then $f = 0$ a.e. in the sense that $f(x) = 0$ for all $x \in I \setminus E$ where $E \subset I$ is of measure zero as a subset of $\mathbb{R}$.
(4) Show that the set of null functions as in the preceeding question is a linear space, denote it $\mathcal{N}(I)$.

(5) Show that $\int_I |f|$ defines a norm on $L^1(I) = \mathcal{L}^1(I)/\mathcal{N}(I)$.
(6) Show that if $f \in \mathcal{L}^1(\mathbb{R})$ then

$$g : I \longrightarrow \mathbb{C},\ g(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I \end{cases} \tag{5.24}$$

is integrable on I.
(7) Show that the preceeding construction gives a *surjective and continuous* linear map 'restriction to I'

$$L^1(\mathbb{R}) \longrightarrow L^1(I). \tag{5.25}$$

(Notice that these are the quotient spaces of integrable functions modulo equality a.e.)

Problem 5.20. Really continuing the previous one.
(1) Show that if $I = [a, b)$ and $f \in L^1(I)$ then the restriction of f to $I_x = [x, b)$ is an element of $L^1(I_x)$ for all $a \le x < b$.
(2) Show that the function

$$F(x) = \int_{I_x} f : [a, b) \longrightarrow \mathbb{C} \tag{5.26}$$

is continuous.
(3) Prove that the function $x^{-1}\cos(1/x)$ is not Lebesgue integrable on the interval $(0, 1]$. Hint: Think about it a bit and use what you have shown above.

Problem 5.21. [Harder but still doable] Suppose $f \in \mathcal{L}^1(\mathbb{R})$.
(1) Show that for each $t \in \mathbb{R}$ the translates

$$f_t(x) = f(x - t) : \mathbb{R} \longrightarrow \mathbb{C} \tag{5.27}$$

are elements of $\mathcal{L}^1(\mathbb{R})$.
(2) Show that

$$\lim_{t \to 0} \int |f_t - f| = 0. \tag{5.28}$$

This is called 'Continuity in the mean for integrable functions'. Hint: I will add one!
(3) Conclude that for each $f \in \mathcal{L}^1(\mathbb{R})$ the map (it is a 'curve')

$$\mathbb{R} \ni t \longmapsto [f_t] \in L^1(\mathbb{R}) \tag{5.29}$$

is continuous.

Problem 5.22. In the last problem set you showed that a continuous function on a compact interval, extended to be zero outside, is Lebesgue integrable. Using this, and the fact that step functions are dense in $L^1(\mathbb{R})$ show that the linear space of continuous functions on $\mathbb{R}$ each of which vanishes outside a compact set (which depends on the function) form a dense subset of $L^1(\mathbb{R})$.

Problem 5.23. (1) If $g : \mathbb{R} \longrightarrow \mathbb{C}$ is bounded and continuous and $f \in \mathcal{L}^1(\mathbb{R})$ show that $gf \in \mathcal{L}^1(\mathbb{R})$ and that

$$\int |gf| \le \sup_{\mathbb{R}} |g| \cdot \int |f|. \tag{5.30}$$

(2) Suppose now that $G \in \mathcal{C}([0,1]\times[0,1])$ is a continuous function (I use $\mathcal{C}(K)$ to denote the continuous functions on a compact metric space). Recall from the preceeding discussion that we have defined $L^1([0,1])$. Now, using the first part show that if $f \in L^1([0,1])$ then

$$F(x) = \int_{[0,1]} G(x,\cdot)f(\cdot) \in \mathbb{C} \tag{5.31}$$

(where $\cdot$ is the variable in which the integral is taken) is well-defined for each $x \in [0,1]$.

(3) Show that for each $f \in L^1([0,1])$, F is a continuous function on $[0,1]$.

(4) Show that

$$L^1([0,1]) \ni f \longmapsto F \in \mathcal{C}([0,1]) \tag{5.32}$$

is a bounded (i.e. continuous) linear map into the Banach space of continuous functions, with supremum norm, on $[0,1]$.

Problem 5.24. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be an element of $\mathcal{L}^1(\mathbb{R})$. Define

$$f_L(x) = \begin{cases} f(x) & x \in [-L,L] \\ 0 & \text{otherwise.} \end{cases} \tag{5.33}$$

Show that $f_L \in \mathcal{L}^1(\mathbb{R})$ and that $\int |f_L - f| \to 0$ as $L \to \infty$.

Problem 5.25. Consider a real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is locally integrable in the sense that

$$g_L(x) = \begin{cases} f(x) & x \in [-L,L] \\ 0 & x \in \mathbb{R} \setminus [-L,L] \end{cases} \tag{5.34}$$

is Lebesgue integrable of each $L \in \mathbb{N}$.

(1) Show that for each fixed L the function

$$g_L^{(N)}(x) = \begin{cases} g_L(x) & \text{if } g_L(x) \in [-N,N] \\ N & \text{if } g_L(x) > N \\ -N & \text{if } g_L(x) < -N \end{cases} \tag{5.35}$$

is Lebesgue integrable.

(2) Show that $\int |g_L^{(N)} - g_L| \to 0$ as $N \to \infty$.

(3) Show that there is a sequence, h_n, of step functions such that

$$h_n(x) \to f(x) \text{ a.e. in } \mathbb{R}. \tag{5.36}$$

(4) Defining

$$h_{n,L}^{(N)} = \begin{cases} 0 & x \notin [-L,L] \\ h_n(x) & \text{if } h_n(x) \in [-N,N],\ x \in [-L,L] \\ N & \text{if } h_n(x) > N,\ x \in [-L,L] \\ -N & \text{if } h_n(x) < -N,\ x \in [-L,L] \end{cases}. \tag{5.37}$$

Show that $\int |h_{n,L}^{(N)} - g_L^{(N)}| \to 0$ as $n \to \infty$.

Problem 5.26. Show that $\mathcal{L}^2(\mathbb{R})$ is a Hilbert space.

First working with real functions, define $\mathcal{L}^2(\mathbb{R})$ as the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which are locally integrable and such that $|f|^2$ is integrable.

(1) For such f choose h_n and define g_L, $g_L^{(N)}$ and $h_n^{(N)}$ by (5.34), (5.35) and (5.37).
(2) Show using the sequence $h_{n,L}^{(N)}$ for fixed N and L that $g_L^{(N)}$ and $(g_L^{(N)})^2$ are in $\mathcal{L}^1(\mathbb{R})$ and that $\int |(h_{n,L}^{(N)})^2 - (g_L^{(N)})^2| \to 0$ as $n \to \infty$.
(3) Show that $(g_L)^2 \in \mathcal{L}^1(\mathbb{R})$ and that $\int |(g_L^{(N)})^2 - (g_L)^2| \to 0$ as $N \to \infty$.
(4) Show that $\int |(g_L)^2 - f^2| \to 0$ as $L \to \infty$.
(5) Show that f, $g \in \mathcal{L}^2(\mathbb{R})$ then $fg \in \mathcal{L}^1(\mathbb{R})$ and that

$$|\int fg| \le \int |fg| \le \|f\|_{L^2}\|g\|_{L^2},\ \|f\|_{L^2}^2 = \int |f|^2. \tag{5.38}$$

(6) Use these constructions to show that $\mathcal{L}^2(\mathbb{R})$ is a linear space.
(7) Conclude that the quotient space $L^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R})/\mathcal{N}$, where $\mathcal{N}$ is the space of null functions, is a real Hilbert space.
(8) Extend the arguments to the case of complex-valued functions.

Problem 5.27. Consider the sequence space

$$h^{2,1} = \left\{ c : \mathbb{N} \ni j \longmapsto c_j \in \mathbb{C}; \sum_j (1+j^2)|c_j|^2 < \infty \right\}. \tag{5.39}$$

(1) Show that

$$h^{2,1} \times h^{2,1} \ni (c,d) \longmapsto \langle c, d\rangle = \sum_j (1+j^2) c_j \overline{d_j} \tag{5.40}$$

is an Hermitian inner form which turns $h^{2,1}$ into a Hilbert space.
(2) Denoting the norm on this space by $\|\cdot\|_{2,1}$ and the norm on l^2 by $\|\cdot\|_2$, show that

$$h^{2,1} \subset l^2,\ \|c\|_2 \le \|c\|_{2,1}\ \forall\ c \in h^{2,1}. \tag{5.41}$$

Problem 5.28. In the separable case, prove Riesz Representation Theorem directly.

Choose an orthonormal basis $\{e_i\}$ of the separable Hilbert space H. Suppose $T : H \longrightarrow \mathbb{C}$ is a bounded linear functional. Define a sequence

$$w_i = \overline{T(e_i)},\ i \in \mathbb{N}. \tag{5.42}$$

(1) Now, recall that $|Tu| \le C\|u\|_H$ for some constant C. Show that for every finite N,

$$\sum_{j=1}^{N} |w_i|^2 \le C^2. \tag{5.43}$$

(2) Conclude that $\{w_i\} \in l^2$ and that

$$w = \sum_i w_i e_i \in H. \tag{5.44}$$

(3) Show that

$$T(u) = \langle u, w\rangle_H\ \forall\ u \in H \text{ and } \|T\| = \|w\|_H. \tag{5.45}$$

PROBLEM 5.29. If $f \in L^1(\mathbb{R}^k \times \mathbb{R}^p)$ show that there exists a set of measure zero $E \subset \mathbb{R}^k$ such that

$$x \in \mathbb{R}^k \setminus E \Longrightarrow g_x(y) = f(x,y) \text{ defines } g_x \in L^1(\mathbb{R}^p), \tag{5.46}$$

that $F(x) = \int g_x$ defines an element $F \in L^1(\mathbb{R}^k)$ and that

$$\int_{\mathbb{R}^k} F = \int_{\mathbb{R}^k \times \mathbb{R}^p} f. \tag{5.47}$$

Note: These identities are usually written out as an equality of an iterated integral and a 'regular' integral:

$$\int_{\mathbb{R}^k} \int_{\mathbb{R}^p} f(x,y) = \int f. \tag{5.48}$$

It is often used to 'exchange the order of integration' since the hypotheses are the same if we exchange the variables.

5. Solutions to problems

PROBLEM 5.30. Suppose that $f \in \mathcal{L}^1(0, 2\pi)$ is such that the constants

$$c_k = \int_{(0,2\pi)} f(x) e^{-ikx},\ k \in \mathbb{Z},$$

satisfy

$$\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty.$$

Show that $f \in \mathcal{L}^2(0, 2\pi)$.

Solution. So, this was a good bit harder than I meant it to be – but still in principle solvable (even though no one quite got to the end).

First, (for half marks in fact!) we know that the c_k exists, since $f \in \mathcal{L}^1(0, 2\pi)$ and e^{-ikx} is continuous so $fe^{-ikx} \in \mathcal{L}^1(0, 2\pi)$ and then the condition $\sum\limits_k |c_k|^2 < \infty$ implies that the Fourier series does converge in $L^2(0, 2\pi)$ so there is a function

$$g = \frac{1}{2\pi} \sum_{k \in \mathbb{C}} c_k e^{ikx}. \tag{5.49}$$

Now, what we want to show is that $f = g$ a .e . since then $f \in \mathcal{L}^2(0, 2\pi)$.

Set $h = f - g \in \mathcal{L}^1(0, 2\pi)$ since $\mathcal{L}^2(0, 2\pi) \subset \mathcal{L}^1(0, 2\pi)$. It follows from (5.49) that f and g have the same Fourier coefficients, and hence that

$$\int_{(0,2\pi)} h(x) e^{ikx} = 0 \ \forall\ k \in \mathbb{Z}. \tag{5.50}$$

So, we need to show that this implies that $h = 0$ a .e . Now, we can recall from class that we showed (in the proof of the completeness of the Fourier basis of L^2) that these exponentials are dense, in the supremum norm, in continuous functions which vanish near the ends of the interval. Thus, by continuity of the integral we know that

$$\int_{(0,2\pi)} hg = 0 \tag{5.51}$$

for all such continuous functions g. We also showed at some point that we can find such a sequence of continuous functions g_n to approximate the characteristic function of any interval χ_I. It is not true that $g_n \to \chi_I$ uniformly, but for any integrable function h, $hg_n \to h\chi_I$ in $\mathcal{L}^1$. So, the upshot of this is that we know a bit more than (5.51), namely we know that

$$\int_{(0,2\pi)} hg = 0 \ \forall \text{ step functions } g. \tag{5.52}$$

So, now the trick is to show that (5.52) implies that $h = 0$ almost everywhere. Well, this would follow if we know that $\int_{(0,2\pi)} |h| = 0$, so let's aim for that. Here is the trick. Since $g \in \mathcal{L}^1$ we know that there is a sequence (the partial sums of an absolutely convergent series) of step functions h_n such that $h_n \to g$ both in $L^1(0, 2\pi)$ and almost everywhere and also $|h_n| \to |h|$ in both these senses. Now, consider the functions

$$s_n(x) = \begin{cases} 0 & \text{if } h_n(x) = 0 \\ \frac{\overline{h_n(x)}}{|h_n(x)|} & \text{otherwise.} \end{cases} \tag{5.53}$$

Clearly s_n is a sequence of step functions, bounded (in absolute value by 1 in fact) and such that $s_n h_n = |h_n|$. Now, write out the wonderful identity

$$|h(x)| = |h(x)| - |h_n(x)| + s_n(x)(h_n(x) - h(x)) + s_n(x)h(x). \tag{5.54}$$

Integrate this identity and then apply the triangle inequality to conclude that

$$\begin{aligned} \int_{(0,2\pi)} |h| &= \int_{(0,2\pi)} (|h(x)| - |h_n(x)| + \int_{(0,2\pi)} s_n(x)(h_n - h) \\ &\le \int_{(0,2\pi)} (||h(x)| - |h_n(x)|| + \int_{(0,2\pi)} |h_n - h| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{5.55}$$

Here on the first line we have used (5.52) to see that the third term on the right in (5.54) integrates to zero. Then the fact that $|s_n| \le 1$ and the convergence properties.

Thus in fact $h = 0$ a .e . so indeed $f = g$ and $f \in \mathcal{L}^2(0, 2\pi)$. Piece of cake, right! Mia culpa.

6. Problems – Chapter 3

Problem 5.31. Let H be a normed space in which the norm satisfies the parallelogram law:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2) \ \forall \ u, v \in H. \tag{5.56}$$

Show that the norm comes from a positive definite sesquilinear (i.e. ermitian) inner product. Big Hint:- Try

$$(u, v) = \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2\right)! \tag{5.57}$$

Problem 5.32. Let H be a finite dimensional (pre)Hilbert space. So, by definition H has a basis $\{v_i\}_{i=1}^n$, meaning that any element of H can be written

$$v = \sum_i c_i v_i \tag{5.58}$$

and there is no dependence relation between the v_i's – the presentation of $v = 0$ in the form (5.58) is unique. Show that H has an orthonormal basis, $\{e_i\}_{i=1}^n$ satisfying

$(e_i, e_j) = \delta_{ij}$ (= 1 if $i = j$ and 0 otherwise). Check that for the orthonormal basis the coefficients in (5.58) are $c_i = (v, e_i)$ and that the map

$$T : H \ni v \longmapsto ((v, e_i)) \in \mathbb{C}^n \tag{5.59}$$

is a linear isomorphism with the properties

$$(u, v) = \sum_i (Tu)_i \overline{(Tv)_i},\ \|u\|_H = \|Tu\|_{\mathbb{C}^n}\ \forall\ u, v \in H. \tag{5.60}$$

Why is a finite dimensional preHilbert space a Hilbert space?

Problem 5.33. : Prove (3.149). The important step is actually the fact that $\operatorname{Spec}(A) \subset [-\|A\|, \|A\|]$ if A is self-adjoint, which is proved somewhere above. Now, if f is a real polynomial, we can assume the leading constant, c, in (3.148) is 1. If $\lambda \notin f([-\|A\|, \|A\|])$ then $f(A)$ is self-adjoint and $\lambda - f(A)$ is invertible – it is enough to check this for each factor in (3.148). Thus $\operatorname{Spec}(f(A)) \subset f([-\|A\|, \|A\|])$ which means that

$$\|f(A)\| \le \sup\{z \in f([-\|A\|, \|A\|])\} \tag{5.61}$$

which is in fact (3.148).

Problem 5.34. Let H be a separable Hilbert space. Show that $K \subset H$ is compact if and only if it is closed, bounded and has the property that any sequence in K which is weakly convergent sequence in H is (strongly) convergent.

Hint (Problem 5.34) In one direction use the result from class that any bounded sequence has a weakly convergent subsequence.

Problem 5.35. Show that, in a separable Hilbert space, a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if the weak limit, v satisfies

$$\|v\|_H = \lim_{n\to\infty} \|v_n\|_H. \tag{5.62}$$

Hint (Problem 5.35) To show that this condition is sufficient, expand

$$(v_n - v, v_n - v) = \|v_n\|^2 - 2\operatorname{Re}(v_n, v) + \|v\|^2. \tag{5.63}$$

Problem 5.36. Show that a subset of a separable Hilbert space is compact if and only if it is closed and bounded and has the property of 'finite dimensional approximation' meaning that for any $\epsilon > 0$ there exists a linear subspace $D_N \subset H$ of finite dimension such that

$$d(K, D_N) = \sup_{u\in K} \inf_{v\in D_N} \{d(u, v)\} \le \epsilon. \tag{5.64}$$

See Hint 6

Hint (Problem 5.36) To prove necessity of this condition use the 'equi-small tails' property of compact sets with respect to an orthonormal basis. To use the finite dimensional approximation condition to show that any weakly convergent sequence in K is strongly convergent, use the convexity result from class to define the sequence $\{v_n'\}$ in D_N where v_n' is the closest point in D_N to v_n. Show that v_n' is weakly, hence strongly, convergent and hence deduce that $\{v_n\}$ is Cauchy.

Problem 5.37. Suppose that $A : H \longrightarrow H$ is a bounded linear operator with the property that $A(H) \subset H$ is finite dimensional. Show that if v_n is weakly convergent in H then Av_n is strongly convergent in H.

PROBLEM 5.38. Suppose that H_1 and H_2 are two different Hilbert spaces and $A : H_1 \longrightarrow H_2$ is a bounded linear operator. Show that there is a unique bounded linear operator (the adjoint) $A^* : H_2 \longrightarrow H_1$ with the property

$$(Au_1, u_2)_{H_2} = (u_1, A^* u_2)_{H_1} \ \forall \ u_1 \in H_1, \ u_2 \in H_2. \tag{5.65}$$

PROBLEM 5.39. Question:- Is it possible to show the completeness of the Fourier basis

$$\exp(ikx)/\sqrt{2\pi}$$

by computation? Maybe, see what you think. These questions are also intended to get you to say things clearly.

(1) Work out the Fourier coefficients $c_k(t) = \int_{(0,2\pi)} f_t e^{-ikx}$ of the step function

$$f_t(x) = \begin{cases} 1 & 0 \le x < t \\ 0 & t \le x \le 2\pi \end{cases} \tag{5.66}$$

for each fixed $t \in (0, 2\pi)$.

(2) Explain why this Fourier series converges to f_t in $L^2(0, 2\pi)$ if and only if

$$2 \sum_{k>0} |c_k(t)|^2 = 2\pi t - t^2, \ t \in (0, 2\pi). \tag{5.67}$$

(3) Write this condition out as a Fourier series and apply the argument again to show that the completeness of the Fourier basis implies identities for the sum of k^{-2} and k^{-4}.

(4) Can you explain how reversing the argument, that knowledge of the sums of these two series should imply the completeness of the Fourier basis? There is a serious subtlety in this argument, and you get full marks for spotting it, without going ahead a using it to prove completeness.

PROBLEM 5.40. Prove that for appropriate choice of constants d_k, the functions $d_k \sin(kx/2)$, $k \in \mathbb{N}$, form an orthonormal basis for $L^2(0, 2\pi)$.

See Hint 6

Hint (Problem 5.40 The usual method is to use the basic result from class plus translation and rescaling to show that $d'_k \exp(ikx/2)$ $k \in \mathbb{Z}$ form an orthonormal basis of $L^2(-2\pi, 2\pi)$. Then extend functions as odd from $(0, 2\pi)$ to $(-2\pi, 2\pi)$.

PROBLEM 5.41. Let e_k, $k \in \mathbb{N}$, be an orthonormal basis in a separable Hilbert space, H. Show that there is a uniquely defined bounded linear operator $S : H \longrightarrow H$, satisfying

$$Se_j = e_{j+1} \ \forall \ j \in \mathbb{N}. \tag{5.68}$$

Show that if $B : H \longrightarrow H$ is a bounded linear operator then $S + \epsilon B$ is *not* invertible if $\epsilon < \epsilon_0$ for some $\epsilon_0 > 0$.

Hint (Problem 5.41)- Consider the linear functional $L : H \longrightarrow \mathbb{C}$, $Lu = (Bu, e_1)$. Show that $B'u = Bu - (Lu)e_1$ is a bounded linear operator from H to the Hilbert space $H_1 = \{u \in H; (u, e_1) = 0\}$. Conclude that $S + \epsilon B'$ is invertible as a linear map from H to H_1 for small ϵ. Use this to argue that $S + \epsilon B$ cannot be an isomorphism from H to H by showing that either e_1 is not in the range or else there is a non-trivial element in the null space.

PROBLEM 5.42. Show that the product of bounded operators on a Hilbert space is strong continuous, in the sense that if A_n and B_n are strong convergent sequences of bounded operators on H with limits A and B then the product A_nB_n is strongly convergent with limit AB.

Hint (Problem 5.42) Be careful! Use the result in class which was deduced from the Uniform Boundedness Theorem.

PROBLEM 5.43. Show that a continuous function $K : [0,1] \longrightarrow L^2(0,2\pi)$ has the property that the Fourier series of $K(x) \in L^2(0,2\pi)$, for $x \in [0,1]$, converges uniformly in the sense that if $K_n(x)$ is the sum of the Fourier series over $|k| \le n$ then $K_n : [0,1] \longrightarrow L^2(0,2\pi)$ is also continuous and

$$\sup_{x\in[0,1]} \|K(x) - K_n(x)\|_{L^2(0,2\pi)} \to 0. \tag{5.69}$$

Hint (Problem 5.43) Use one of the properties of compactness in a Hilbert space that you proved earlier.

PROBLEM 5.44. Consider an integral operator acting on $L^2(0,1)$ with a kernel which is continuous – $K \in \mathcal{C}([0,1]^2)$. Thus, the operator is

$$Tu(x) = \int_{(0,1)} K(x,y)u(y). \tag{5.70}$$

Show that T is bounded on L^2 (I think we did this before) and that it is in the norm closure of the finite rank operators.

Hint (Problem 5.43) Use the previous problem! Show that a continuous function such as K in this Problem defines a continuous map $[0,1] \ni x \longmapsto K(x,\cdot) \in \mathcal{C}([0,1])$ and hence a continuous function $K : [0,1] \longrightarrow L^2(0,1)$ then apply the previous problem with the interval rescaled.

Here is an even more expanded version of the hint: You can think of $K(x,y)$ as a continuous function of x with values in $L^2(0,1)$. Let $K_n(x,y)$ be the continuous function of x and y given by the previous problem, by truncating the Fourier series (in y) at some point n. Check that this defines a finite rank operator on $L^2(0,1)$ – yes it maps into continuous functions but that is fine, they are Lebesgue square integrable. Now, the idea is the difference $K - K_n$ defines a bounded operator with small norm as n becomes large. It might actually be clearer to do this the other way round, exchanging the roles of x and y.

PROBLEM 5.45. Although we have concentrated on the Lebesgue integral in one variable, you proved at some point the covering lemma in dimension 2 and that is pretty much all that was needed to extend the discussion to 2 dimensions. Let's just assume you have assiduously checked everything and so you know that $L^2((0,2\pi)^2)$ is a Hilbert space. Sketch a proof – noting anything that you are not sure of – that the functions $\exp(ikx+ily)/2\pi$, $k,l \in \mathbb{Z}$, form a complete orthonormal basis.

PROBLEM 5.46. Let H be a separable (partly because that is mostly what I have been talking about) Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Say that a sequence u_n in H converges weakly if (u_n,v) is Cauchy in $\mathbb{C}$ for each $v \in H$.

(1) Explain why the sequence $\|u_n\|_H$ is bounded.

Solution: Each u_n defines a continuous linear functional on H by

$$T_n(v) = (v, u_n),\ \|T_n\| = \|u_n\|, T_n : H \longrightarrow \mathbb{C}. \tag{5.71}$$

For fixed v the sequence $T_n(v)$ is Cauchy, and hence bounded, in $\mathbb{C}$ so by the 'Uniform Boundedness Principle' the $\|T_n\|$ are bounded, hence $\|u_n\|$ is bounded in $\mathbb{R}$.

(2) Show that there exists an element $u \in H$ such that $(u_n, v) \to (u, v)$ for each $v \in H$.

Solution: Since (v, u_n) is Cauchy in $\mathbb{C}$ for each fixed $v \in H$ it is convergent. Set

$$Tv = \lim_{n\to\infty}(v, u_n) \text{ in } \mathbb{C}. \tag{5.72}$$

This is a linear map, since

$$T(c_1v_1 + c_2v_2) = \lim_{n\to\infty} c_1(v_1, u_n) + c_2(v_2, u) = c_1Tv_1 + c_2Tv_2 \tag{5.73}$$

and is bounded since $|Tv| \le C\|v\|$, $C = \sup_n \|u_n\|$. Thus, by Riesz' theorem there exists $u \in H$ such that $Tv = (v, u)$. Then, by definition of T,

$$(u_n, v) \to (u, v)\ \forall\ v \in H. \tag{5.74}$$

(3) If e_i, $i \in \mathbb{N}$, is an orthonormal sequence, give, with justification, an example of a sequence u_n which is *not* weakly convergent in H but is such that (u_n, e_j) converges for each j.

Solution: One such example is $u_n = ne_n$. Certainly $(u_n, e_i) = 0$ for all $i > n$, so converges to 0. However, $\|u_n\|$ is not bounded, so the sequence cannot be weakly convergent by the first part above.

(4) Show that if the e_i form an orthonormal basis, $\|u_n\|$ is bounded and (u_n, e_j) converges for each j then u_n converges weakly.

Solution: By the assumption that (u_n, e_j) converges for all j it follows that (u_n, v) converges as $n \to \infty$ for all v which is a finite linear combination of the e_i. For general $v \in H$ the convergence of the Fourier-Bessell series for v with respect to the orthonormal basis e_j

$$v = \sum_k (v, e_k)e_k \tag{5.75}$$

shows that there is a sequence $v_k \to v$ where each v_k is in the finite span of the e_j. Now, by Cauchy's inequality

$$|(u_n, v) - (u_m, v)| \le |(u_nv_k) - (u_m, v_k)| + |(u_n, v - v_k)| + |(u_m, v - v_k)|. \tag{5.76}$$

Given $\epsilon > 0$ the boundedness of $\|u_n\|$ means that the last two terms can be arranged to be each less than $\epsilon/4$ by choosing k sufficiently large. Having chosen k the first term is less than $\epsilon/4$ if $n, m > N$ by the fact that (u_n, v_k) converges as $n \to \infty$. Thus the sequence (u_n, v) is Cauchy in $\mathbb{C}$ and hence convergent.

Problem 5.47. Consider the two spaces of sequences

$$h_{\pm 2} = \{c : \mathbb{N} \longmapsto \mathbb{C}; \sum_{j=1}^{\infty} j^{\pm 4}|c_j|^2 < \infty\}.$$

Show that both $h_{\pm 2}$ are Hilbert spaces and that any linear functional satisfying

$$T : h_2 \longrightarrow \mathbb{C},\ |Tc| \le C\|c\|_{h_2}$$

for some constant C is of the form

$$Tc = \sum_{j=1}^{\infty} c_i d_i$$

where $d : \mathbb{N} \longrightarrow \mathbb{C}$ is an element of h_{-2}.

Solution: Many of you hammered this out by parallel with l^2. This is fine, but to prove that $h_{\pm 2}$ are Hilbert spaces we can actually use l^2 itself. Thus, consider the maps on complex sequences

$$(T^{\pm} c)_j = c_j j^{\pm 2}. \tag{5.77}$$

Without knowing anything about $h_{\pm 2}$ this is a bijection between the sequences in $h_{\pm 2}$ and those in l^2 which takes the norm

$$\|c\|_{h_{\pm 2}} = \|Tc\|_{l^2}. \tag{5.78}$$

It is also a linear map, so it follows that $h_{\pm}$ are linear, and that they are indeed Hilbert spaces with $T^{\pm}$ isometric isomorphisms onto l^2; The inner products on $h_{\pm 2}$ are then

$$(c, d)_{h_{\pm 2}} = \sum_{j=1}^{\infty} j^{\pm 4} c_j \overline{d_j}. \tag{5.79}$$

Don't feel bad if you wrote it all out, it is good for you!

Now, once we know that h_2 is a Hilbert space we can apply Riesz' theorem to see that any continuous linear functional $T : h_2 \longrightarrow \mathbb{C}$, $|Tc| \le C\|c\|_{h_2}$ is of the form

$$Tc = (c, d')_{h_2} = \sum_{j=1}^{\infty} j^4 c_j \overline{d'_j},\ d' \in h_2. \tag{5.80}$$

Now, if $d' \in h_2$ then $d_j = j^4 d'_j$ defines a sequence in h_{-2}. Namely,

$$\sum_j j^{-4} |d_j|^2 = \sum_j j^4 |d'_j|^2 < \infty. \tag{5.81}$$

Inserting this in (5.80) we find that

$$Tc = \sum_{j=1}^{\infty} c_j d_j,\ d \in h_{-2}. \tag{5.82}$$

(1) In P9.2 (2), and elsewhere, $\mathcal{C}^{\infty}(\mathbb{S})$ should be $\mathcal{C}^0(\mathbb{S})$, the space of continuous functions on the circle – with supremum norm.
(2) In (5.95) it should be $u = Fv$, not $u = Sv$.
(3) Similarly, before (5.96) it should be $u = Fv$.
(4) Discussion around (5.98) clarified.
(5) Last part of P10.2 clarified.

This week I want you to go through the invertibility theory for the operator

$$Qu = (-\frac{d^2}{dx^2} + V(x))u(x) \tag{5.83}$$

acting on periodic functions. Since we have not developed the theory to handle this directly we need to approach it through integral operators.

PROBLEM 5.48. Let $\mathbb{S}$ be the circle of radius 1 in the complex plane, centered at the origin, $\mathbb{S} = \{z; |z| = 1\}$.

(1) Show that there is a 1-1 correspondence

$$\begin{aligned}\mathcal{C}^0(\mathbb{S}) = \{u : \mathbb{S} \longrightarrow \mathbb{C},\ \text{continuous}\} \longrightarrow \\ \{u : \mathbb{R} \longrightarrow \mathbb{C};\ \text{continuous and satisfying } u(x + 2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}.\end{aligned} \tag{5.84}$$

(2) Show that there is a 1-1 correspondence

$$\begin{aligned}L^2(0, 2\pi) \longleftrightarrow \{u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}); u\big|_{(0,2\pi)} \in \mathcal{L}^2(0, 2\pi) \\ \text{and } u(x + 2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}/\mathcal{N}_P\end{aligned} \tag{5.85}$$

where $\mathcal{N}_P$ is the space of null functions on $\mathbb{R}$ satisfying $u(x + 2\pi) = u(x)$ for all $x \in \mathbb{R}$.

(3) If we denote by $L^2(\mathbb{S})$ the space on the left in (5.85) show that there is a dense inclusion

$$\mathcal{C}^0(\mathbb{S}) \longrightarrow L^2(\mathbb{S}). \tag{5.86}$$

So, the idea is that we can think of functions on $\mathbb{S}$ as 2π-periodic functions on $\mathbb{R}$.

Next are some problems dealing with Schrödinger's equation, or at least it is an example thereof:

$$-\frac{d^2u(x)}{dx^2} + V(x)u(x) = f(x),\ x \in \mathbb{R}, \tag{5.87}$$

(1) First we will consider the special case $V = 1$. Why not $V = 0$? – Don't try to answer this until the end!

(2) Recall how to solve the differential equation

$$-\frac{d^2u(x)}{dx^2} + u(x) = f(x),\ x \in \mathbb{R}, \tag{5.88}$$

where $f(x) \in \mathcal{C}^0(\mathbb{S})$ is a continuous, 2π-periodic function on the line. Show that there is a unique 2π-periodic and twice continuously differentiable function, u, on $\mathbb{R}$ satisfying (5.88) and that this solution can be written in the form

$$u(x) = (Sf)(x) = \int_{0,2\pi} A(x, y)f(y) \tag{5.89}$$

where $A(x, y) \in \mathcal{C}^0(\mathbb{R}^2)$ satisfies $A(x+2\pi, y+2\pi) = A(x, y)$ for all $(x, y) \in \mathbb{R}$.

Extended hint: In case you managed to avoid a course on differential equations! First try to find a solution, igonoring the periodicity issue. To do so one can (for example, there are other ways) factorize the differential operator involved, checking that

$$-\frac{d^2u(x)}{dx^2} + u(x) = -(\frac{dv}{dx} + v) \text{ if } v = \frac{du}{dx} - u \tag{5.90}$$

since the cross terms cancel. Then recall the idea of integrating factors to see that

$$\begin{aligned}\frac{du}{dx}-u&=e^{x}\frac{d\phi}{dx},\ \phi=e^{-x}u,\\ \frac{dv}{dx}+v&=e^{-x}\frac{d\psi}{dx},\ \psi=e^{x}v.\end{aligned} \tag{5.91}$$

Now, solve the problem by integrating twice from the origin (say) and hence get a solution to the differential equation (5.88). Write this out explicitly as a double integral, and then change the order of integration to write the solution as

$$u'(x)=\int_{0,2\pi}A'(x,y)f(y)dy \tag{5.92}$$

where A' is continuous on $\mathbb{R}\times[0,2\pi]$. Compute the difference $u'(2\pi)-u'(0)$ and $\frac{du'}{dx}(2\pi)-\frac{du'}{dx}(0)$ as integrals involving f. Now, add to u' as solution to the *homogeneous* equation, for $f=0$, namely $c_1e^{x}+c_2e^{-x}$, so that the new solution to (5.88) satisfies $u(2\pi)=u(0)$ and $\frac{du}{dx}(2\pi)=\frac{du}{dx}(0)$. Now, check that u is given by an integral of the form (5.89) with A as stated.

(3) Check, either directly or indirectly, that $A(y,x)=A(x,y)$ and that A is real.

(4) Conclude that the operator S extends by continuity to a bounded operator on $L^2(\mathbb{S})$.

(5) Check, probably indirectly rather than directly, that

$$S(e^{ikx})=(k^2+1)^{-1}e^{ikx},\ k\in\mathbb{Z}. \tag{5.93}$$

(6) Conclude, either from the previous result or otherwise that S is a compact self-adjoint operator on $L^2(\mathbb{S})$.

(7) Show that if $g\in\mathcal{C}^0(\mathbb{S}))$ then Sg is twice continuously differentiable. Hint: Proceed directly by differentiating the integral.

(8) From (5.93) conclude that $S=F^2$ where F is also a compact self-adjoint operator on $L^2(\mathbb{S})$ with eigenvalues $(k^2+1)^{-\frac{1}{2}}$.

(9) Show that $F:L^2(\mathbb{S})\longrightarrow\mathcal{C}^0(\mathbb{S})$.

(10) Now, going back to the real equation (5.87), we assume that V is continuous, real-valued and 2π-periodic. Show that if u is a twice-differentiable 2π-periodic function satisfying (5.87) for a given $f\in\mathcal{C}^0(\mathbb{S})$ then

$$u+S((V-1)u)=Sf \text{ and hence } u=-F^2((V-1)u)+F^2f \tag{5.94}$$

and hence conclude that

$$u=Fv \text{ where } v\in L^2(\mathbb{S}) \text{ satisfies } v+(F(V-1)F)v=Ff \tag{5.95}$$

where $V-1$ is the operator defined by multiplication by $V-1$.

(11) Show the converse, that if $v\in L^2(\mathbb{S})$ satisfies

$$v+(F(V-1)F)v=Ff,\ f\in\mathcal{C}^0(\mathbb{S}) \tag{5.96}$$

then $u=Fv$ is 2π-periodic and twice-differentiable on $\mathbb{R}$ and satisfies (5.87).

(12) Apply the Spectral theorem to $F(V-1)F$ (including why it applies) and show that there is a sequence λ_j in $\mathbb{R}\setminus\{0\}$ with $|\lambda_j|\to 0$ such that for all $\lambda\in\mathbb{C}\setminus\{0\}$, the equation

$$\lambda v+(F(V-1)F)v=g,\ g\in L^2(\mathbb{S}) \tag{5.97}$$

has a unique solution for every $g\in L^2(\mathbb{S})$ if and only if $\lambda\neq\lambda_j$ for any j.

(13) Show that for the λ_j the solutions of

$$\lambda_j v+(F(V-1)F)v=0, v\in L^2(\mathbb{S}), \tag{5.98}$$

are all continuous 2π-periodic functions on $\mathbb{R}$.

(14) Show that the corresponding functions $u=Fv$ where v satisfies (5.98) are all twice continuously differentiable, 2π-periodic functions on $\mathbb{R}$ satisfying

$$-\frac{d^2u}{dx^2}+(1-s_j+s_jV(x))u(x)=0,\ s_j=1/\lambda_j. \tag{5.99}$$

(15) Conversely, show that if u is a twice continuously differentiable and 2π-periodic function satisfying

$$-\frac{d^2u}{dx^2}+(1-s+sV(x))u(x)=0,\ s\in\mathbb{C}, \tag{5.100}$$

and u is not identically 0 then $s=s_j$ for some j.

(16) Finally, conclude that Fredholm's alternative holds for the equation (5.87)

Theorem 23. *For a given real-valued, continuous 2π-periodic function V on $\mathbb{R}$, either* (5.87) *has a unique twice continuously differentiable, 2π-periodic, solution for each f which is continuous and 2π-periodic or else there exists a finite, but positive, dimensional space of twice continuously differentiable 2π-periodic solutions to the homogeneous equation*

$$-\frac{d^2w(x)}{dx^2}+V(x)w(x)=0,\ x\in\mathbb{R}, \tag{5.101}$$

and (5.87) *has a solution if and only if $\int_{(0,2\pi)}fw=0$ for every 2π-periodic solution, w, to* (5.101).

Problem 5.49. Check that we really can understand all the 2π periodic eigenfunctions of the Schrödinger operator using the discussion above. First of all, there was nothing sacred about the addition of 1 to $-d^2/dx^2$, we could add any positive number and get a similar result – the problem with 0 is that the constants satisfy the homogeneous equation $d^2u/dx^2=0$. What we have shown is that the operator

$$u\longmapsto Qu=-\frac{d^2u}{dx^2}u+Vu \tag{5.102}$$

applied to twice continuously differentiable functions has at least a left inverse unless there is a non-trivial solution of

$$-\frac{d^2u}{dx^2}u+Vu=0. \tag{5.103}$$

Namely, the left inverse is $R=F(\mathrm{Id}+F(V-1)F)^{-1}F$. This is a compact self-adjoint operator. Show – and there is still a bit of work to do – that (twice continuously differentiable) eigenfunctions of Q, meaning solutions of $Qu=\tau u$ are precisely the non-trivial solutions of $Ru=\tau^{-1}u$.

What to do in case (5.103) does have a non-trivial solution? Show that the space of these is finite dimensional and conclude that essentially the same result holds by working on the orthocomplement in $L^2(\mathbb{S})$.

By now you should have become reasonably comfortable with a separable Hilbert space such as l_2. However, it is worthwhile checking once again that it is rather large – if you like, let me try to make you uncomfortable for one last time. An important result in this direction is Kuiper's theorem, which I will *not* ask you to prove[1]. However, I want you to go through the closely related result sometimes known as *Eilenberg's swindle*. Perhaps you will appreciate the little bit of trickery. First some preliminary results. Note that everything below is a closed curve in the $x \in [0,1]$ variable – you might want to identify this with a circle instead, I just did it the primitive way.

PROBLEM 5.50. Let H be a separable, infinite dimensional Hilbert space. Show that the direct sum of two copies of H is a Hilbert space with the norm

$$H \oplus H \ni (u_1, u_2) \longmapsto (\|u_1\|_H^2 + \|u_2\|_H^2)^{\frac{1}{2}} \tag{5.104}$$

either by constructing an isometric isomorphism

$$T : H \longrightarrow H \oplus H,\ \text{1-1 and onto},\ \|u\|_H = \|Tu\|_{H\oplus H} \tag{5.105}$$

or otherwise. In any case, construct a map as in (5.105).

PROBLEM 5.51. One can repeat the preceding construction any finite number of times. Show that it can be done 'countably often' in the sense that if H is a separable, infinite dimensional, Hilbert space then

$$l_2(H) = \{u : \mathbb{N} \longrightarrow H; \|u\|_{l_2(H)}^2 = \sum_i \|u_i\|_H^2 < \infty\} \tag{5.106}$$

has a Hilbert space structure and construct an explicit isometric isomorphism from $l_2(H)$ to H.

PROBLEM 5.52. Recall, or perhaps learn about, the winding number of a closed curve with values in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We take as given the following fact:[2] If $Q = [0,1]^N$ and $f : Q \longrightarrow \mathbb{C}^*$ is continuous then for each choice of $b \in \mathbb{C}$ satisfying $\exp(2\pi i b) = f(0)$, there exists a unique continuous function $F : Q \longrightarrow \mathbb{C}$ satisfying

$$\exp(2\pi i F(q)) = f(q),\ \forall\ q \in Q \text{ and } F(0) = b. \tag{5.107}$$

Of course, you are free to change b to $b + n$ for any $n \in \mathbb{Z}$ but then F changes to $F + n$, just shifting by the same integer.

(1) Now, suppose $c : [0,1] \longrightarrow \mathbb{C}^*$ is a closed curve – meaning it is continuous and $c(1) = c(0)$. Let $C : [0,1] \longrightarrow \mathbb{C}$ be a choice of F for $N = 1$ and $f = c$. Show that the winding number of the closed curve c may be defined *unambiguously* as

$$\operatorname{wn}(c) = C(1) - C(0) \in \mathbb{Z}. \tag{5.108}$$

[1]Kuiper's theorem says that for any (norm) continuous map, say from any compact metric space, $g : M \longrightarrow \mathrm{GL}(H)$ with values in the invertible operators on a separable infinite-dimensional Hilbert space there exists a continuous map, an homotopy, $h : M \times [0,1] \longrightarrow \mathrm{GL}(H)$ such that $h(m,0) = g(m)$ and $h(m,1) = \mathrm{Id}_H$ for all $m \in M$.

[2]Of course, you are free to give a proof – it is not hard.

(2) Show that $\operatorname{wn}(c)$ is *constant under homotopy*. That is if $c_i : [0,1] \longrightarrow \mathbb{C}^*$, $i = 1, 2$, are two closed curves so $c_i(1) = c_i(0)$, $i = 1, 2$, which are *homotopic* through closed curves in the sense that there exists $f : [0,1]^2 \longrightarrow \mathbb{C}^*$ continuous and such that $f(0,x) = c_1(x)$, $f(1,x) = c_2(x)$ for all $x \in [0,1]$ and $f(y,0) = f(y,1)$ for all $y \in [0,1]$, then $\operatorname{wn}(c_1) = \operatorname{wn}(c_2)$.

(3) Consider the closed curve $L_n : [0,1] \ni x \longmapsto e^{2\pi ix} \operatorname{Id}_{n\times n}$ of $n\times n$ matrices. Using the standard properties of the determinant, show that this curve is not homotopic to the identity through closed curves in the sense that there does not exist a continuous map $G : [0,1]^2 \longrightarrow \operatorname{GL}(n)$, with values in the invertible $n \times n$ matrices, such that $G(0,x) = L_n(x)$, $G(1,x) \equiv \operatorname{Id}_{n\times n}$ for all $x \in [0,1]$, $G(y,0) = G(y,1)$ for all $y \in [0,1]$.

Problem 5.53. Consider the closed curve corresponding to L_n above in the case of a separable but now infinite dimensional Hilbert space:

$$L : [0,1] \ni x \longmapsto e^{2\pi ix} \operatorname{Id}_H \in \operatorname{GL}(H) \subset \mathcal{B}(H) \tag{5.109}$$

taking values in the invertible operators on H. Show that after identifying H with $H \oplus H$ as above, there is a continuous map

$$M : [0,1]^2 \longrightarrow \operatorname{GL}(H \oplus H) \tag{5.110}$$

with values in the invertible operators and satisfying

$$M(0,x) = L(x),\ M(1,x)(u_1,u_2) = (e^{4\pi ix}u_1, u_2),\ M(y,0) = M(y,1),\ \forall\, x,y \in [0,1]. \tag{5.111}$$

Hint: So, think of $H \oplus H$ as being 2-vectors (u_1, u_2) with entries in H. This allows one to think of 'rotation' between the two factors. Indeed, show that

$$U(y)(u_1,u_2) = (\cos(\pi y/2)u_1 + \sin(\pi y/2)u_2, -\sin(\pi y/2)u_1 + \cos(\pi y/2)u_2) \tag{5.112}$$

defines a continuous map $[0,1] \ni y \longmapsto U(y) \in \operatorname{GL}(H \oplus H)$ such that $U(0) = \operatorname{Id}$, $U(1)(u_1,u_2) = (u_2, -u_1)$. Now, consider the 2-parameter family of maps

$$U^{-1}(y)V_2(x)U(y)V_1(x) \tag{5.113}$$

where $V_1(x)$ and $V_2(x)$ are defined on $H \oplus H$ as multiplication by $\exp(2\pi ix)$ on the first and the second component respectively, leaving the other fixed.

Problem 5.54. Using a rotation similar to the one in the preceeding problem (or otherwise) show that there is a continuous map

$$G : [0,1]^2 \longrightarrow \operatorname{GL}(H \oplus H) \tag{5.114}$$

such that

$$\begin{aligned} G(0,x)(u_1,u_2) &= (e^{2\pi ix}u_1, e^{-2\pi ix}u_2),\\ G(1,x)(u_1,u_2) &= (u_1,u_2),\ G(y,0) = G(y,1)\ \forall\, x,y \in [0,1]. \end{aligned} \tag{5.115}$$

Problem 5.55. Now, think about combining the various constructions above in the following way. Show that on $l_2(H)$ there is an homotopy like (5.114), $\tilde{G} : [0,1]^2 \longrightarrow \operatorname{GL}(l_2(H))$, (very like in fact) such that

$$\begin{aligned} \tilde{G}(0,x)\,\{u_k\}_{k=1}^{\infty} &= \left\{\exp((-1)^k 2\pi ix)u_k\right\}_{k=1}^{\infty},\\ \tilde{G}(1,x) &= \operatorname{Id},\ \tilde{G}(y,0) = \tilde{G}(y,1)\ \forall\, x,y \in [0,1]. \end{aligned} \tag{5.116}$$

PROBLEM 5.56. "Eilenberg's swindle" For an infinite dimenisonal separable Hilbert space, construct an homotopy – meaning a continuous map $G : [0,1]^2 \longrightarrow \mathrm{GL}(H)$ – with $G(0,x) = L(x)$ in (5.109) and $G(1,x) = \mathrm{Id}$ and of course $G(y,0) = G(y,1)$ for all $x, y \in [0,1]$.

Hint: Just put things together – of course you can rescale the interval at the end to make it all happen over $[0,1]$. First 'divide H into 2 copies of itself' and deform from L to $M(1,x)$ in (5.111). Now, 'divide the second H up into $l_2(H)$' and apply an argument just like the preceding problem to turn the identity on this factor into alternating terms multiplying by $\exp(\pm 4\pi i x)$ – starting with $-$. Now, you are on $H \oplus l_2(H)$, 'renumbering' allows you to regard this as $l_2(H)$ again and when you do so your curve has become alternate multiplication by $\exp(\pm 4\pi i x)$ (with $+$ first). Finally then, apply the preceding problem again, to deform to the identity (always of course through closed curves). Presto, Eilenberg's swindle!

PROBLEM 5.57. Check that we really can understand all the 2π periodic eigenfunctions of the Schrödinger operator using the discussion above. First of all, there was nothing sacred about the addition of 1 to $-d^2/dx^2$, we could add any positive number and get a similar result – the problem with 0 is that the constants satisfy the homogeneous equation $d^2u/dx^2 = 0$. What we have shown is that the operator

$$u \longmapsto Qu = -\frac{d^2u}{dx^2}u + Vu \tag{5.117}$$

applied to twice continuously differentiable functions has at least a left inverse unless there is a non-trivial solution of

$$-\frac{d^2u}{dx^2}u + Vu = 0. \tag{5.118}$$

Namely, the left inverse is $R = F(\mathrm{Id} + F(V-1)F)^{-1}F$. This is a compact self-adjoint operator. Show – and there is still a bit of work to do – that (twice continuously differentiable) eigenfunctions of Q, meaning solutions of $Qu = \tau u$ are precisely the non-trivial solutions of $Ru = \tau^{-1}u$.

What to do in case (5.118) does have a non-trivial solution? Show that the space of these is finite dimensional and conclude that essentially the same result holds by working on the orthocomplement in $L^2(\mathbb{S})$.

7. Exam Preparation Problems

EP.1 Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and suppose that

$$B : H \times H \longleftrightarrow \mathbb{C} \tag{5.119}$$

is a(nother) sesquilinear form – so for all c_1, $c_2 \in \mathbb{C}$, u, u_1, u_2 and $v \in H$,

$$B(c_1u_1 + c_2u_2, v) = c_1B(u_1, v) + c_2B(u_2, v),\ B(u, v) = \overline{B(v, u)}. \tag{5.120}$$

Show that B is continuous, with respect to the norm $\|(u, v)\| = \|u\|_H + \|v\|_H$ on $H \times H$ if and only if it is bounded, in the sense that for some $C > 0$,

$$|B(u, v)| \le C\|u\|_H\|v\|_H. \tag{5.121}$$

EP.2 A continuous linear map $T : H_1 \longrightarrow H_2$ between two, possibly different, Hilbert spaces is said to be *compact* if the image of the unit ball in H_1 under T is precompact in H_2. Suppose $A : H_1 \longrightarrow H_2$ is a continuous linear operator which is injective and surjective and $T : H_1 \longrightarrow H_2$ is compact. Show that there is a compact operator $K : H_2 \longrightarrow H_2$ such that $T = KA$.

EP.3 Suppose $P \subset H$ is a (non-trivial, i.e. not $\{0\}$) closed linear subspace of a Hilbert space. Deduce from a result done in class that each u in H has a unique decomposition

$$u = v + v',\ v \in P,\ v' \perp P \tag{5.122}$$

and that the map $\pi_P : H \ni u \longmapsto v \in P$ has the properties

$$(\pi_P)^* = \pi_P,\ (\pi_P)^2 = \pi_P,\ \|\pi_P\|_{\mathcal{B}(H)} = 1. \tag{5.123}$$

EP.4 Show that for a sequence of non-negative step functions f_j, defined on $\mathbb{R}$, which is absolutely summable, meaning $\sum\limits_j \int f_j < \infty$, the series $\sum\limits_j f_j(x)$ cannot diverge for all $x \in (a, b)$, for any $a < b$.

EP.5 Let $A_j \subset [-N, N] \subset \mathbb{R}$ (for N fixed) be a sequence of subsets with the property that the characteristic function, χ_j of A_j, is integrable for each j. Show that the characteristic function of

$$A = \bigcup_j A_j \tag{5.124}$$

is integrable.

EP.6 Let $e_j = c_j C^j e^{-x^2/2}$, $c_j > 0$, $C = -\frac{d}{dx} + x$ the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. Define an operator on $L^2(\mathbb{R})$ by

$$Au = \sum_{j=0}^{\infty} (2j+1)^{-\frac{1}{2}} (u, e_j)_{L^2} e_j. \tag{5.125}$$

(1) Show that A is compact as an operator on $L^2(\mathbb{R})$.
(2) Suppose that $V \in \mathcal{C}^0_\infty(\mathbb{R})$ is a bounded, real-valued, continuous function on $\mathbb{R}$. What can you say about the eigenvalues τ_j, and eigenfunctions v_j, of $K = AVA$, where V is acting by multiplication on $L^2(\mathbb{R})$?
(3) Show that for $C > 0$ a large enough constant, $\operatorname{Id} + A(V + C)A$ is invertible (with bounded inverse on $L^2(\mathbb{R})$).
(4) Show that $L^2(\mathbb{R})$ has an orthonormal basis of eigenfunctions of $J = A(\operatorname{Id} + A(V + C)A)^{-1}A$.
(5) What would you need to show to conclude that these eigenfunctions of J satisfy

$$-\frac{d^2 v_j(x)}{dx^2} + x^2 v_j(x) + V(x) v_j(x) = \lambda_j v_j? \tag{5.126}$$

(6) What would you need to show to check that all the square-integrable, twice continuously differentiable, solutions of (5.126), for some $\lambda_j \in \mathbb{C}$, are eigenfunctions of K?

EP.7 Test 1 from last year (N.B. There may be some confusion between $\mathcal{L}^1$ and L^1 here, just choose the correct interpretation!):-

Q1. Recall Lebesgue's Dominated Convergence Theorem and use it to show that if $u \in \mathcal{L}^2(\mathbb{R})$ and $v \in \mathcal{L}^1(\mathbb{R})$ then

$$\begin{aligned}\lim_{N\to\infty}\int_{|x|>N}|u|^2=0,\ \lim_{N\to\infty}\int|C_Nu-u|^2=0,\\ \lim_{N\to\infty}\int_{|x|>N}|v|=0 \text{ and } \lim_{N\to\infty}\int|C_Nv-v|=0.\end{aligned} \tag{Eq1}$$

where

$$C_Nf(x)=\begin{cases}N & \text{if } f(x)>N\\ -N & \text{if } f(x)<-N\\ f(x) & \text{otherwise.}\end{cases} \tag{Eq2}$$

Q2. Show that step functions are dense in $L^1(\mathbb{R})$ and in $L^2(\mathbb{R})$ (Hint:- Look at Q1 above and think about $f-f_N$, $f_N=C_Nf\chi_{[-N,N]}$ and its square. So it suffices to show that f_N is the limit in L^2 of a sequence of step functions. Show that if g_n is a sequence of step functions converging to f_N in L^1 then $C_Ng_n\chi_{[-N,N]}$ is converges to f_N in L^2.) and that if $f\in L^1(\mathbb{R})$ then there is a sequence of step functions u_n and an element $g\in L^1(\mathbb{R})$ such that $u_n\to f$ a.e. and $|u_n|\le g$.

Q3. Show that $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ are separable, meaning that each has a countable dense subset.

Q4. Show that the minimum and the maximum of two locally integrable functions is locally integrable.

Q5. A subset of $\mathbb{R}$ is said to be (Lebesgue) measurable if its characteristic function is locally integrable. Show that a countable union of measurable sets is measurable. Hint: Start with two!

Q6. Define $\mathcal{L}^\infty(\mathbb{R})$ as consisting of the locally integrable functions which are bounded, $\sup_{\mathbb{R}}|u|<\infty$. If $\mathcal{N}_\infty\subset L^\infty(\mathbb{R})$ consists of the bounded functions which vanish outside a set of measure zero show that

$$\|u+\mathcal{N}_\infty\|_{L^\infty}=\inf_{h\in\mathcal{N}_\infty}\sup_{x\in\mathbb{R}}|u(x)+h(x)| \tag{Eq3}$$

is a norm on $L^\infty(\mathbb{R})=L^\infty(\mathbb{R})/\mathcal{N}_\infty$.

Q7. Show that if $u\in L^\infty(\mathbb{R})$ and $v\in L^1(\mathbb{R})$ then $uv\in L^1(\mathbb{R})$ and that

$$|\int uv|\le\|u\|_{L^\infty}\|v\|_{L^1}. \tag{Eq4}$$

Q8. Show that each $u\in L^2(\mathbb{R})$ is continuous in the mean in the sense that $T_zu(x)=u(x-z)\in L^2(\mathbb{R})$ for all $z\in\mathbb{R}$ and that

$$\lim_{|z|\to0}\int|T_zu-u|^2=0. \tag{Eq5}$$

Q9. If $\{u_j\}$ is a Cauchy sequence in $L^2(\mathbb{R})$ show that both (Eq5) and (Eq1) are uniform in j, so given $\epsilon>0$ there exists $\delta>0$ such that

$$\int|T_zu_j-u_j|^2<\epsilon,\ \int_{|x|>1/\delta}|u_j|^2<\epsilon\ \forall\ |z|<\delta \text{ and all } j. \tag{Eq6}$$

Q10. Construct a sequence in $L^2(\mathbb{R})$ for which the uniformity in (Eq6) does not hold.

EP.8 Test 2 from last year.

(1) Recall the discussion of the Dirichlet problem for d^2/dx^2 from class and carry out an analogous discussion for the Neumann problem to arrive at a complete orthonormal basis of $L^2([0,1])$ consisting of $\psi_n \in \mathcal{C}^2$ functions which are all eigenfunctions in the sense that

$$\frac{d^2\psi_n(x)}{dx^2} = \gamma_n \psi_n(x) \ \forall \ x \in [0,1], \ \frac{d\psi_n}{dx}(0) = \frac{d\psi_n}{dx}(1) = 0. \tag{NeuEig}$$

This is actually a little harder than the Dirichlet problem which I did in class, because there is an eigenfunction of norm 1 with $\gamma = 0$. Here are some individual steps which may help you along the way!

What is the eigenfunction with eigenvalue 0 for (NeuEig)?

What is the operator of orthogonal projection onto this function?

What is the operator of orthogonal projection onto the orthocomplement of this function?

The crucual part. Find an integral operator $A_N = B - B_N$, where B is the operator from class,

$$(Bf)(x) = \int_0^x (x-s)f(s)ds \tag{B-Def}$$

and B_N is of finite rank, such that if f is continuous then $u = A_N f$ is twice continuously differentiable, satisfies $\int_0^1 u(x)dx = 0$, $A_N 1 = 0$ (where 1 is the constant function) and

$$\begin{gathered}\int_0^1 f(x)dx = 0 \Longrightarrow \\ \frac{d^2u}{dx^2} = f(x) \ \forall \ x \in [0,1], \ \frac{du}{dx}(0) = \frac{du}{dx}(1) = 0.\end{gathered} \tag{GI}$$

Show that A_N is compact and self-adjoint.

Work out what the spectrum of A_N is, including its null space.

Deduce the desired conclusion.

(2) Show that these two orthonormal bases of $L^2([0,1])$ (the one above and the one from class) can each be turned into an orthonormal basis of $L^2([0,\pi])$ by change of variable.

(3) Construct an orthonormal basis of $L^2([-\pi,\pi])$ by dividing each element into its odd and even parts, resticting these to $[0,\pi]$ and using the Neumann basis above on the even part and the Dirichlet basis from class on the odd part.

(4) Prove the basic theorem of Fourier series, namely that for any function $u \in L^2([-\pi,\pi])$ there exist unique constants $c_k \in \mathbb{C}$, $k \in \mathbb{Z}$ such that

$$u(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx} \text{ converges in } L^2([-\pi,\pi]) \tag{FS}$$

and give an integral formula for the constants.

EP.9 Let $B \in \mathcal{C}([0,1]^2)$ be a continuous function of two variables. Explain why the integral operator

$$Tu(x) = \int_{[0,1]} B(x,y)u(y)$$

defines a bounded linear map $L^1([0,1]) \longrightarrow \mathcal{C}([0,1])$ and hence a bounded operator on $L^2([0,1])$.

(a) Explain why T is not surjective as a bounded operator on $L^2([0,1])$.
(b) Explain why $\mathrm{Id}-T$ has finite dimensional null space $N \subset L^2([0,1])$ as an operator on $L^2([0,1])$
(c) Show that $N \subset \mathcal{C}([0,1])$.
(d) Show that $\mathrm{Id}-T$ has closed range $R \subset L^2([0,1]))$ as a bounded operator on $L^2([0,1])$.
(e) Show that the orthocomplement of R is a subspace of $\mathcal{C}([0,1])$.

EP.10 Let $c : \mathbb{N}^2 \longrightarrow \mathbb{C}$ be an 'infinite matrix' of complex numbers satisfying

$$\sum_{i,j=1}^{\infty} |c_{ij}|^2 < \infty. \tag{5.127}$$

If $\{e_i\}_{i=1}^{\infty}$ is an orthornomal basis of a (separable of course) Hilbert space $\mathcal{H}$, show that

$$Au = \sum_{i,j=1}^{\infty} c_{ij}(u,e_j)e_i \tag{5.128}$$

defines a compact operator on $\mathcal{H}$.

8. Solutions to problems

Solution 5.14 (Problem 5.1). Write out a proof (you can steal it from one of many places but at least write it out in your own hand) either for $p=2$ or for each p with $1 \le p < \infty$ that

$$l^p = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_{j=1}^{\infty} |a_j|^p < \infty,\ a_j = a(j)\}$$

is a normed space with the norm

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}}.$$

This means writing out the proof that this is a linear space and that the three conditions required of a norm hold.

Solution:- We know that the functions from any set with values in a linear space form a linear space – under addition of values (don't feel bad if you wrote this out, it is a good thing to do once). So, to see that l^p is a linear space it suffices to see that it is closed under addition and scalar multiplication. For scalar multiples this is clear:-

$$|ta_i| = |t||a_i| \text{ so } \|ta\|_p = |t|\|a\|_p \tag{5.129}$$

which is part of what is needed for the proof that $\|\cdot\|_p$ is a norm anyway. The fact that $a,\ b \in l^p$ imples $a+b \in l^p$ follows once we show the triangle inequality or we can be a little cruder and observe that

$$\begin{gathered}|a_i+b_i|^p \le (2\max(|a|_i,|b_i|))^p = 2^p\max(|a|_i^p,|b_i|^p) \le 2^p(|a_i|+|b_i|)\\ \|a+b\|_p^p = \sum_j |a_i+b_i|^p \le 2^p(\|a\|^p+\|b\|^p),\end{gathered} \tag{5.130}$$

where we use the fact that t^p is an increasing function of $t \ge 0$.

Now, to see that l^p is a normed space we need to check that $\|a\|_p$ is indeed a norm. It is non-negative and $\|a\|_p = 0$ implies $a_i = 0$ for all i which is to say $a = 0$. So, only the triangle inequality remains. For $p = 1$ this is a direct consequence of the usual triangle inequality:

$$\|a+b\|_1 = \sum_i |a_i + b_i| \le \sum_i (|a_i| + |b_i|) = \|a\|_1 + \|b\|_1. \tag{5.131}$$

For $1 < p < \infty$ it is known as Minkowski's inequality. This in turn is deduced from Hölder's inequality – which follows from Young's inequality! The latter says if $1/p + 1/q = 1$, so $q = p/(p-1)$, then

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q} \ \forall\ \alpha, \beta \ge 0. \tag{5.132}$$

To check it, observe that as a function of $\alpha = x$,

$$f(x) = \frac{x^p}{p} - x\beta + \frac{\beta^q}{q} \tag{5.133}$$

if non-negative at $x = 0$ and clearly positive when $x >> 0$, since x^p grows faster than $x\beta$. Moreover, it is differentiable and the derivative only vanishes at $x^{p-1} = \beta$, where it must have a global minimum in $x > 0$. At this point $f(x) = 0$ so Young's inequality follows. Now, applying this with $\alpha = |a_i|/\|a\|_p$ and $\beta = |b_i|/\|b\|_q$ (assuming both are non-zero) and summing over i gives Hölder's inequality

$$\begin{gathered} |\sum_i a_i b_i|/\|a\|_p\|b\|_q \le \sum_i |a_i||b_i|/\|a\|_p\|b\|_q \le \sum_i \left(\frac{|a_i|^p}{\|a\|_p^p p} + \frac{|b_i|^q}{\|b\|_q^q q} \right) = 1 \\ \Longrightarrow |\sum_i a_i b_i| \le \|a\|_p \|b\|_q. \end{gathered} \tag{5.134}$$

Of course, if either $\|a\|_p = 0$ or $\|b\|_q = 0$ this inequality holds anyway.

Now, from this Minkowski's inequality follows. Namely from the ordinary triangle inequality and then Minkowski's inequality (with q power in the first factor)

$$\begin{aligned} \sum_i |a_i + b_i|^p &= \sum_i |a_i + b_i|^{(p-1)} |a_i + b_i| \\ &\le \sum_i |a_i + b_i|^{(p-1)} |a_i| + \sum_i |a_i + b_i|^{(p-1)} |b_i| \\ &\le \left(\sum_i |a_i + b_i|^p \right)^{1/q} (\|a\|_p + \|b\|_p) \end{aligned} \tag{5.135}$$

gives after division by the first factor on the right

$$\|a+b\|_p \le \|a\|_p + \|b\|_p. \tag{5.136}$$

Thus, l^p is indeed a normed space.

I did not necessarily expect you to go through the proof of Young-Hölder-Minkowksi, but I think you should do so at some point since I will not do it in class.

Solution 5.15. The 'tricky' part in Problem 1.1 is the triangle inequality. Suppose you knew – meaning I tell you – that for each N

$$\left(\sum_{j=1}^{N}|a_j|^p\right)^{\frac{1}{p}} \text{ is a norm on } \mathbb{C}^N$$

would that help?

Solution. Yes indeed it helps. If we know that for each N

$$\left(\sum_{j=1}^{N}|a_j+b_j|^p\right)^{\frac{1}{p}} \le \left(\sum_{j=1}^{N}|a_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{N}|b_j|^p\right)^{\frac{1}{p}} \tag{5.137}$$

then for elements of l^p the norms always bounds the right side from above, meaning

$$\left(\sum_{j=1}^{N}|a_j+b_j|^p\right)^{\frac{1}{p}} \le \|a\|_p + \|b\|_p. \tag{5.138}$$

Since the left side is increasing with N it must converge and be bounded by the right, which is independent of N. That is, the triangle inequality follows. Really this just means it is enough to go through the discussion in the first problem for finite, but arbitrary, N. □

Solution 5.16. Prove directly that each l^p as defined in Problem 1.1 – or just l^2 – is complete, i.e. it is a Banach space. At the risk of offending some, let me say that this means showing that each Cauchy sequence converges. The problem here is to find the limit of a given Cauchy sequence. Show that for each N the sequence in $\mathbb{C}^N$ obtained by truncating each of the elements at point N is Cauchy with respect to the norm in Problem 1.2 on $\mathbb{C}^N$. Show that this is the same as being Cauchy in $\mathbb{C}^N$ in the usual sense (if you are doing $p = 2$ it is already the usual sense) and hence, this cut-off sequence converges. Use this to find a putative limit of the Cauchy sequence and then check that it works.

Solution. So, suppose we are given a Cauchy sequence $a^{(n)}$ in l^p. Thus, each element is a sequence $\{a_j^{(n)}\}_{j=1}^{\infty}$ in l^p. From the continuity of the norm in Problem 1.5 below, $\|a^{(n)}\|$ must be Cauchy in $\mathbb{R}$ and so converges. In particular the sequence is norm bounded, there exists A such that $\|a^{(n)}\|_p \le A$ for all n. The Cauchy condition itself is that given $\epsilon > 0$ there exists M such that for all $m, n > M$,

$$\|a^{(n)} - a^{(m)}\|_p = \left(\sum_i |a_i^{(n)} - a_i^{(m)}|^p\right)^{\frac{1}{p}} < \epsilon/2. \tag{5.139}$$

Now for each i, $|a_i^{(n)} - a_i^{(m)}| \le \|a^{(n)} - a^{(m)}\|_p$ so each of the sequences $a_i^{(n)}$ must be Cauchy in $\mathbb{C}$. Since $\mathbb{C}$ is complete

$$\lim_{n\to\infty} a_i^{(n)} = a_i \text{ exists for each } i = 1, 2, \ldots. \tag{5.140}$$

So, our putative limit is a, the sequence $\{a_i\}_{i=1}^{\infty}$. The boundedness of the norms shows that

$$\sum_{i=1}^{N} |a_i^{(n)}|^p \le A^p \tag{5.141}$$

and we can pass to the limit here as $n \to \infty$ since there are only finitely many terms. Thus

$$\sum_{i=1}^{N} |a_i|^p \leq A^p \; \forall \; N \Longrightarrow \|a\|_p \leq A. \tag{5.142}$$

Thus, $a \in l^p$ as we hoped. Similarly, we can pass to the limit as $m \to \infty$ in the finite inequality which follows from the Cauchy conditions

$$(\sum_{i=1}^{N} |a_i^{(n)} - a_i^{(m)}|^p)^{\frac{1}{p}} < \epsilon/2 \tag{5.143}$$

to see that for each N

$$(\sum_{i=1}^{N} |a_i^{(n)} - a_i|^p)^{\frac{1}{p}} \leq \epsilon/2 \tag{5.144}$$

and hence

$$\|a^{(n)} - a\| < \epsilon \; \forall \; n > M. \tag{5.145}$$

Thus indeed, $a^{(n)} \to a$ in l^p as we were trying to show.

Notice that the trick is to 'back off' to finite sums to avoid any issues of interchanging limits. $\square$

SOLUTION 5.17. Consider the 'unit sphere' in l^p – where if you want you can set $p = 2$. This is the set of vectors of length 1 :

$$S = \{a \in l^p; \|a\|_p = 1\}.$$

(1) Show that S is closed.
(2) Recall the sequential (so not the open covering definition) characterization of compactness of a set in a metric space (e .g . by checking in Rudin).
(3) Show that S is not compact by considering the sequence in l^p with kth element the sequence which is all zeros except for a 1 in the kth slot. Note that the main problem is not to get yourself confused about sequences of sequences!

SOLUTION. By the next problem, the norm is continuous as a function, so

$$S = \{a; \|a\| = 1\} \tag{5.146}$$

is the inverse image of the closed subset $\{1\}$, hence closed.

Now, the standard result on metric spaces is that a subset is compact if and only if every sequence with values in the subset has a convergent subsequence with limit in the subset (if you drop the last condition then the closure is compact).

In this case we consider the sequence (of sequences)

$$a_i^{(n)} = \begin{cases} 0 & i \neq n \\ 1 & i = n \end{cases}. \tag{5.147}$$

This has the property that $\|a^{(n)} - a^{(m)}\|_p = 2^{\frac{1}{p}}$ whenever $n \neq m$. Thus, it cannot have any Cauchy subsequence, and hence cannot have a convergent subsequence, so S is not compact.

This is important. In fact it is a major difference between finite-dimensional and infinite-dimensional normed spaces. In the latter case the unit sphere cannot be compact whereas in the former it is. $\square$

SOLUTION 5.18. Show that the norm on any normed space is continuous. Solution:- Right, so I should have put this problem earlier!

The triangle inequality shows that for any u, v in a normed space

$$\|u\| \le \|u-v\| + \|v\|,\ \|v\| \le \|u-v\| + \|u\| \tag{5.148}$$

which implies that

$$|\|u\| - \|v\|| \le \|u-v\|. \tag{5.149}$$

This shows that $\|\cdot\|$ is continuous, indeed it is Lipschitz continuous.

SOLUTION 5.19. Finish the proof of the completeness of the space B constructed in lecture on February 10. The description of that construction can be found in the notes to Lecture 3 as well as an indication of one way to proceed.

SOLUTION. The proof could be shorter than this, I have tried to be fairly complete.

To recap. We start of with a normed space V. From this normed space we construct the new linear space $\tilde{V}$ with points the absolutely summable series in V. Then we consider the subspace $S \subset \tilde{V}$ of those absolutely summable series which converge to 0 in V. We are interested in the quotient space

$$B = \tilde{V}/S. \tag{5.150}$$

What we know already is that this is a normed space where the norm of $b = \{v_n\}+S$ – where $\{v_n\}$ is an absolutely summable series in V is

$$\|b\|_B = \lim_{N\to\infty} \|\sum_{n=1}^{N} v_n\|_V. \tag{5.151}$$

This is independent of which series is used to represent b – i.e. is the same if an element of S is added to the series.

Now, what is an absolutely summable series in B? It is a sequence $\{b_n\}$, thought of a series, with the property that

$$\sum_n \|b_n\|_B < \infty. \tag{5.152}$$

We have to show that it converges in B. The first task is to guess what the limit should be. The idea is that it should be a series which adds up to 'the sum of the b_n's'. Each b_n is represented by an absolutely summable series $v_k^{(n)}$ in V. So, we can just look for the usual diagonal sum of the double series and set

$$w_j = \sum_{n+k=j} v_k^{(n)}. \tag{5.153}$$

The problem is that this will not in generall be absolutely summable as a series in V. What we want is the estimate

$$\sum_j \|w_j\| = \sum_j \|\sum_{j=n+k} v_k^{(n)}\| < \infty. \tag{5.154}$$

The only way we can really estimate this is to use the triangle inequality and conclude that

$$\sum_{j=1}^{\infty} \|w_j\| \le \sum_{k,n} \|v_k^{(n)}\|_V. \tag{5.155}$$

Each of the sums over k on the right is finite, but we do not know that the sum over k is then finite. This is where the first suggestion comes in:-

We can *choose* the absolutely summable series $v_k^{(n)}$ representing b_n such that

$$\sum_k \|v_k^{(n)}\| \le \|b_n\|_B + 2^{-n}. \tag{5.156}$$

Suppose an initial choice of absolutely summable series representing b_n is u_k, so $\|b_n\| = \lim_{N\to\infty} \|\sum\limits_{k=1}^{N} u_k\|$ and $\sum\limits_k \|u_k\|_V < \infty$. Choosing M large it follows that

$$\sum_{k>M} \|u_k\|_V \le 2^{-n-1}. \tag{5.157}$$

With this choice of M set $v_1^{(n)} = \sum\limits_{k=1}^{M} u_k$ and $v_k^{(n)} = u_{M+k-1}$ for all $k \ge 2$. This does still represent b_n since the difference of the sums,

$$\sum_{k=1}^{N} v_k^{(n)} - \sum_{k=1}^{N} u_k = -\sum_{k=N}^{N+M-1} u_k \tag{5.158}$$

for all N. The sum on the right tends to 0 in V (since it is a fixed number of terms). Moreover, because of (5.157),

$$\sum_k \|v_k^{(n)}\|_V = \|\sum_{j=1}^{M} u_j\|_V + \sum_{k>M} \|u_k\| \le \|\sum_{j=1}^{N} u_j\| + 2\sum_{k>M} \|u_k\| \le \|\sum_{j=1}^{N} u_j\| + 2^{-n} \tag{5.159}$$

for all N. Passing to the limit as $N \to \infty$ gives (5.156).

Once we have chosen these 'nice' representatives of each of the b_n's if we define the w_j's by (5.153) then (5.154) means that

$$\sum_j \|w_j\|_V \le \sum_n \|b_n\|_B + \sum_n 2^{-n} < \infty \tag{5.160}$$

because the series b_n is absolutely summable. Thus $\{w_j\}$ defines an element of $\tilde{V}$ and hence $b \in B$.

Finally then we want to show that $\sum\limits_n b_n = b$ in B. This just means that we need to show

$$\lim_{N\to\infty} \|b - \sum_{n=1}^{N} b_n\|_B = 0. \tag{5.161}$$

The norm here is itself a limit – $b - \sum\limits_{n=1}^{N} b_n$ is represented by the summable series with nth term

$$w_k - \sum_{n=1}^{N} v_k^{(n)} \tag{5.162}$$

and the norm is then

$$\lim_{p\to\infty} \|\sum_{k=1}^{p} (w_k - \sum_{n=1}^{N} v_k^{(n)})\|_V. \tag{5.163}$$

Then we need to understand what happens as $N \to \infty$! Now, w_k is the diagonal sum of the $v_j^{(n)}$'s so sum over k gives the difference of the sum of the $v_j^{(n)}$ over the first p anti-diagonals minus the sum over a square with height N (in n) and width p. So, using the triangle inequality the norm of the difference can be estimated by the sum of the norms of all the 'missing terms' and then some so

$$\|\sum_{k=1}^{p}(w_k - \sum_{n=1}^{N} v_k^{(n)})\|_V \le \sum_{l+m\ge L} \|v_l^{(m)}\|_V \tag{5.164}$$

where $L = \min(p, N)$. This sum is finite and letting $p \to \infty$ is replaced by the sum over $l + m \ge N$. Then letting $N \to \infty$ it tends to zero by the absolute (double) summability. Thus

$$\lim_{N\to\infty} \|b - \sum_{n=1}^{N} b_n\|B = 0 \tag{5.165}$$

which is the statelent we wanted, that $\sum_n b_n = b$. □

Problem 5.58. Let's consider an example of an absolutely summable sequence of step functions. For the interval $[0, 1)$ (remember there is a strong preference for left-closed but right-open intervals for the moment) consider a variant of the construction of the standard Cantor subset based on 3 proceeding in steps. Thus, remove the 'central interval $[1/3, 2/3)$. This leave $C_1 = [0, 1/3) \cup [2/3, 1)$. Then remove the central interval from each of the remaining two intervals to get $C_2 = [0, 1/9) \cup [2/9, 1/3) \cup [2/3, 7/9) \cup [8/9, 1)$. Carry on in this way to define successive sets $C_k \subset C_{k-1}$, each consisting of a finite union of semi-open intervals. Now, consider the *series* of step functions f_k where $f_k(x) = 1$ on C_k and 0 otherwise.

(1) Check that this is an absolutely summable series.
(2) For which $x \in [0, 1)$ does $\sum_k |f_k(x)|$ converge?
(3) Describe a function on $[0, 1)$ which is shown to be Lebesgue integrable (as defined in Lecture 4) by the existence of this series and compute its Lebesgue integral.
(4) Is this function Riemann integrable (this is easy, not hard, if you check the definition of Riemann integrability)?
(5) Finally consider the function g which is equal to one on the union of all the subintervals of $[0, 1)$ which are *removed* in the construction and zero elsewhere. Show that g is Lebesgue integrable and compute its integral.

Solution. (1) The total length of the intervals is being reduced by a factor of 1/3 each time. Thus $l(C_k) = \frac{2^k}{3^k}$. Thus the integral of f, which is non-negative, is actually

$$\int f_k = \frac{2^k}{3^k} \Longrightarrow \sum_k \int |f_k| = \sum_{k=1}^{\infty} \frac{2^k}{3^k} = 2 \tag{5.166}$$

Thus the series is absolutely summable.

(2) Since the C_k are decreasing, $C_k \supset C_{k+1}$, only if

$$x \in E = \bigcap_k C_k \tag{5.167}$$

does the series $\sum_k |f_k(x)|$ diverge (to $+\infty$) otherwise it converges.

(3) The function defined as the sum of the series where it converges and zero otherwise

$$f(x) = \begin{cases} \sum_k f_k(x) & x \in \mathbb{R} \setminus E \\ 0 & x \in E \end{cases} \tag{5.168}$$

is integrable by definition. Its integral is by definition

$$\int f = \sum_k \int f_k = 2 \tag{5.169}$$

from the discussion above.

(4) The function f is not Riemann integrable since it is not bounded – and this is part of the definition. In particular for $x \in C_k \setminus C_{k+1}$, which is not an empty set, $f(x) = k$.

(5) The set F, which is the union of the intervals removed is $[0, 1) \setminus E$. Taking step functions equal to 1 on each of the intervals removed gives an absolutely summable series, since they are non-negative and the kth one has integral $1/3 \times (2/3)^{k-1}$ for $k = 1, \dots$. This series converges to g on F so g is Lebesgue integrable and hence

$$\int g = 1. \tag{5.170}$$

□

Problem 5.59. The covering lemma for $\mathbb{R}^2$. By a rectangle we will mean a set of the form $[a_1, b_1) \times [a_2, b_2)$ in $\mathbb{R}^2$. The area of a rectangle is $(b_1 - a_1) \times (b_2 - a_2)$.

(1) We may subdivide a rectangle by subdividing either of the intervals – replacing $[a_1, b_1)$ by $[a_1, c_1) \cup [c_1, b_1)$. Show that the sum of the areas of rectangles made by any repeated subdivision is always the same as that of the original.
(2) Suppose that a finite collection of disjoint rectangles has union a rectangle (always in this same half-open sense). Show, and I really mean prove, that the sum of the areas is the area of the whole rectange. Hint:- proceed by subdivision.
(3) Now show that for any countable collection of disjoint rectangles contained in a given rectange the sum of the areas is less than or equal to that of the containing rectangle.
(4) Show that if a finite collection of rectangles has union *containing* a given rectange then the sum of the areas of the rectangles is at least as large of that of the rectangle contained in the union.
(5) Prove the extension of the preceeding result to a countable collection of rectangles with union containing a given rectangle.

Solution. (1) For the subdivision of one rectangle this is clear enough. Namely we either divide the first side in two or the second side in two at an intermediate point c. After subdivision the area of the two rectanges is either

$$\begin{gathered}(c - a_1)(b_2 - a_2) + (b_1 - c)(b_2 - a_2) = (b_1 - c_1)(b_2 - a_2) \text{ or} \\ (b_1 - a_1)(c - a_2) + (b_1 - a_1)(b_2 - c) = (b_1 - c_1)(b_2 - a_2).\end{gathered} \tag{5.171}$$

this shows by induction that the sum of the areas of any the rectangles made by repeated subdivision is always the same as the original.

(2) If a finite collection of disjoint rectangles has union a rectangle, say $[a_1, b_2) \times [a_2, b_2)$ then the same is true after any subdivision of any of the rectangles. Moreover, by the preceeding result, after such subdivision the sum of the areas is always the same. Look at all the points $C_1 \subset [a_1, b_1)$ which occur as an endpoint of the first interval of one of the rectangles. Similarly let C_2 be the corresponding set of end-points of the second intervals of the rectangles. Now divide each of the rectangles repeatedly using the finite number of points in C_1 and the finite number of points in C_2. The total area remains the same and now the rectangles covering $[a_1, b_1) \times [A_2, b_2)$ are precisely the $A_i \times B_j$ where the A_i are a set of disjoint intervals covering $[a_1, b_1)$ and the B_j are a similar set covering $[a_2, b_2)$. Applying the one-dimensional result from class we see that the sum of the areas of the rectangles with first interval A_i is the product

$$\text{length of } A_i \times (b_2 - a_2). \tag{5.172}$$

Then we can sum over i and use the same result again to prove what we want.

(3) For any finite collection of disjoint rectangles contained in $[a_1, b_1) \times [a_2, b_2)$ we can use the same division process to show that we can add more disjoint rectangles to cover the whole big rectangle. Thus, from the preceeding result the sum of the areas must be less than or equal to $(b_1 - a_1)(b_2 - a_2)$. For a countable collection of disjoint rectangles the sum of the areas is therefore bounded above by this constant.

(4) Let the rectangles be D_i, $i = 1, \dots, N$ the union of which contains the rectangle D. Subdivide D_1 using all the endpoints of the intervals of D. Each of the resulting rectangles is either contained in D or is disjoint from it. Replace D_1 by the (one in fact) subrectangle contained in D. Proceeding by induction we can suppose that the first $N - k$ of the rectangles are disjoint and all contained in D and together all the rectangles cover D. Now look at the next one, D_{N-k+1}. Subdivide it using all the endpoints of the intervals for the earlier rectangles $D_1, \dots, D_k$ and D. After subdivision of D_{N-k+1} each resulting rectangle is either contained in one of the D_j, $j \le N - k$ or is *not* contained in D. All these can be discarded and the result is to decrease k by 1 (maybe increasing N but that is okay). So, by induction we can decompose and throw away rectangles until what is left are disjoint and individually contained in D but still cover. The sum of the areas of the remaining rectangles is precisely the area of D by the previous result, so the sum of the areas must originally have been at least this large.

(5) Now, for a countable collection of rectangles covering $D = [a_1, b_1) \times [a_2, b_2)$ we proceed as in the one-dimensional case. First, we can assume that there is a fixed upper bound C on the lengths of the sides. Make the kth rectangle a little larger by extending both the upper limits by $2^{-k}\delta$ where $\delta > 0$. The area increases, but by no more than $2C2^{-k}$. After extension the interiors of the countable collection cover the compact set $[a_1, b_1 - \delta] \times [a_2, b_1 - \delta]$. By compactness, a finite number of these open

rectangles cover, and hence there semi-closed version, with the same endpoints, covers $[a_1, b_1-\delta)\times[a_2, b_1-\delta)$. Applying the preceeding finite result we see that

$$\text{Sum of areas} + 2C\delta \geq \text{Area } D - 2C\delta. \tag{5.173}$$

Since this is true for all $\delta > 0$ the result follows.

$\square$

I encourage you to go through the discussion of integrals of step functions – now based on rectangles instead of intervals – and see that everything we have done can be extended to the case of two dimensions. In fact if you want you can go ahead and see that everything works in $\mathbb{R}^n$!

Problem 2.4

(1) Show that any continuous function on $[0,1]$ is the *uniform limit* on $[0,1)$ of a sequence of step functions. Hint:- Reduce to the real case, divide the interval into 2^n equal pieces and define the step functions to take infimim of the continuous function on the corresponding interval. Then use uniform convergence.

(2) By using the 'telescoping trick' show that any continuous function on $[0,1)$ can be written as the sum

$$\sum_i f_j(x)\ \forall\ x \in [0,1) \tag{5.174}$$

where the f_j are step functions and $\sum\limits_j |f_j(x)| < \infty$ for all $x \in [0,1)$.

(3) Conclude that any continuous function on $[0,1]$, extended to be 0 outside this interval, is a Lebesgue integrable function on $\mathbb{R}$.

SOLUTION. (1) Since the real and imaginary parts of a continuous function are continuous, it suffices to consider a real continous function f and then add afterwards. By the *uniform* continuity of a continuous function on a compact set, in this case $[0,1]$, given n there exists N such that $|x-y| \leq 2^{-N} \Longrightarrow |f(x)-f(y)| \leq 2^{-n}$. So, if we divide into 2^N equal intervals, where N depends on n and we insist that it be non-decreasing as a function of n and take the step function f_n on each interval which is equal to $\min f = \inf f$ on the closure of the interval then

$$|f(x) - F_n(x)| \leq 2^{-n}\ \forall\ x \in [0,1) \tag{5.175}$$

since this even works at the endpoints. Thus $F_n \to f$ uniformly on $[0,1)$.

(2) Now just define $f_1 = F_1$ and $f_k = F_k - F_{k-1}$ for all $k > 1$. It follows that these are step functions and that

$$\sum_{k=1}^{n} = f_n. \tag{5.176}$$

Moreover, each interval for F_{n+1} is a subinterval for F_n. Since f can varying by no more than 2^{-n} on each of the intervals for F_n it follows that

$$|f_n(x)| = |F_{n+1}(x) - F_n(x)| \leq 2^{-n}\ \forall\ n > 1. \tag{5.177}$$

Thus $\int |f_n| \leq 2^{-n}$ and so the series is absolutely summable. Moreover, it actually converges everywhere on $[0,1)$ and uniformly to f by (5.175).

(3) Hence f is Lebesgue integrable.
(4) For some reason I did not ask you to check that

$$\int f = \int_0^1 f(x)dx \tag{5.178}$$

where on the right is the Riemann integral. However this follows from the fact that

$$\int f = \lim_{n\to\infty} \int F_n \tag{5.179}$$

and the integral of the step function is between the Riemann upper and lower sums for the corresponding partition of $[0,1]$.

$\square$

SOLUTION 5.20. If f and $g \in \mathcal{L}^1(\mathbb{R})$ are Lebesgue integrable functions on the line show that

(1) If $f(x) \ge 0$ a.e. then $\int f \ge 0$.
(2) If $f(x) \le g(x)$ a.e. then $\int f \le \int g$.
(3) If f is complex valued then its real part, $\operatorname{Re} f$, is Lebesgue integrable and $|\int \operatorname{Re} f| \le \int |f|$.
(4) For a general complex-valued Lebesgue integrable function

$$|\int f| \le \int |f|. \tag{5.180}$$

Hint: You can look up a proof of this easily enough, but the usual trick is to choose $\theta \in [0, 2\pi)$ so that $e^{i\theta}\int f = \int (e^{i\theta} f) \ge 0$. Then apply the preceeding estimate to $g = e^{i\theta} f$.
(5) Show that the integral is a continuous linear functional

$$\int : L^1(\mathbb{R}) \longrightarrow \mathbb{C}. \tag{5.181}$$

SOLUTION. (1) If f is real and f_n is a real-valued absolutely summable series of step functions converging to f where it is absolutely convergent (if we only have a complex-valued sequence use part (3)). Then we know that

$$g_1 = |f_1|,\ g_j = |f_j| - |f_{j-1}|,\ f \ge 1 \tag{5.182}$$

is an absolutely convergent sequence converging to $|f|$ almost everywhere. It follows that $f_+ = \frac{1}{2}(|f|+f) = f$, if $f \ge 0$, is the limit almost everywhere of the series obtained by interlacing $\frac{1}{2}g_j$ and $\frac{1}{2}f_j$:

$$h_n = \begin{cases} \frac{1}{2}g_k & n = 2k-1 \\ f_k & n = 2k. \end{cases} \tag{5.183}$$

Thus f_+ is Lebesgue integrable. Moreover we know that

$$\int f_+ = \lim_{k\to\infty} \sum_{n\le 2k} \int h_k = \lim_{k\to\infty} \int \left(|\sum_{j=1}^k f_j| + \sum_{j=1}^k f_j \right) \tag{5.184}$$

where each term is a non-negative step function, so $\int f_+ \ge 0$.

(2) Apply the preceeding result to $g-f$ which is integrable and satisfies

$$\int g - \int f = \int (g-f) \geq 0. \tag{5.185}$$

(3) Arguing from first principles again, if f_n is now complex valued and an absolutely summable series of step functions converging a .e . to f then define

$$h_n = \begin{cases} \operatorname{Re} f_k & n = 3k-2 \\ \operatorname{Im} f_k & n = 3k-1 \\ -\operatorname{Im} f_k & n = 3k. \end{cases} \tag{5.186}$$

This series of step functions is absolutely summable and

$$\sum_n |h_n(x)| < \infty \Longleftrightarrow \sum_n |f_n(x)| < \infty \Longrightarrow \sum_n h_n(x) = \operatorname{Re} f. \tag{5.187}$$

Thus $\operatorname{Re} f$ is integrable. Since $\pm \operatorname{Re} f \leq |f|$

$$\pm \int \operatorname{Re} f \leq \int |f| \Longrightarrow |\int \operatorname{Re} f| \leq \int |f|. \tag{5.188}$$

(4) For a complex-valued f proceed as suggested. Choose $z \in \mathbb{C}$ with $|z| = 1$ such that $z \int f \in [0, \infty)$ which is possible by the properties of complex numbers. Then by the linearity of the integral

$$z \int f = \int (zf) = \int \operatorname{Re}(zf) \leq \int |z \operatorname{Re} f| \leq \int |f| \Longrightarrow |\int f| = z \int f \leq \int |f|. \tag{5.189}$$

(where the second equality follows from the fact that the integral is equal to its real part).

(5) We know that the integral defines a linear map

$$I : L^1(\mathbb{R}) \ni [f] \longmapsto \int f \in \mathbb{C} \tag{5.190}$$

since $\int f = \int g$ if $f = g$ a.e. are two representatives of the same class in $L^1(\mathbb{R})$. To say this is continuous is equivalent to it being bounded, which follows from the preceeding estimate

$$|I([f])| = |\int f| \leq \int |f| = \|[f]\|_{L^1} \tag{5.191}$$

(Note that writing $[f]$ instead of $f \in L^1(\mathbb{R})$ is correct but would normally be considered pedantic – at least after you are used to it!)

(6) I should have asked – and might do on the test: What is the norm of I as an element of the dual space of $L^1(\mathbb{R})$. It is 1 – better make sure that you can prove this.

□

Problem 3.2 If $I \subset \mathbb{R}$ is an interval, including possibly $(-\infty, a)$ or (a, ∞), we *define* Lebesgue integrability of a function $f : I \longrightarrow \mathbb{C}$ to mean the Lebesgue integrability of

$$\tilde{f} : \mathbb{R} \longrightarrow \mathbb{C},\ \tilde{f}(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I. \end{cases} \tag{5.192}$$

The integral of f on I is then defined to be

$$\int_I f = \int \tilde{f}. \tag{5.193}$$

(1) Show that the space of such integrable functions on I is linear, denote it $\mathcal{L}^1(I)$.
(2) Show that is f is integrable on I then so is $|f|$.
(3) Show that if f is integrable on I and $\int_I |f| = 0$ then $f = 0$ a.e. in the sense that $f(x) = 0$ for all $x \in I \setminus E$ where $E \subset I$ is of measure zero as a subset of $\mathbb{R}$.
(4) Show that the set of null functions as in the preceeding question is a linear space, denote it $\mathcal{N}(I)$.
(5) Show that $\int_I |f|$ defines a norm on $L^1(I) = \mathcal{L}^1(I)/\mathcal{N}(I)$.
(6) Show that if $f \in \mathcal{L}^1(\mathbb{R})$ then

$$g : I \longrightarrow \mathbb{C},\ g(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I \end{cases} \tag{5.194}$$

is in $\mathcal{L}^1(\mathbb{R})$ an hence that f is integrable on I.
(7) Show that the preceeding construction gives a *surjective and continuous* linear map 'restriction to I'

$$L^1(\mathbb{R}) \longrightarrow L^1(I). \tag{5.195}$$

(Notice that these are the quotient spaces of integrable functions modulo equality a.e.)

Solution:

(1) If f and g are both integrable on I then setting $h = f + g$, $\tilde{h} = \tilde{f} + \tilde{g}$, directly from the definitions, so $f + g$ is integrable on I if f and g are by the linearity of $\mathcal{L}^1(\mathbb{R})$. Similarly if $h = cf$ then $\tilde{h} = c\tilde{f}$ is integrable for any constant c if $\tilde{f}$ is integrable. Thus $\mathcal{L}^1(I)$ is linear.
(2) Again from the definition, $|\tilde{f}| = \tilde{h}$ if $h = |f|$. Thus f integrable on I implies $\tilde{f} \in \mathcal{L}^1(\mathbb{R})$, which, as we know, implies that $|\tilde{f}| \in \mathcal{L}^1(\mathbb{R})$. So in turn $\tilde{h} \in \mathcal{L}^1(\mathbb{R})$ where $h = |f|$, so $|f| \in \mathcal{L}^1(I)$.
(3) If $f \in \mathcal{L}^1(I)$ and $\int_I |f| = 0$ then $\int_{\mathbb{R}} |\tilde{f}| = 0$ which implies that $\tilde{f} = 0$ on $\mathbb{R} \setminus E$ where $E \subset \mathbb{R}$ is of measure zero. Now, $E_I = E \cap I \subset E$ is also of measure zero (as a subset of a set of measure zero) and f vanishes outside E_I.
(4) If $f, g : I \longrightarrow \mathbb{C}$ are both of measure zero in this sense then $f + g$ vanishes on $I \setminus (E_f \cup E_g)$ where $E_f \subset I$ and $E_f \subset I$ are of measure zero. The union of two sets of measure zero (in $\mathbb{R}$) is of measure zero so this shows $f + g$ is null. The same is true of $cf + dg$ for constant c and d, so $\mathcal{N}(I)$ is a linear space.
(5) If $f \in \mathcal{L}^1(I)$ and $g \in \mathcal{N}(I)$ then $|f + g| - |f| \in \mathcal{N}(I)$, since it vanishes where g vanishes. Thus

$$\int_I |f + g| = \int_I |f| \ \forall\ f \in \mathcal{L}^1(I),\ g \in \mathcal{N}(I). \tag{5.196}$$

Thus

$$\|[f]\|_I = \int_I |f| \tag{5.197}$$

is a well-defined function on $L^1(I) = \mathcal{L}^1(\mathbb{R})/\mathcal{N}(I)$ since it is constant on equivalence classes. Now, the norm properties follow from the same properties on the whole of $\mathbb{R}$.

(6) Suppose $f \in \mathcal{L}^1(\mathbb{R})$ and g is defined in (5.194) above by restriction to I. We need to show that $g \in \mathcal{L}^1(\mathbb{R})$. If f_n is an absolutely summable series of step functions converging to f wherever, on $\mathbb{R}$, it converges absolutely consider

$$g_n(x) = \begin{cases} f_n(x) & \text{on } \tilde{I} \\ 0 & \text{on } \mathbb{R} \setminus \tilde{I} \end{cases} \tag{5.198}$$

where $\tilde{I}$ is I made half-open if it isn't already – by adding the lower end-point (if there is one) and removing the upper end-point (if there is one). Then g_n is a step function (which is why we need $\tilde{I}$). Moreover, $\int |g_n| \le \int |f_n|$ so the series g_n is absolutely summable and converges to g_n outside I and at all points inside I where the series is absolutely convergent (since it is then the same as f_n). Thus g is integrable, and since $\tilde{f}$ differs from g by its values at two points, at most, it too is integrable so f is integrable on I by definition.

(7) First we check we do have a map. Namely if $f \in \mathcal{N}(\mathbb{R})$ then g in (5.194) is certainly an element of $\mathcal{N}(I)$. We have already seen that 'restriction to I' maps $\mathcal{L}^1(\mathbb{R})$ into $\mathcal{L}^1(I)$ and since this is clearly a linear map it defines (5.195) – the image only depends on the equivalence class of f. It is clearly linear and to see that it is surjective observe that if $g \in \mathcal{L}^1(I)$ then extending it as zero outside I gives an element of $\mathcal{L}^1(\mathbb{R})$ and the class of this function maps to $[g]$ under (5.195).

Problem 3.3 Really continuing the previous one.

(1) Show that if $I = [a, b)$ and $f \in L^1(I)$ then the restriction of f to $I_x = [x, b)$ is an element of $L^1(I_x)$ for all $a \le x < b$.

(2) Show that the function

$$F(x) = \int_{I_x} f : [a, b) \longrightarrow \mathbb{C} \tag{5.199}$$

is continuous.

(3) Prove that the function $x^{-1} \cos(1/x)$ is not Lebesgue integrable on the interval $(0, 1]$. Hint: Think about it a bit and use what you have shown above.

Solution:

(1) This follows from the previous question. If $f \in L^1([a, b))$ with f' a representative then extending f' as zero outside the interval gives an element of $\mathcal{L}^1(\mathbb{R})$, by defintion. As an element of $L^1(\mathbb{R})$ this does not depend on the choice of f' and then (5.195) gives the restriction to $[x, b)$ as an element of $L^1([x, b))$. This is a linear map.

(2) Using the discussion in the preceeding question, we now that if f_n is an absolutely summable series converging to f' (a representative of f) where it converges absolutely, then for any $a \le x \le b$, we can define

$$f_n' = \chi([a, x))f_n, \;\; f_n'' = \chi([x, b))f_n \tag{5.200}$$

where $\chi([a,b))$ is the characteristic function of the interval. It follows that f_n' converges to $f\chi([a,x))$ and f_n'' to $f\chi([x,b))$ where they converge absolutely. Thus

$$\int_{[x,b)} f = \int f\chi([x,b)) = \sum_n \int f_n'', \int_{[a,x)} f = \int f\chi([a,x)) = \sum_n \int f_n'. \tag{5.201}$$

Now, for step functions, we know that $\int f_n = \int f_n' + \int f_n''$ so

$$\int_{[a,b)} f = \int_{[a,x)} f + \int_{[x,b)} f \tag{5.202}$$

as we have every right to expect. Thus it suffices to show (by moving the end point from a to a general point) that

$$\lim_{x\to a} \int_{[a,x)} f = 0 \tag{5.203}$$

for any f integrable on $[a,b)$. Thus can be seen in terms of a defining absolutely summable sequence of step functions using the usual estimate that

$$|\int_{[a,x)} f| \le \int_{[a,x)} |\sum_{n\le N} f_n| + \sum_{n>N} \int_{[a,x)} |f_n|. \tag{5.204}$$

The last sum can be made small, independent of x, by choosing N large enough. On the other hand as $x \to a$ the first integral, for fixed N, tends to zero by the definition for step functions. This proves (5.204) and hence the continuity of F.

(3) If the function $x^{-1}\cos(1/x)$ were Lebesgue integrable on the interval $(0,1]$ (on which it is defined) then it would be integrable on $[0,1)$ if we define it arbitrarily, say to be 0, at 0. The same would be true of the absolute value and Riemann integration shows us easily that

$$\lim_{t\downarrow 0} \int_t^1 x|\cos(1/x)|dx = \infty. \tag{5.205}$$

This is contrary to the continuity of the integral as a function of the limits just shown.

Problem 3.4 [Harder but still doable] Suppose $f \in \mathcal{L}^1(\mathbb{R})$.

(1) Show that for each $t \in \mathbb{R}$ the translates

$$f_t(x) = f(x-t) : \mathbb{R} \longrightarrow \mathbb{C} \tag{5.206}$$

are elements of $\mathcal{L}^1(\mathbb{R})$.

(2) Show that

$$\lim_{t\to 0} \int |f_t - f| = 0. \tag{5.207}$$

This is called 'Continuity in the mean for integrable functions'. Hint: I will add one!

(3) Conclude that for each $f \in \mathcal{L}^1(\mathbb{R})$ the map (it is a 'curve')

$$\mathbb{R} \ni t \longmapsto [f_t] \in L^1(\mathbb{R}) \tag{5.208}$$

is continuous.

Solution:

(1) If f_n is an absolutely summable series of step functions converging to f where it converges absolutely then $f_n(\cdot - t)$ is such a series converging to $f(\cdot - t)$ for each $t \in \mathbb{R}$. Thus, each of the $f(x-t)$ is Lebesgue integrable, i.e. are elements of $\mathcal{L}^1(\mathbb{R})$

(2) Now, we know that if f_n is a series converging to f as above then

$$\int |f| \le \sum_n \int |f_n|. \tag{5.209}$$

We can sum the first terms and then start the series again and so it follows that for any N,

$$\int |f| \le \int |\sum_{n\le N} f_n| + \sum_{n>N} \int |f_n|. \tag{5.210}$$

Applying this to the series $f_n(\cdot - t) - f_n(\cdot)$ we find that

$$\int |f_t - f| \le \int |\sum_{n\le N} f_n(\cdot - t) - f_n(\cdot)| + \sum_{n>N} \int |f_n(\cdot - t) - f_n(\cdot)| \tag{5.211}$$

The second sum here is bounded by $2 \sum\limits_{n>N} \int |f_n|$. Given $\delta > 0$ we can choose N so large that this sum is bounded by $\delta/2$, by the absolute convergence. So the result is reduce to proving that if $|t|$ is small enough then

$$\int |\sum_{n\le N} f_n(\cdot - t) - f_n(\cdot)| \le \delta/2. \tag{5.212}$$

This however is a finite sum of step functions. So it suffices to show that

$$|\int g(\cdot - t) - g(\cdot)| \to 0 \text{ as } t \to 0 \tag{5.213}$$

for each component, i.e. a constant, c, times the characteristic function of an interval $[a,b)$ where it is bounded by $2|c||t|$.

(3) For the 'curve' f_t which is a map

$$\mathbb{R} \ni t \longmapsto f_t \in \mathcal{L}^1(\mathbb{R}) \tag{5.214}$$

it follows that $f_{t+s} = (f_t)_s$ so we can apply the argument above to show that for each s,

$$\lim_{t\to s} \int |f_t - f_s| = 0 \Longrightarrow \lim_{t\to s} \|[f_t] - [f_s]\|_{L^1} = 0 \tag{5.215}$$

which proves continuity of the map (5.214).

Problem 3.5 In the last problem set you showed that a continuous function on a compact interval, extended to be zero outside, is Lebesgue integrable. Using this, and the fact that step functions are dense in $L^1(\mathbb{R})$ show that the linear space of continuous functions on $\mathbb{R}$ each of which vanishes outside a compact set (which depends on the function) form a dense subset of $L^1(\mathbb{R})$.

Solution: Since we know that step functions (really of course the equivalence classes of step functions) are dense in $L^1(\mathbb{R})$ we only need to show that any step function is the limit of a sequence of continuous functions each vanishing outside a compact set, with respect to L^1. So, it suffices to prove this for the charactertistic function of an interval $[a, b)$ and then multiply by constants and add. The sequence

$$g_n(x) = \begin{cases} 0 & x < a - 1/n \\ n(x - a + 1/n) & a - 1/n \leq x \leq a \\ 0 & a < x < b \\ n(b + 1/n - x) & b \leq x \leq b + 1/n \\ 0 & x > b + 1/n \end{cases} \tag{5.216}$$

is clearly continuous and vanishes outside a compact set. Since

$$\int |g_n - \chi([a, b))| = \int_{a-1/n}^{1} g_n + \int_{b}^{b+1/n} g_n \leq 2/n \tag{5.217}$$

it follows that $[g_n] \to [\chi([a, b))]$ in $L^1(\mathbb{R})$. This proves the density of continuous functions with compact support in $L^1(\mathbb{R})$.

Problem 3.6

(1) If $g : \mathbb{R} \longrightarrow \mathbb{C}$ is bounded and continuous and $f \in \mathcal{L}^1(\mathbb{R})$ show that $gf \in \mathcal{L}^1(\mathbb{R})$ and that

$$\int |gf| \leq \sup_{\mathbb{R}} |g| \cdot \int |f|. \tag{5.218}$$

(2) Suppose now that $G \in \mathcal{C}([0, 1] \times [0, 1])$ is a continuous function (I use $\mathcal{C}(K)$ to denote the continuous functions on a compact metric space). Recall from the preceeding discussion that we have defined $L^1([0, 1])$. Now, using the first part show that if $f \in L^1([0, 1])$ then

$$F(x) = \int_{[0,1]} G(x, \cdot)f(\cdot) \in \mathbb{C} \tag{5.219}$$

(where $\cdot$ is the variable in which the integral is taken) is well-defined for each $x \in [0, 1]$.

(3) Show that for each $f \in L^1([0, 1])$, F is a continuous function on $[0, 1]$.

(4) Show that

$$L^1([0, 1]) \ni f \longmapsto F \in \mathcal{C}([0, 1]) \tag{5.220}$$

is a bounded (i.e. continuous) linear map into the Banach space of continuous functions, with supremum norm, on $[0, 1]$.

Solution:

(1) Let's first assume that $f = 0$ outside $[-1, 1]$. Applying a result form Problem set there exists a sequence of step functions g_n such that for any R, $g_n \to g$ *uniformly* on $[0, 1)$. By passing to a subsequence we can arrange that $\sup_{[-1,1]} |g_n(x) - g_{n-1}(x)| < 2^{-n}$. If f_n is an absolutly summable series of step functions converging a .e . to f we can replace it by $f_n\chi([-1, 1])$ as discussed above, and still have the same conclusion. Thus, from the uniform convergence of g_n,

$$g_n(x) \sum_{k=1}^{n} f_k(x) \to g(x)f(x) \text{ a.e. on } \mathbb{R}. \tag{5.221}$$

So define $h_1 = g_1 f_1$, $h_n = g_n(x) \sum\limits_{k=1}^{n} f_k(x) - g_{n-1}(x) \sum\limits_{k=1}^{n-1} f_k(x)$. This series of step functions converges to $gf(x)$ almost everywhere and since

$$|h_n| \le A|f_n(x)| + 2^{-n} \sum_{k<n} |f_k(x)|, \ \sum_n \int |h_n| \le A \sum_n \int |f_n| + 2 \sum_n \int |f_n| < \infty \tag{5.222}$$

it is absolutely summable. Here A is a bound for $|g_n|$ independent of n. Thus $gf \in \mathcal{L}^1(\mathbb{R})$ under the assumption that $f = 0$ outside $[0, 1)$ and

$$\int |gf| \le \sup |g| \int |f| \tag{5.223}$$

follows from the limiting argument. Now we can apply this argument to f_p which is the restriction of p to the interval $[p, p+1)$, for each $p \in \mathbb{Z}$. Then we get gf as the limit a .e . of the absolutely summable series gf_p where (5.223) provides the absolute summablitly since

$$\sum_p \int |gf_p| \le \sup |g| \sum_p \int_{[p,p+1)} |f| < \infty. \tag{5.224}$$

Thus, $gf \in \mathcal{L}^1(\mathbb{R})$ by a theorem in class and

$$\int |gf| \le \sup |g| \int |f|. \tag{5.225}$$

(2) If $f \in L^1[(0, 1])$ has a representative f' then $G(x, \cdot)f'(\cdot) \in \mathcal{L}^1([0, 1))$ so

$$F(x) = \int_{[0,1]} G(x, \cdot)f(\cdot) \in \mathbb{C} \tag{5.226}$$

is well-defined, since it is indpendent of the choice of f', changing by a null function if f' is changed by a null function.

(3) Now by the uniform continuity of continuous functions on a compact metric space such as $S = [0, 1] \times [0, 1]$ given $\delta > 0$ there exist $\epsilon > 0$ such that

$$\sup_{y \in [0,1]} |G(x, y) - G(x', y)| < \delta \text{ if } |x - x'| < \epsilon. \tag{5.227}$$

Then if $|x - x'| < \epsilon$,

$$|F(x) - F(x')| = |\int_{[0,1]} (G(x, \cdot) - G(x', \cdot))f(\cdot)| \le \delta \int |f|. \tag{5.228}$$

Thus $F \in \mathcal{C}([0, 1])$ is a continuous function on $[0, 1]$. Moreover the map $f \longmapsto F$ is linear and

$$\sup_{[0,1]} |F| \le \sup_S |G| \int_{[0,1]} ||f| \tag{5.229}$$

which is the desired boundedness, or continuity, of the map

$$I : L^1([0, 1]) \longrightarrow \mathcal{C}([0, 1]), \ F(f)(x) = \int G(x, \cdot)f(\cdot), \\ \|I(f)\|_{\sup} \le \sup |G| \|f\|_{L^1}. \tag{5.230}$$

You should be thinking about using Lebesgue's dominated convergence at several points below.

Problem 5.1

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be an element of $\mathcal{L}^1(\mathbb{R})$. Define

$$f_L(x) = \begin{cases} f(x) & x \in [-L, L] \\ 0 & \text{otherwise.} \end{cases} \tag{5.231}$$

Show that $f_L \in \mathcal{L}^1(\mathbb{R})$ and that $\int |f_L - f| \to 0$ as $L \to \infty$.

Solution. If χ_L is the characteristic function of $[-N, N]$ then $f_L = f\chi_L$. If f_n is an absolutely summable series of step functions converging a.e. to f then $f_n\chi_L$ is absolutely summable, since $\int |f_n\chi_L| \le \int |f_n|$ and converges a.e. to f_L, so $f_L \int \mathcal{L}^1(\mathbb{R})$. Certainly $|f_L(x) - f(x)| \to 0$ for each x as $L \to \infty$ and $|f_L(x) - f(x)| \le |f_l(x)| + |f(x)| \le 2|f(x)|$ so by Lebesgue's dominated convergence, $\int |f - f_L| \to 0$.

Problem 5.2 Consider a real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is locally integrable in the sense that

$$g_L(x) = \begin{cases} f(x) & x \in [-L, L] \\ 0 & x \in \mathbb{R} \setminus [-L, L] \end{cases} \tag{5.232}$$

is Lebesgue integrable of each $L \in \mathbb{N}$.

(1) Show that for each fixed L the function

$$g_L^{(N)}(x) = \begin{cases} g_L(x) & \text{if } g_L(x) \in [-N, N] \\ N & \text{if } g_L(x) > N \\ -N & \text{if } g_L(x) < -N \end{cases} \tag{5.233}$$

is Lebesgue integrable.

(2) Show that $\int |g_L^{(N)} - g_L| \to 0$ as $N \to \infty$.

(3) Show that there is a sequence, h_n, of step functions such that

$$h_n(x) \to f(x) \text{ a.e. in } \mathbb{R}. \tag{5.234}$$

(4) Defining

$$h_{n,L}^{(N)} = \begin{cases} 0 & x \notin [-L, L] \\ h_n(x) & \text{if } h_n(x) \in [-N, N],\ x \in [-L, L] \\ N & \text{if } h_n(x) > N,\ x \in [-L, L] \\ -N & \text{if } h_n(x) < -N,\ x \in [-L, L] \end{cases}. \tag{5.235}$$

Show that $\int |h_{n,L}^{(N)} - g_L^{(N)}| \to 0$ as $n \to \infty$.

Solution:

(1) By definition $g_L^{(N)} = \max(-N\chi_L, \min(N\chi_L, g_L))$ where χ_L is the characteristic funciton of $-[L, L]$, thus it is in $\mathcal{L}^1(\mathbb{R})$.

(2) Clearly $g_L^{(N)}(x) \to g_L(x)$ for every x and $|g_L^{(N)}(x)| \le |g_L(x)|$ so by Dominated Convergence, $g_L^{(N)} \to g_L$ in L^1, i.e. $\int |g_L^{(N)} - g_L| \to 0$ as $N \to \infty$ since the sequence converges to 0 pointwise and is bounded by $2|g(x)|$.

(3) Let $S_{L,n}$ be a sequence of step functions converging a.e. to g_L – for example the sequence of partial sums of an absolutely summable series of step functions converging to g_L which exists by the assumed integrability.

Then replacing $S_{L,n}$ by $S_{L,n}\chi_L$ we can assume that the elements all vanish outside $[-N,N]$ but still have convergence a.e. to g_L. Now take the sequence

$$h_n(x) = \begin{cases} S_{k,n-k} & \text{on } [k,-k]\setminus[(k-1),-(k-1)],\ 1\le k\le n, \\ 0 & \text{on } \mathbb{R}\setminus[-n,n]. \end{cases} \tag{5.236}$$

This is certainly a sequence of step functions – since it is a finite sum of step functions for each n – and on $[-L,L]\setminus[-(L-1),(L-1)]$ for large integral L is just $S_{L,n-L}\to g_L$. Thus $h_n(x)\to f(x)$ outside a countable union of sets of measure zero, so also almost everywhere.

(4) This is repetition of the first problem, $h^{(N)}_{n,L}(x)\to g^{(N)}_L$ almost everywhere and $|h^{(N)}_{n,L}|\le N\chi_L$ so $g^{(N)}_L\in\mathcal{L}^1(\mathbb{R})$ and $\int|h^{(N)}_{n,L}-g^{(N)}_L|\to 0$ as $n\to\infty$.

Problem 5.3 Show that $\mathcal{L}^2(\mathbb{R})$ is a Hilbert space – since it is rather central to the course I wanted you to go through the details carefully!

First working with real functions, define $\mathcal{L}^2(\mathbb{R})$ as the set of functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ which are locally integrable and such that $|f|^2$ is integrable.

(1) For such f choose h_n and define g_L, $g^{(N)}_L$ and $h^{(N)}_n$ by (5.232), (5.233) and (5.235).
(2) Show using the sequence $h^{(N)}_{n,L}$ for fixed N and L that $g^{(N)}_L$ and $(g^{(N)}_L)^2$ are in $\mathcal{L}^1(\mathbb{R})$ and that $\int|(h^{(N)}_{n,L})^2-(g^{(N)}_L)^2|\to 0$ as $n\to\infty$.
(3) Show that $(g_L)^2\in\mathcal{L}^1(\mathbb{R})$ and that $\int|(g^{(N)}_L)^2-(g_L)^2|\to 0$ as $N\to\infty$.
(4) Show that $\int|(g_L)^2-f^2|\to 0$ as $L\to\infty$.
(5) Show that $f,\ g\in\mathcal{L}^2(\mathbb{R})$ then $fg\in\mathcal{L}^1(\mathbb{R})$ and that

$$|\int fg|\le\int|fg|\le\|f\|_{L^2}\|g\|_{L^2},\ \|f\|^2_{L^2}=\int|f|^2. \tag{5.237}$$

(6) Use these constructions to show that $\mathcal{L}^2(\mathbb{R})$ is a linear space.
(7) Conclude that the quotient space $L^2(\mathbb{R})=\mathcal{L}^2(\mathbb{R})/\mathcal{N}$, where $\mathcal{N}$ is the space of null functions, is a real Hilbert space.
(8) Extend the arguments to the case of complex-valued functions.

Solution:

(1) Done. I think it should have been $h^{(N)}_{n,L}$.
(2) We already checked that $g^{(N)}_L\in\mathcal{L}^1(\mathbb{R})$ and the same argument applies to $(g^{(N)}_L)$, namely $(h^{(N)}_{n,L})^2\to g^{(N)}_L$ almost everywhere and both are bounded by $N^2\chi_L$ so by dominated convergence

$$\begin{gathered}(h^{(N)}_{n,L})^2\to g^{(N)}_L)^2\le N^2\chi_L \text{ a.e. } \Longrightarrow g^{(N)}_L)^2\in\mathcal{L}^1(\mathbb{R}) \text{ and}\\ |h^{(N)}_{n,L})^2-g^{(N)}_L)^2|\to 0 \text{ a.e. },\\ |h^{(N)}_{n,L})^2-g^{(N)}_L)^2|\le 2N^2\chi_L\Longrightarrow\int|h^{(N)}_{n,L})^2-g^{(N)}_L)^2|\to 0.\end{gathered} \tag{5.238}$$

(3) Now, as $N\to\infty$, $(g^{(N)}_L)^2\to(g_L)^2$ a .e . and $(g^{(N)}_L)^2\to(g_L)^2\le f^2$ so by dominated convergence, $(g_L)^2\in\mathcal{L}^1$ and $\int|(g^{(N)}_L)^2-(g_L)^2|\to 0$ as $N\to\infty$.
(4) The same argument of dominated convergence shows now that $g^2_L\to f^2$ and $\int|g^2_L-f^2|\to 0$ using the bound by $f^2\in\mathcal{L}^1(\mathbb{R})$.

(5) What this is all for is to show that $fg \in \mathcal{L}^1(\mathbb{R})$ if f, $F = g \in \mathcal{L}^2(\mathbb{R})$ (for easier notation). Approximate each of them by sequences of step functions as above, $h^{(N)}_{n,L}$ for f and $H^{(N)}_{n,L}$ for g. Then the product sequence is in $\mathcal{L}^1$ – being a sequence of step functions – and

$$h^{(N)}_{n,L}(x)H^{(N)}_{n,L}(x) \to g^{(N)}_L(x)G^{(N)}_L(x) \tag{5.239}$$

almost everywhere and with absolute value bounded by $N^2\chi_L$. Thus by dominated convergence $g^{(N)}_L G^{(N)}_L \in \mathcal{L}^1(\mathbb{R})$. Now, let $N \to \infty$; this sequence converges almost everywhere to $g_L(x)G_L(x)$ and we have the bound

$$|g^{(N)}_L(x)G^{(N)}_L(x)| \le |f(x)F(x)|\frac{1}{2}(f^2+F^2) \tag{5.240}$$

so as always by dominated convergence, the limit $g_LG_L \in \mathcal{L}^1$. Finally, letting $L \to \infty$ the same argument shows that $fF \in \mathcal{L}^1(\mathbb{R})$. Moreover, $|fF| \in \mathcal{L}^1(\mathbb{R})$ and

$$|\int fF| \le \int |fF| \le \|f\|_{L^2}\|F\|_{L^2} \tag{5.241}$$

where the last inequality follows from Cauchy's inequality – if you wish, first for the approximating sequences and then taking limits.

(6) So if f, $g \in \mathcal{L}^2(\mathbb{R})$ are real-value, $f+g$ is certainly locally integrable and

$$(f+g)^2 = f^2 + 2fg + g^2 \in \mathcal{L}^1(\mathbb{R}) \tag{5.242}$$

by the discussion above. For constants $f \in \mathcal{L}^2(\mathbb{R})$ implies $cf \in \mathcal{L}^2(\mathbb{R})$ is directly true.

(7) The argument is the same as for $\mathcal{L}^1$ versus L^1. Namely $\int f^2 = 0$ implies that $f^2 = 0$ almost everywhere which is equivalent to $f = 0$ a@ė. Then the norm is the same for all $f+h$ where h is a null function since fh and h^2 are null so $(f+h)^2 = f^2 + 2fh + h^2$. The same is true for the inner product so it follows that the quotient by null functions

$$L^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R})/\mathcal{N} \tag{5.243}$$

is a preHilbert space.

However, it remains to show completeness. Suppose $\{[f_n]\}$ is an absolutely summable series in $L^2(\mathbb{R})$ which means that $\sum\limits_n \|f_n\|_{L^2} < \infty$. It follows that the cut-off series $f_n\chi_L$ is absolutely summable in the L^1 sense since

$$\int |f_n\chi_L| \le L^{\frac{1}{2}}(\int f_n^2)^{\frac{1}{2}} \tag{5.244}$$

by Cauchy's inequality. Thus if we set $F_n = \sum\limits_{k-1}^{n} f_k$ then $F_n(x)\chi_L$ converges almost everywhere for each L so in fact

$$F_n(x) \to f(x) \text{ converges almost everywhere.} \tag{5.245}$$

We want to show that $f \in \mathcal{L}^2(\mathbb{R})$ where it follows already that f is locally integrable by the completeness of L^1. Now consider the series

$$g_1 = F_1^2,\ g_n = F_n^2 - F_{n-1}^2. \tag{5.246}$$

The elements are in $\mathcal{L}^1(\mathbb{R})$ and by Cauchy's inequality for $n > 1$,

$$\begin{aligned}\int |g_n| = \int |F_n^2 - F_{n-1}|^2 &\le \|F_n - F_{n-1}\|_{L^2}\|F_n + F_{n-1}\|_{L^2} \\ &\le \|f_n\|_{L^2} 2\sum_k \|f_k\|_{L^2}\end{aligned} \tag{5.247}$$

where the triangle inequality has been used. Thus in fact the series g_n is absolutely summable in $\mathcal{L}^1$

$$\sum_n \int |g_n| \le 2(\sum_n \|f_n\|_{L^2})^2. \tag{5.248}$$

So indeed the sequence of partial sums, the F_n^2 converge to $f^2 \in \mathcal{L}^1(\mathbb{R})$. Thus $f \in \mathcal{L}^2(\mathbb{R})$ and moroever

$$\int (F_n - f)^2 = \int F_n^2 + \int f^2 - 2\int F_n f \to 0 \text{ as } n \to \infty. \tag{5.249}$$

Indeed the first term converges to $\int f^2$ and, by Cauchys inequality, the series of products $f_n f$ is absulutely summable in L^1 with limit f^2 so the third term converges to $-2\int f^2$. Thus in fact $[F_n] \to [f]$ in $L^2(\mathbb{R})$ and we have proved completeness.

(8) For the complex case we need to check linearity, assuming f is locally integrable and $|f|^2 \in \mathcal{L}^1(\mathbb{R})$. The real part of f is locally integrable and the approximation $F_L^{(N)}$ discussed above is square integrable with $(F_L^{(N)})^2 \le |f|^2$ so by dominated convergence, letting first $N \to \infty$ and then $L \to \infty$ the real part is in $\mathcal{L}^2(\mathbb{R})$. Now linearity and completeness follow from the real case.

Problem 5.4

Consider the sequence space

$$h^{2,1} = \left\{ c : \mathbb{N} \ni j \longmapsto c_j \in \mathbb{C}; \sum_j (1+j^2)|c_j|^2 < \infty \right\}. \tag{5.250}$$

(1) Show that

$$h^{2,1} \times h^{2,1} \ni (c,d) \longmapsto \langle c, d\rangle = \sum_j (1+j^2) c_j \overline{d_j} \tag{5.251}$$

is an Hermitian inner form which turns $h^{2,1}$ into a Hilbert space.

(2) Denoting the norm on this space by $\|\cdot\|_{2,1}$ and the norm on l^2 by $\|\cdot\|_2$, show that

$$h^{2,1} \subset l^2,\ \|c\|_2 \le \|c\|_{2,1}\ \forall\ c \in h^{2,1}. \tag{5.252}$$

Solution:

(1) The inner product is well defined since the series defining it converges absolutely by Cauchy's inequality:

$$\begin{gathered}\langle c, d\rangle = \sum_j (1+j^2)^{\frac12} c_j \overline{(1+j^2)^{\frac12} d_j}, \\ \sum_j |(1+j^2)^{\frac12} c_j \overline{(1+j^2)^{\frac12} d_j}| \le (\sum_j (1+j^2)|c_j|^2)^{\frac12} (\sum_j (1+j^2)|d_j|^2)^{\frac12}.\end{gathered} \tag{5.253}$$

It is sesquilinear and positive definite since

$$\|c\|_{2,1} = (\sum_j (1+j^2)|c_j|^2)^{\frac{1}{2}} \tag{5.254}$$

only vanishes if all c_j vanish. Completeness follows as for l^2 – if $c^{(n)}$ is a Cauchy sequence then each component $c_j^{(n)}$ converges, since $(1+j)^{\frac{1}{2}} c_j^{(n)}$ is Cauchy. The limits c_j define an element of $h^{2,1}$ since the sequence is bounded and

$$\sum_{j=1}^{N} (1+j^2)^{\frac{1}{2}} |c_j|^2 = \lim_{n\to\infty} \sum_{j=1}^{N} (1+j^2)|c_j^{(n)}|^2 \le A \tag{5.255}$$

where A is a bound on the norms. Then from the Cauchy condition $c^{(n)} \to c$ in $h^{2,1}$ by passing to the limit as $m \to \infty$ in $\|c^{(n)} - c^{(m)}\|_{2,1} \le \epsilon$.

(2) Clearly $h^{2,2} \subset l^2$ since for any finite N

$$\sum_{j=1}^{N} |c_j|^2 \sum_{j=1}^{N} (1+j)^2 |c_j|^2 \le \|c\|_{2,1}^2 \tag{5.256}$$

and we may pass to the limit as $N \to \infty$ to see that

$$\|c\|_{l^2} \le \|c\|_{2,1}. \tag{5.257}$$

Problem 5.5 In the separable case, prove Riesz Representation Theorem directly.

Choose an orthonormal basis $\{e_i\}$ of the separable Hilbert space H. Suppose $T : H \longrightarrow \mathbb{C}$ is a bounded linear functional. Define a sequence

$$w_i = \overline{T(e_i)},\ i \in \mathbb{N}. \tag{5.258}$$

(1) Now, recall that $|Tu| \le C\|u\|_H$ for some constant C. Show that for every finite N,

$$\sum_{j=1}^{N} |w_i|^2 \le C^2. \tag{5.259}$$

(2) Conclude that $\{w_i\} \in l^2$ and that

$$w = \sum_i w_i e_i \in H. \tag{5.260}$$

(3) Show that

$$T(u) = \langle u, w\rangle_H \ \forall\ u \in H \text{ and } \|T\| = \|w\|_H. \tag{5.261}$$

Solution:

(1) The finite sum $w_N = \sum\limits_{i=1}^{N} w_i e_i$ is an element of the Hilbert space with norm $\|w_N\|_N^2 = \sum\limits_{i=1}^{N} |w_i|^2$ by Bessel's identity. Expanding out

$$T(w_N) = T(\sum_{i=1}^{N} w_i e_i) = \sum_{i=1}^{n} w_i T(e_i) = \sum_{i=1}^{N} |w_i|^2 \tag{5.262}$$

and from the continuity of T,

$$|T(w_N)| \le C\|w_N\|_H \Longrightarrow \|w_N\|_H^2 \le C\|w_N\|_H \Longrightarrow \|w_N\|^2 \le C^2 \tag{5.263}$$

which is the desired inequality.

(2) Letting $N \to \infty$ it follows that the infinite sum converges and

$$\sum_i |w_i|^2 \le C^2 \Longrightarrow w = \sum_i w_i e_i \in H \tag{5.264}$$

since $\|w_N - w\| \le \sum_{j>N} |w_i|^2$ tends to zero with N.

(3) For any $u \in H$ $u_N = \sum_{i=1}^{N} \langle u, e_i \rangle e_i$ by the completness of the $\{e_i\}$ so from the continuity of T

$$\begin{aligned} T(u) = \lim_{N\to\infty} T(u_N) &= \lim_{N\to\infty} \sum_{i=1}^{N} \langle u, e_i \rangle T(e_i) \\ &= \lim_{N\to\infty} \sum_{i=1}^{N} \langle u, w_i e_i \rangle = \lim_{N\to\infty} \langle u, w_N \rangle = \langle u, w \rangle \end{aligned} \tag{5.265}$$

where the continuity of the inner product has been used. From this and Cauchy's inequality it follows that $\|T\| = \sup_{\|u\|_H=1} |T(u)| \le \|w\|$. The converse follows from the fact that $T(w) = \|w\|_H^2$.

Solution 5.21. If $f \in L^1(\mathbb{R}^k \times \mathbb{R}^p)$ show that there exists a set of measure zero $E \subset \mathbb{R}^k$ such that

$$x \in \mathbb{R}^k \setminus E \Longrightarrow g_x(y) = f(x, y) \text{ defines } g_x \in L^1(\mathbb{R}^p), \tag{5.266}$$

that $F(x) = \int g_x$ defines an element $F \in L^1(\mathbb{R}^k)$ and that

$$\int_{\mathbb{R}^k} F = \int_{\mathbb{R}^k \times \mathbb{R}^p} f. \tag{5.267}$$

Note: These identities are usually written out as an equality of an iterated integral and a 'regular' integral:

$$\int_{\mathbb{R}^k} \int_{\mathbb{R}^p} f(x, y) = \int f. \tag{5.268}$$

It is often used to 'exchange the order of integration' since the hypotheses are the same if we exchange the variables.

Solution. This is not hard but is a little tricky (I believe Fubini never understood what the fuss was about).

Certainly this result holds for step functions, since ultimately it reduces to the case of the characterisitic function for a 'rectrangle'.

In the general case we can take an absolutely summable sequence f_j of step functions summing to f

$$f(x, y) = \sum_j f_j(x, y) \text{ whenever } \sum_j |f_j(x, y)| < \infty. \tag{5.269}$$

This, after all, is our definition of integrability.

Now, consider the functions

$$h_j(x) = \int_{\mathbb{R}^p} |f_j(x, \cdot)| \tag{5.270}$$

which are step functions. Moreover this series is absolutely summable since

$$\sum_j \int_{\mathbb{R}^k} |h_j| = \sum_j \int_{\mathbb{R}^k\times\mathbb{R}^p} |f_j|. \tag{5.271}$$

Thus the series $\sum_j h_j(x)$ converges (absolutely) on the complement of a set $E \subset \mathbb{R}^k$ of measure zero. It follows that the series of step functions

$$F_j(x) = \int_{\mathbb{R}^p} f_j(x,\cdot) \tag{5.272}$$

converges absolutely on $\mathbb{R}^k \setminus E$ since $|f_j(x)| \le h_j(x)$. Thus,

$$F(x) = \sum_j F_j(x) \text{ converges absolutely on } \mathbb{R}^k \setminus E \tag{5.273}$$

defines $F \in L^1(\mathbb{R}^k)$ with

$$\int_{\mathbb{R}^k} F = \sum_j \int_{\mathbb{R}^k} F_j = \sum_j \int_{\mathbb{R}^k\times\mathbb{R}^p} f_j = \int_{\mathbb{R}^k\times\mathbb{R}^p} f. \tag{5.274}$$

The absolute convergence of $\sum_j h_j(x)$ for a given x is precisely the absolutely summability of $f_k(x,y)$ as a series of functions of y,

$$\sum_j \int_{\mathbb{R}^p} |f_j(x,\cdot)| = \sum_j h_j(x). \tag{5.275}$$

Thus for each $x \notin E$ the series $\sum_j f_k(x,y)$ must converge absolutely for $y \in (\mathbb{R}^p \setminus E_x)$ where E_x is a set of measure zero. But (5.269) shows that the sum is $g_x(y) = f(x,y)$ at all such points, so for $x \notin E$, $f(x,\cdot) \in L^1(\mathbb{R}^p)$ (as the limit of an absolutely summable series) and

$$F(x) = \int_{\mathbb{R}^p} g_x. \tag{5.276}$$

With (5.274) this is what we wanted to show. $\square$

Problem 4.1

Let H be a normed space in which the norm satisfies the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)\ \forall\ u,v \in H. \tag{5.277}$$

Show that the norm comes from a positive definite sesquilinear (i.e. Hermitian) inner product. Big Hint:- Try

$$(u,v) = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2\right)! \tag{5.278}$$

Solution: Setting $u = v$, even without the parallelogram law,

$$(u,u) = \frac{1}{4}\left\|2u\|^2 + i\|(1+i)u\|^2 - i\|(1-i)u\|^2\right) = \|u\|^2. \tag{5.279}$$

So the point is that the parallelogram law shows that (u,v) is indeed an Hermitian inner product. Taking complex conjugates and using properties of the norm, $\|u + iv\| = \|v - iu\|$ etc

$$\overline{(u,v)} = \frac{1}{4}\left(\|v+u\|^2 - \|v-u\|^2 - i\|v-iu\|^2 + i\|v+iu\|^2\right) = (v,u). \tag{5.280}$$

Thus we only need check the linearity in the first variable. This *is* a little tricky! First compute away. Directly from the identity $(u, -v) = -(u, v)$ so $(-u, v) = -(u, v)$ using (5.280). Now,

(5.281)

$$\begin{aligned}(2u, v) = &\frac{1}{4}\big(\|u + (u + v)\|^2 - \|u + (u - v)\|^2\\ &+ i\|u + (u + iv)\|^2 - i\|u + (u - iv)\|^2\big)\\ =&\frac{1}{2}\big(\|u + v\|^2 + \|u\|^2 - \|u - v\|^2 - \|u\|^2\\ &+ i\|(u + iv)\|^2 + i\|u\|^2 - i\|u - iv\|^2 - i\|u\|^2\big)\\ &- \frac{1}{4}\left(\|u - (u + v)\|^2 - \|u - (u - v)\|^2 + i\|u - (u + iv)\|^2 - i\|u - (u - iv)\|^2\right)\\ =&2(u, v).\end{aligned}$$

Using this and (5.280), for any u, u' and v,

$$\begin{aligned}(u + u', v) = &\frac{1}{2}(u + u', 2v)\\ =&\frac{1}{2}\frac{1}{4}\big(\|(u + v) + (u' + v)\|^2 - \|(u - v) + (u' - v)\|^2\\ &+ i\|(u + iv) + (u - iv)\|^2 - i\|(u - iv) + (u' - iv)\|^2\big)\\ =&\frac{1}{4}\big(\|u + v\| + \|u' + v\|^2 - \|u - v\| - \|u' - v\|^2\\ &+ i\|(u + iv)\|^2 + i\|u - iv\|^2 - i\|u - iv\| - i\|u' - iv\|^2\big)\\ &- \frac{1}{2}\frac{1}{4}\big(\|(u + v) - (u' + v)\|^2 - \|(u - v) - (u' - v)\|^2\\ &+ i\|(u + iv) - (u - iv)\|^2 - i\|(u - iv) = (u' - iv)\|^2\big)\\ &= (u, v) + (u', v).\end{aligned} \tag{5.282}$$

Using the second identity to iterate the first it follows that $(ku, v) = k(u, v)$ for any u and v and any positive integer k. Then setting $nu' = u$ for any other positive integer and $r = k/n$, it follows that

$$(ru, v) = (ku', v) = k(u', v) = rn(u', v) = r(u, v) \tag{5.283}$$

where the identity is reversed. Thus it follows that $(ru, v) = r(u, v)$ for any rational r. Now, from the definition both sides are continuous in the first element, with respect to the norm, so we can pass to the limit as $r \to x$ in $\mathbb{R}$. Also directly from the definition,

$$(iu, v) = \frac{1}{4}\left(\|iu + v\|^2 - \|iu - v\|^2 + i\|iu + iv\|^2 - i\|iu - iv\|^2\right) = i(u, v) \tag{5.284}$$

so now full linearity in the first variable follows and that is all we need.

Problem 4.2

Let H be a finite dimensional (pre)Hilbert space. So, by definition H has a basis $\{v_i\}_{i=1}^n$, meaning that any element of H can be written

$$v = \sum_i c_i v_i \tag{5.285}$$

and there is no dependence relation between the v_i's – the presentation of $v = 0$ in the form (5.285) is unique. Show that H has an orthonormal basis, $\{e_i\}_{i=1}^n$ satisfying $(e_i, e_j) = \delta_{ij}$ ($= 1$ if $i = j$ and 0 otherwise). Check that for the orthonormal basis the coefficients in (5.285) are $c_i = (v, e_i)$ and that the map

$$T : H \ni v \longmapsto ((v, e_i)) \in \mathbb{C}^n \tag{5.286}$$

is a linear isomorphism with the properties

$$(u, v) = \sum_i (Tu)_i \overline{(Tv)_i}, \ \|u\|_H = \|Tu\|_{\mathbb{C}^n} \ \forall \ u, v \in H. \tag{5.287}$$

Why is a finite dimensional preHilbert space a Hilbert space?

Solution: Since H is assumed to be finite dimensional, it has a basis v_i, $i = 1, \dots, n$. This basis can be replaced by an orthonormal basis in n steps. First replace v_1 by $e_1 = v_1/\|v_1\|$ where $\|v_1\| \neq 0$ by the linear indepedence of the basis. Then replace v_2 by

$$e_2 = w_2/\|w_2\|, \ w_2 = v_2 - (v_2, e_1)e_1. \tag{5.288}$$

Here $w_2 \perp e_1$ as follows by taking inner products; w_2 cannot vanish since v_2 and e_1 must be linearly independent. Proceeding by finite induction we may assume that we have replaced $v_1, v_2, \dots, v_k$, $k < n$, by $e_1, e_2, \dots, e_k$ which are orthonormal and span the same subspace as the v_i's $i = 1, \dots, k$. Then replace v_{k+1} by

$$e_{k+1} = w_{k+1}/\|w_{k+1}\|, \ w_{k+1} = v_{k+1} - \sum_{i=1}^{k} (v_{k+1}, e_i)e_i. \tag{5.289}$$

By taking inner products, $w_{k+1} \perp e_i$, $i = 1, \dots, k$ and $w_{k+1} \neq 0$ by the linear independence of the v_i's. Thus the orthonormal set has been increased by one element preserving the same properties and hence the basis can be orthonormalized.

Now, for each $u \in H$ set

$$c_i = (u, e_i). \tag{5.290}$$

It follows that $U = u - \sum\limits_{i=1}^{n} c_i e_i$ is orthogonal to all the e_i since

$$(u, e_j) = (u, e_j) - \sum_i c_i(e_i, e_j) = (u.e_j) - c_j = 0. \tag{5.291}$$

This implies that $U = 0$ since writing $U = \sum\limits_i d_i e_i$ it follows that $d_i = (U, e_i) = 0$.

Now, consider the map (5.286). We have just shown that this map is injective, since $Tu = 0$ implies $c_i = 0$ for all i and hence $u = 0$. It is linear since the c_i depend linearly on u by the linearity of the inner product in the first variable. Moreover it is surjective, since for any $c_i \in \mathbb{C}$, $u = \sum\limits_i c_i e_i$ reproduces the c_i through (5.290). Thus T is a linear isomorphism and the first identity in (5.287) follows by direct computation:-

$$\begin{aligned} \sum_{i=1}^{n} (Tu)_i \overline{(Tv)_i} &= \sum_i (u, e_i) \\ &= (u, \sum_i (v, e_i)e_i) \\ &= (u, v). \end{aligned} \tag{5.292}$$

Setting $u = v$ shows that $\|Tu\|_{\mathbb{C}^n} = \|u\|_H$.

Now, we know that $\mathbb{C}^n$ is complete with its standard norm. Since T is an isomorphism, it carries Cauchy sequences in H to Cauchy sequences in $\mathbb{C}^n$ and T^{-1} carries convergent sequences in $\mathbb{C}^n$ to convergent sequences in H, so every Cauchy sequence in H is convergent. Thus H is complete.

Hint: Don't pay too much attention to my hints, sometimes they are a little off-the-cuff and may not be very helpfult. An example being the old hint for Problem 6.2!

Problem 6.1 Let H be a separable Hilbert space. Show that $K \subset H$ is compact if and only if it is closed, bounded and has the property that any sequence in K which is weakly convergent sequence in H is (strongly) convergent.

Hint:- In one direction use the result from class that any bounded sequence has a weakly convergent subsequence.

Problem 6.2 Show that, in a separable Hilbert space, a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if the weak limit, v satisfies

$$\|v\|_H = \lim_{n\to\infty} \|v_n\|_H. \tag{5.293}$$

Hint:- To show that this condition is sufficient, expand

$$(v_n - v, v_n - v) = \|v_n\|^2 - 2\operatorname{Re}(v_n, v) + \|v\|^2. \tag{5.294}$$

Problem 6.3 Show that a subset of a separable Hilbert space is compact if and only if it is closed and bounded and has the property of 'finite dimensional approximation' meaning that for any $\epsilon > 0$ there exists a linear subspace $D_N \subset H$ of finite dimension such that

$$d(K, D_N) = \sup_{u\in K} \inf_{v\in D_N} \{d(u, v)\} \le \epsilon. \tag{5.295}$$

Hint:- To prove necessity of this condition use the 'equi-small tails' property of compact sets with respect to an orthonormal basis. To use the finite dimensional approximation condition to show that any weakly convergent sequence in K is strongly convergent, use the convexity result from class to define the sequence $\{v'_n\}$ in D_N where v'_n is the closest point in D_N to v_n. Show that v'_n is weakly, hence strongly, convergent and hence deduce that $\{v_n\}$ is Cauchy.

Problem 6.4 Suppose that $A : H \longrightarrow H$ is a bounded linear operator with the property that $A(H) \subset H$ is finite dimensional. Show that if v_n is weakly convergent in H then Av_n is strongly convergent in H.

Problem 6.5 Suppose that H_1 and H_2 are two different Hilbert spaces and $A : H_1 \longrightarrow H_2$ is a bounded linear operator. Show that there is a unique bounded linear operator (the adjoint) $A^* : H_2 \longrightarrow H_1$ with the property

$$(Au_1, u_2)_{H_2} = (u_1, A^*u_2)_{H_1} \ \forall\ u_1 \in H_1,\ u_2 \in H_2. \tag{5.296}$$

Problem 8.1 Show that a continuous function $K : [0, 1] \longrightarrow L^2(0, 2\pi)$ has the property that the Fourier series of $K(x) \in L^2(0, 2\pi)$, for $x \in [0, 1]$, converges uniformly in the sense that if $K_n(x)$ is the sum of the Fourier series over $|k| \le n$ then $K_n : [0, 1] \longrightarrow L^2(0, 2\pi)$ is also continuous and

$$\sup_{x\in[0,1]} \|K(x) - K_n(x)\|_{L^2(0,2\pi)} \to 0. \tag{5.297}$$

Hint. Use one of the properties of compactness in a Hilbert space that you proved earlier.

Problem 8.2

Consider an integral operator acting on $L^2(0,1)$ with a kernel which is continuous – $K \in \mathcal{C}([0,1]^2)$. Thus, the operator is

$$Tu(x) = \int_{(0,1)} K(x,y)u(y). \tag{5.298}$$

Show that T is bounded on L^2 (I think we did this before) and that it is in the norm closure of the finite rank operators.

Hint. Use the previous problem! Show that a continuous function such as K in this Problem defines a continuous map $[0,1] \ni x \longmapsto K(x,\cdot) \in \mathcal{C}([0,1])$ and hence a continuous function $K : [0,1] \longrightarrow L^2(0,1)$ then apply the previous problem with the interval rescaled.

Here is an even more expanded version of the hint: You can think of $K(x,y)$ as a continuous function of x with values in $L^2(0,1)$. Let $K_n(x,y)$ be the continuous function of x and y given by the previous problem, by truncating the Fourier series (in y) at some point n. Check that this defines a finite rank operator on $L^2(0,1)$ – yes it maps into continuous functions but that is fine, they are Lebesgue square integrable. Now, the idea is the difference $K - K_n$ defines a bounded operator with small norm as n becomes large. It might actually be clearer to do this the other way round, exchanging the roles of x and y.

Problem 8.3 Although we have concentrated on the Lebesgue integral in one variable, you proved at some point the covering lemma in dimension 2 and that is pretty much all that was needed to extend the discussion to 2 dimensions. Let's just assume you have assiduously checked everything and so you know that $L^2((0,2\pi)^2)$ is a Hilbert space. Sketch a proof – noting anything that you are not sure of – that the functions $\exp(ikx + ily)/2\pi$, $k, l \in \mathbb{Z}$, form a complete orthonormal basis.

P9.1: Periodic functions

Let $\mathbb{S}$ be the circle of radius 1 in the complex plane, centered at the origin, $\mathbb{S} = \{z; |z| = 1\}$.

(1) Show that there is a 1-1 correspondence

$$\begin{aligned}\mathcal{C}^0(\mathbb{S}) = \{u : \mathbb{S} \longrightarrow \mathbb{C},\ \text{continuous}\} \longrightarrow \\ \{u : \mathbb{R} \longrightarrow \mathbb{C};\ \text{continuous and satisfying } u(x+2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}.\end{aligned} \tag{5.299}$$

Solution: The map $E : \mathbb{R} \ni \theta \longmapsto e^{2\pi i\theta} \in \mathbb{S}$ is continuous, surjective and 2π-periodic and the inverse image of any point of the circle is precisly of the form $\theta + 2\pi\mathbb{Z}$ for some $\theta \in \mathbb{R}$. Thus composition defines a map

$$E^* : \mathcal{C}^0(\mathbb{S}) \longrightarrow \mathcal{C}^0(\mathbb{R}),\ E^*f = f \circ E. \tag{5.300}$$

This map is a linear bijection.

(2) Show that there is a 1-1 correspondence

$$\begin{aligned}L^2(0,2\pi) \longleftrightarrow \{u \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}); u\big|_{(0,2\pi)} \in \mathcal{L}^2(0,2\pi) \\ \text{and } u(x+2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}/\mathcal{N}_P\end{aligned} \tag{5.301}$$

where $\mathcal{N}_P$ is the space of null functions on $\mathbb{R}$ satisfying $u(x+2\pi) = u(x)$ for all $x \in \mathbb{R}$.

Solution: Our original definition of $L^2(0,2\pi)$ is as functions on $\mathbb{R}$ which are square-integrable and vanish outside $(0,2\pi)$. Given such a function u we can define an element of the right side of (5.301) by assigning a value at 0 and then extending by periodicity

$$\tilde{u}(x) = u(x - 2n\pi),\ n \in \mathbb{Z} \tag{5.302}$$

where for each $x \in \mathbb{R}$ there is a unique integer n so that $x - 2n\pi \in [0, 2\pi)$. Null functions are mapped to null functions his way and changing the choice of value at 0 changes $\tilde{u}$ by a null function. This gives a map as in (5.301) and restriction to $(0, 2\pi)$ is a 2-sided invese.

(3) If we denote by $L^2(\mathbb{S})$ the space on the left in (5.301) show that there is a dense inclusion

$$\mathcal{C}^0(\mathbb{S}) \longrightarrow L^2(\mathbb{S}). \tag{5.303}$$

Solution: Combining the first map and the inverse of the second gives an inclusion. We know that continuous functions vanishing near the end-points of $(0, 2\pi)$ are dense in $L^2(0, 2\pi)$ so density follows.

So, the idea is that we can freely think of functions on $\mathbb{S}$ as 2π-periodic functions on $\mathbb{R}$ and conversely.

P9.2: Schrödinger's operator

Since that is what it is, or at least it is an example thereof:

$$-\frac{d^2u(x)}{dx^2} + V(x)u(x) = f(x),\ x \in \mathbb{R}, \tag{5.304}$$

(1) First we will consider the special case $V = 1$. Why not $V = 0$? – Don't try to answer this until the end!

Solution: The reason we take $V = 1$, or at least some other positive constant is that there is 1-d space of periodic solutions to $d^2u/dx^2 = 0$, namely the constants.

(2) Recall how to solve the differential equation

$$-\frac{d^2u(x)}{dx^2} + u(x) = f(x),\ x \in \mathbb{R}, \tag{5.305}$$

where $f(x) \in \mathcal{C}^0(\mathbb{S})$ is a continuous, 2π-periodic function on the line. Show that there is a unique 2π-periodic and twice continuously differentiable function, u, on $\mathbb{R}$ satisfying (5.305) and that this solution can be written in the form

$$u(x) = (Sf)(x) = \int_{0,2\pi} A(x,y)f(y) \tag{5.306}$$

where $A(x,y) \in \mathcal{C}^0(\mathbb{R}^2)$ satisfies $A(x+2\pi, y+2\pi) = A(x,y)$ for all $(x,y) \in \mathbb{R}$.

Extended hint: In case you managed to avoid a course on differential equations! First try to find a solution, igonoring the periodicity issue. To do so one can (for example, there are other ways) factorize the differential operator involved, checking that

$$-\frac{d^2u(x)}{dx^2} + u(x) = -(\frac{dv}{dx} + v) \text{ if } v = \frac{du}{dx} - u \tag{5.307}$$

since the cross terms cancel. Then recall the idea of integrating factors to see that

$$\begin{aligned}\frac{du}{dx} - u &= e^x \frac{d\phi}{dx},\ \phi = e^{-x}u,\\ \frac{dv}{dx} + v &= e^{-x}\frac{d\psi}{dx},\ \psi = e^x v.\end{aligned} \tag{5.308}$$

Now, solve the problem by integrating twice from the origin (say) and hence get a solution to the differential equation (5.305). Write this out explicitly as a double integral, and then change the order of integration to write the solution as

$$u'(x) = \int_{0,2\pi} A'(x,y)f(y)dy \tag{5.309}$$

where A' is continuous on $\mathbb{R}\times[0,2\pi]$. Compute the difference $u'(2\pi)-u'(0)$ and $\frac{du'}{dx}(2\pi) - \frac{du'}{dx}(0)$ as integrals involving f. Now, add to u' as solution to the *homogeneous* equation, for $f=0$, namely $c_1e^x + c_2e^{-x}$, so that the new solution to (5.305) satisfies $u(2\pi) = u(0)$ and $\frac{du}{dx}(2\pi) = \frac{du}{dx}(0)$. Now, check that u is given by an integral of the form (5.306) with A as stated.

Solution: Integrating once we find that if $v = \frac{du}{dx} - u$ then

$$v(x) = -e^{-x}\int_0^x e^s f(s)ds,\ u'(x) = e^x\int_0^x e^{-t}v(t)dt \tag{5.310}$$

gives a solution of the equation $-\frac{d^2u'}{dx^2} + u'(x) = f(x)$ so combinging these two and changing the order of integration

$$\begin{gathered} u'(x) = \int_0^x \tilde{A}(x,y)f(y)dy,\ \tilde{A}(x,y) = \frac{1}{2}\left(e^{y-x} - e^{x-y}\right)\\ u'(x) = \int_{(0,2\pi)} A'(x,y)f(y)dy,\ A'(x,y) = \begin{cases}\frac{1}{2}\left(e^{y-x}-e^{x-y}\right) & x\ge y\\ 0 & x\le y.\end{cases}\end{gathered} \tag{5.311}$$

Here A' is continuous since $\tilde{A}$ vanishes at $x=y$ where there might otherwise be a discontinuity. This is the only solution which vanishes with its derivative at 0. If it is to extend to be periodic we need to add a solution of the homogeneous equation and arrange that

$$u = u' + u'',\ u'' = ce^x + de^{-x},\ u(0) = u(2\pi),\ \frac{du}{dx}(0) = \frac{du}{dx}(2\pi). \tag{5.312}$$

So, computing away we see that

$$u'(2\pi) = \int_0^{2\pi}\frac{1}{2}\left(e^{y-2\pi} - e^{2\pi-y}\right)f(y),\ \frac{du'}{dx}(2\pi) = -\int_0^{2\pi}\frac{1}{2}\left(e^{y-2\pi}+e^{2\pi-y}\right)f(y). \tag{5.313}$$

Thus there is a unique solution to (5.312) which must satify

$$\begin{gathered} c(e^{2\pi}-1) + d(e^{-2\pi}-1) = -u'(2\pi),\ c(e^{2\pi}-1) - d(e^{-2\pi}-1) = -\frac{du'}{dx}(2\pi)\\ (e^{2\pi}-1)c = \frac{1}{2}\int_0^{2\pi}\left(e^{2\pi-y}\right)f(y),\ (e^{-2\pi}-1)d = -\frac{1}{2}\int_0^{2\pi}\left(e^{y-2\pi}\right)f(y).\end{gathered} \tag{5.314}$$

Putting this together we get the solution in the desired form:

$$u(x) = \int_{(0.2\pi)} A(x,y)f(y),\ A(x,y) = A'(x,y) + \frac{1}{2}\frac{e^{2\pi-y+x}}{e^{2\pi}-1} - \frac{1}{2}\frac{e^{-2\pi+y-x}}{e^{-2\pi}-1} \Longrightarrow$$
$$A(x,y) = \frac{\cosh(|x-y|-\pi)}{e^{\pi}-e^{-\pi}}. \tag{5.315}$$

(3) Check, either directly or indirectly, that $A(y,x) = A(x,y)$ and that A is real.

Solution: Clear from (5.315).

(4) Conclude that the operator S extends by continuity to a bounded operator on $L^2(\mathbb{S})$.

Solution. We know that $\|S\| \le \sqrt{2\pi}\sup|A|$.

(5) Check, probably indirectly rather than directly, that

$$S(e^{ikx}) = (k^2+1)^{-1}e^{ikx},\ k \in \mathbb{Z}. \tag{5.316}$$

Solution. We know that Sf is the unique solution with periodic boundary conditions and e^{ikx} satisfies the boundary conditions and the equation with $f = (k^2+1)e^{ikx}$.

(6) Conclude, either from the previous result or otherwise that S is a compact self-adjoint operator on $L^2(\mathbb{S})$.

Soluion: Self-adjointness and compactness follows from (5.316) since we know that the $e^{ikx}/\sqrt{2\pi}$ form an orthonormal basis, so the eigenvalues of S tend to 0. (Myabe better to say it is approximable by finite rank operators by truncating the sum).

(7) Show that if $g \in \mathcal{C}^0(\mathbb{S}))$ then Sg is twice continuously differentiable. Hint: Proceed directly by differentiating the integral.

Solution: Clearly Sf is continuous. Going back to the formula in terms of $u' + u''$ we see that both terms are twice continuously differentiable.

(8) From (5.316) conclude that $S = F^2$ where F is also a compact self-adjoint operator on $L^2(\mathbb{S})$ with eigenvalues $(k^2+1)^{-\frac{1}{2}}$.

Solution: Define $F(e^{ikx}) = (k^2+1)^{-\frac{1}{2}}e^{ikx}$. Same argument as above applies to show this is compact and self-adjoint.

(9) Show that $F : L^2(\mathbb{S}) \longrightarrow \mathcal{C}^0(\mathbb{S})$.

Solution. The series for Sf

$$Sf(x) = \frac{1}{2\pi}\sum_k (2k^2+1)^{-\frac{1}{2}}(f, e^{ikx})e^{ikx} \tag{5.317}$$

converges absolutely and uniformly, using Cauchy's inequality – for instance it is Cauchy in the supremum norm:

$$\|\sum_{|k|>p}(2k^2+1)^{-\frac{1}{2}}(f, e^{ikx})e^{ikx}| \le \epsilon\|f\|_{L^2} \tag{5.318}$$

for p large since the sum of the squares of the eigenvalues is finite.

(10) Now, going back to the real equation (5.304), we assume that V is continuous, real-valued and 2π-periodic. Show that if u is a twice-differentiable 2π-periodic function satisfying (5.304) for a given $f \in \mathcal{C}^0(\mathbb{S})$ then

$$u + S((V-1)u) = Sf \text{ and hence } u = -F^2((V-1)u) + F^2 f \tag{5.319}$$

and hence conclude that

$$u = Fv \text{ where } v \in L^2(\mathbb{S}) \text{ satisfies } v + (F(V-1)F)v = Ff \tag{5.320}$$

where $V-1$ is the operator defined by multiplication by $V-1$.

Solution: If u satisfies (5.304) then

$$-\frac{d^2u(x)}{dx^2} + u(x) = -(V(x)-1)u(x) + f(x) \tag{5.321}$$

so by the uniqueness of the solution with periodic boundary conditions, $u = -S(V-1)u + Sf$ so $u = F(-F(V-1)u + Ff)$. Thus indeed $u = Fv$ with $v = -F(V-1)u + Ff$ which means that v satisfies

$$v + F(V-1)Fv = Ff. \tag{5.322}$$

(11) Show the converse, that if $v \in L^2(\mathbb{S})$ satisfies

$$v + (F(V-1)F)v = Ff, \ f \in \mathcal{C}^0(\mathbb{S}) \tag{5.323}$$

then $u = Fv$ is 2π-periodic and twice-differentiable on $\mathbb{R}$ and satisfies (5.304).

Solution. If $v \in L^2(0,2\pi)$ satisfies (5.323) then $u = Fv \in \mathcal{C}^0(\mathbb{S})$ satisfies $u + F^2(V-1)u = F^2 f$ and since $F^2 = S$ maps $\mathcal{C}^0(\mathbb{S})$ into twice continuously differentiable functions it follows that u satisfies (5.304).

(12) Apply the Spectral theorem to $F(V-1)F$ (including why it applies) and show that there is a sequence λ_j in $\mathbb{R}\setminus\{0\}$ with $|\lambda_j| \to 0$ such that for all $\lambda \in \mathbb{C}\setminus\{0\}$, the equation

$$\lambda v + (F(V-1)F)v = g, \ g \in L^2(\mathbb{S}) \tag{5.324}$$

has a unique solution for every $g \in L^2(\mathbb{S})$ if and only if $\lambda \neq \lambda_j$ for any j.

Solution: We know that $F(V-1)F$ is self-adjoint and compact so $L^2(0.2\pi)$ has an orthonormal basis of eigenfunctions of $-F(V-1)F$ with eigenvalues λ_j. This sequence tends to zero and (5.324), for given $\lambda \in \mathbb{C}\setminus\{0\}$, if and only if has a solution if and only if it is an isomorphism, meaning $\lambda \neq \lambda_j$ is not an eigenvalue of $-F(V-1)F$.

(13) Show that for the λ_j the solutions of

$$\lambda_j v + (F(V-1)F)v = 0, v \in L^2(\mathbb{S}), \tag{5.325}$$

are all continuous 2π-periodic functions on $\mathbb{R}$.

Solution: If v satisfies (5.325) with $\lambda_j \neq 0$ then $v = -F(V-1)F/\lambda_j \in \mathcal{C}^0(\mathbb{S})$.

(14) Show that the corresponding functions $u = Fv$ where v satisfies (5.325) are all twice continuously differentiable, 2π-periodic functions on $\mathbb{R}$ satisfying

$$-\frac{d^2u}{dx^2} + (1 - s_j + s_jV(x))u(x) = 0, \ s_j = 1/\lambda_j. \tag{5.326}$$

Solution: Then $u = Fv$ satisfies $u = -S(V-1)u/\lambda_j$ so is twice continuously differentiable and satisfies (5.326).

(15) Conversely, show that if u is a twice continuously differentiable and 2π-periodic function satisfying

$$-\frac{d^2u}{dx^2} + (1 - s + sV(x))u(x) = 0, \ s \in \mathbb{C}, \tag{5.327}$$

and u is not identically 0 then $s = s_j$ for some j.

Solution: From the uniquess of periodic solutions $u = -S(V-1)u/\lambda_j$ as before.

(16) Finally, conclude that Fredholm's alternative holds for the equation in (5.304)

THEOREM 24. *For a given real-valued, continuous 2π-periodic function V on $\mathbb{R}$, either* (5.304) *has a unique twice continuously differentiable, 2π-periodic, solution for each f which is continuous and 2π-periodic or else there exists a finite, but positive, dimensional space of twice continuously differentiable 2π-periodic solutions to the homogeneous equation*

$$-\frac{d^2w(x)}{dx^2} + V(x)w(x) = 0,\ x \in \mathbb{R}, \tag{5.328}$$

and (5.304) *has a solution if and only if $\int_{(0,2\pi)} fw = 0$ for every 2π-periodic solution, w, to* (5.328).

Solution: This corresponds to the special case $\lambda_j = 1$ above. If λ_j is not an eigenvalue of $-F(V-1)F$ then

$$v + F(V-1)Fv = Ff \tag{5.329}$$

has a unque solution for all f, otherwise the necessary and sufficient condition is that $(v, Ff) = 0$ for all v' satisfying $v' + F(V-1)Fv' = 0$. Correspondingly either (5.304) has a unique solution for all f or the necessary and sufficient condition is that $(Fv', f) = 0$ for all $w = Fv'$ (remember that F is injetive) satisfying (5.328).

Problem P10.1 Let H be a separable, infinite dimensional Hilbert space. Show that the direct sum of two copies of H is a Hilbert space with the norm

$$H \oplus H \ni (u_1, u_2) \longmapsto (\|u_1\|_H^2 + \|u_2\|_H^2)^{\frac{1}{2}} \tag{5.330}$$

either by constructing an isometric isomorphism

$$T : H \longrightarrow H \oplus H,\ \text{1-1 and onto},\ \|u\|_H = \|Tu\|_{H\oplus H} \tag{5.331}$$

or otherwise. In any case, construct a map as in (5.331).

Solution: Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of H, which exists by virtue of the fact that it is an infinite-dimensional but separable Hilbert space. Define the map

$$T : H \ni u \longrightarrow (\sum_{i=1}^{\infty}(u, e_{2i-1})e_i, \sum_{i=1}^{\infty}(u, e_{2i})e_i) \in H \oplus H \tag{5.332}$$

The convergence of the Fourier Bessel series shows that this map is well-defined and linear. Injectivity similarly follows from the fact that $Tu = 0$ in the image implies that $(u, e_i) = 0$ for all i and hence $u = 0$. Surjectivity is also clear from the fact that

$$S : H \oplus H \ni (u_1, u_2) \longmapsto \sum_{i=1}^{\infty} ((u_1, e_i)e_{2i-1} + (u_2, e_i)e_{2i}) \in H \tag{5.333}$$

is a 2-sided inverse and Bessel's identity implies isometry since $\|S(u_1, u_2)\|^2 = \|u_1\|^2 + \|u_2\|^2$

Problem P10.2 One can repeat the preceding construction any finite number of times. Show that it can be done 'countably often' in the sense that if H is a separable, infinite dimensional, Hilbert space then

$$l_2(H) = \{u : \mathbb{N} \longrightarrow H; \|u\|^2_{l_2(H)} = \sum_i \|u_i\|^2_H < \infty\} \tag{5.334}$$

has a Hilbert space structure and construct an explicit isometric isomorphism from $l_2(H)$ to H.

Solution: A similar argument as in the previous problem works. Take an orthormal basis e_i for H. Then the elements $E_{i,j} \in l_2(H)$, which for each i, i consist of the sequences with 0 entries except the jth, which is e_i, given an orthonromal basis for $l_2(H)$. Orthormality is clear, since with the inner product is

$$(u, v)_{l_2(H)} = \sum_j (u_j, v_j)_H. \tag{5.335}$$

Completeness follows from completeness of the orthonormal basis of H since if $v = \{v_j\}$ $(v, E_{j,i}) = 0$ for all j implies $v_j = 0$ in H. Now, to construct an isometric isomorphism just choose an isomorphism $m : \mathbb{N}^2 \longrightarrow \mathbb{N}$ then

$$Tu = v,\ v_j = \sum_i (u, e_{m(i,j)})e_i \in H. \tag{5.336}$$

I would expect you to go through the argument to check injectivity, surjectivity and that the map is isometric.

Problem P10.3 Recall, or perhaps learn about, the winding number of a closed curve with values in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We take as given the following fact:[3] If $Q = [0,1]^N$ and $f : Q \longrightarrow \mathbb{C}^*$ is continuous then for each choice of $b \in \mathbb{C}$ satisfying $\exp(2\pi i b) = f(0)$, there exists a unique continuous function $F : Q \longrightarrow \mathbb{C}$ satisfying

$$\exp(2\pi i F(q)) = f(q),\ \forall\ q \in Q \text{ and } F(0) = b. \tag{5.337}$$

Of course, you are free to change b to $b + n$ for any $n \in \mathbb{Z}$ but then F changes to $F + n$, just shifting by the same integer.

(1) Now, suppose $c : [0,1] \longrightarrow \mathbb{C}^*$ is a closed curve – meaning it is continuous and $c(1) = c(0)$. Let $C : [0,1] \longrightarrow \mathbb{C}$ be a choice of F for $N = 1$ and $f = c$. Show that the winding number of the closed curve c may be defined *unambiguously* as

$$\mathrm{wn}(c) = C(1) - C(0) \in \mathbb{Z}. \tag{5.338}$$

Solution: Let C', be another choice of F in this case. Now, $g(t) = C'(t) - C(t)$ is continuous and satisfies $\exp(2\pi g(t)) = 1$ for all $t \in [0,1]$ so by the uniqueness must be constant, thus $C'(1) - C'(0) = C(1) - C(0)$ and the winding number is well-defined.

(2) Show that $\mathrm{wn}(c)$ is *constant under homotopy*. That is if $c_i : [0,1] \longrightarrow \mathbb{C}^*$, $i = 1, 2$, are two closed curves so $c_i(1) = c_i(0)$, $i = 1, 2$, which are *homotopic* through closed curves in the sense that there exists $f : [0,1]^2 \longrightarrow \mathbb{C}^*$ continuous and such that $f(0, x) = c_1(x)$, $f(1, x) = c_2(x)$ for all $x \in [0,1]$ and $f(y, 0) = f(y, 1)$ for all $y \in [0,1]$, then $\mathrm{wn}(c_1) = \mathrm{wn}(c_2)$.

Solution: Choose F using the 'fact' corresponding to this homotopy f. Since f is periodic in the second variable – the two curves $f(y, 0)$, and $f(y, 1)$ are the same – so by the uniquess $F(y, 0) - F(y, 1)$ must be constant, hence $\mathrm{wn}(c_2) = F(1, 1) - F(1, 0) = F(0, 1) - F(0, 0) = \mathrm{wn}(c_1)$.

[3] Of course, you are free to give a proof – it is not hard.

(3) Consider the closed curve $L_n : [0,1] \ni x \longmapsto e^{2\pi i x}\,\mathrm{Id}_{n\times n}$ of $n \times n$ matrices. Using the standard properties of the determinant, show that this curve is not homotopic to the identity through closed curves in the sense that there does not exist a continuous map $G : [0,1]^2 \longrightarrow \mathrm{GL}(n)$, with values in the invertible $n \times n$ matrices, such that $G(0,x) = L_n(x)$, $G(1,x) \equiv \mathrm{Id}_{n\times n}$ for all $x \in [0,1]$, $G(y,0) = G(y,1)$ for all $y \in [0,1]$.

Solution: The determinant is a continuous (actually it is analytic) map which vanishes precisely on non-invertible matrices. Moreover, it is given by the product of the eigenvalues so

$$\det(L_n) = \exp(2\pi i x n). \tag{5.339}$$

This is a periodic curve with winding number n since it has the 'lift' xn. Now, if there were to exist such an homotopy of periodic curves of matrices, always invertible, then by the previous result the winding number of the determinant would have to remain constant. Since the winding number for the constant curve with value the identity is 0 such an homotopy cannot exist.

Problem P10.4 Consider the closed curve corresponding toL_n above in the case of a separable but now infinite dimensional Hilbert space:

$$L : [0,1] \ni x \longmapsto e^{2\pi i x}\,\mathrm{Id}_H \in \mathrm{GL}(H) \subset \mathcal{B}(H) \tag{5.340}$$

taking values in the invertible operators on H. Show that after identifying H with $H \oplus H$ as above, there is a continuous map

$$M : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H) \tag{5.341}$$

with values in the invertible operators and satisfying

$$M(0,x) = L(x),\ M(1,x)(u_1,u_2) = (e^{4\pi i x}u_1, u_2),\ M(y,0) = M(y,1),\ \forall\, x, y \in [0,1]. \tag{5.342}$$

Hint: So, think of $H \oplus H$ as being 2-vectors (u_1, u_2) with entries in H. This allows one to think of 'rotation' between the two factors. Indeed, show that

$$U(y)(u_1,u_2) = (\cos(\pi y/2)u_1 + \sin(\pi y/2)u_2, -\sin(\pi y/2)u_1 + \cos(\pi y/2)u_2) \tag{5.343}$$

defines a continuous map $[0,1] \ni y \longmapsto U(y) \in \mathrm{GL}(H \oplus H)$ such that $U(0) = \mathrm{Id}$, $U(1)(u_1,u_2) = (u_2, -u_1)$. Now, consider the 2-parameter family of maps

$$U^{-1}(y)V_2(x)U(y)V_1(x) \tag{5.344}$$

where $V_1(x)$ and $V_2(x)$ are defined on $H \oplus H$ as multiplication by $\exp(2\pi i x)$ on the first and the second component respectively, leaving the other fixed.

Solution: Certainly $U(y)$ is invertible since its inverse is $U(-y)$ as follows in the two dimensional case. Thus the map $W(x,y)$ on $[0,1]^2$ in (5.344) consists of invertible and bounded operators on $H \oplus H$, meaning a continuous map $W : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H)$. When $x = 0$ or $x = 1$, both $V_1(x)$ and $v_2(x)$ reduce to the identiy, and hence $W(0,y) = W(1,y)$ for all y, so W is periodic in x. Moreove at $y = 0$ $W(x,0) = V_2(x)V_1(x)$ is exactly $L(x)$, a multiple of the identity. On the other hand, at $x = 1$ we can track composite as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} e^{2\pi i x}u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} u_2 \\ -e^{2\pi x}u_1 \end{pmatrix} \longmapsto \begin{pmatrix} u_2 \\ -e^{4\pi x}u_1 \end{pmatrix} \longmapsto \begin{pmatrix} e^{4\pi x}u_1 \\ u_2 \end{pmatrix}. \tag{5.345}$$

This is what is required of M in (5.342).

Problem P10.5 Using a rotation similar to the one in the preceeding problem (or otherwise) show that there is a continuous map

$$G : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H) \tag{5.346}$$

such that

$$\begin{aligned} G(0,x)(u_1,u_2) &= (e^{2\pi ix}u_1, e^{-2\pi ix}u_2), \\ G(1,x)(u_1,u_2) &= (u_1,u_2),\ G(y,0) = G(y,1)\ \forall\ x,y \in [0,1]. \end{aligned} \tag{5.347}$$

Solution: We can take

$$G(y,x) = U(-y)\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & e^{-2\pi ix} \end{pmatrix} U(y) \begin{pmatrix} e^{2\pi ix} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{5.348}$$

By the same reasoning as above, this is an homotopy of closed curves of invertible operators on $H \oplus H$ which satisfies (5.347).

Problem P10.6 Now, think about combining the various constructions above in the following way. Show that on $l_2(H)$ there is an homotopy like (5.346), $\tilde{G} : [0,1]^2 \longrightarrow \mathrm{GL}(l_2(H))$, (very like in fact) such that

$$\begin{aligned} \tilde{G}(0,x)\{u_k\}_{k=1}^\infty &= \left\{\exp((-1)^k 2\pi ix)u_k\right\}_{k=1}^\infty, \\ \tilde{G}(1,x) &= \mathrm{Id},\ \tilde{G}(y,0) = \tilde{G}(y,1)\ \forall\ x,y \in [0,1]. \end{aligned} \tag{5.349}$$

Solution: We can divide $l_2(H)$ into its odd an even parts

$$D : l_2(H) \ni v \longmapsto (\{v_{2i-1}\}, \{v_{2i}\}) \in l_2(H) \oplus l_2(H) \longleftrightarrow H \oplus H. \tag{5.350}$$

and then each copy of $l_2(H)$ on the right with H (using the same isometric isomorphism). Then the homotopy in the previous problem is such that

$$\tilde{G}(x,y) = D^{-1}G(y,x)D \tag{5.351}$$

accomplishes what we want.

Problem P10.7: Eilenberg's swindle For any separable, infinite-dimensional, Hilbert space, construct an homotopy – meaning a continuous map $G : [0,1]^2 \longrightarrow \mathrm{GL}(H)$ – with $G(0,x) = L(x)$ in (5.340) and $G(1,x) = \mathrm{Id}$ and of course $G(y,0) = G(y,1)$ for all $x,y \in [0,1]$.

Hint: Just put things together – of course you can rescale the interval at the end to make it all happen over $[0,1]$. First 'divide H into 2 copies of itself' and deform from L to $M(1,x)$ in (5.342). Now, 'divide the second H up into $l_2(H)$' and apply an argument just like the preceding problem to turn the identity on this factor into alternating terms multiplying by $\exp(\pm 4\pi ix)$ – starting with $-$. Now, you are on $H \oplus l_2(H)$, 'renumbering' allows you to regard this as $l_2(H)$ again and when you do so your curve has become alternate multiplication by $\exp(\pm 4\pi ix)$ (with $+$ first). Finally then, apply the preceding problem again, to deform to the identity (always of course through closed curves). Presto, Eilenberg's swindle!

Solution: By rescaling the variables above, we now have three homotopies, always through periodic families. On $H \oplus H$ between $L(x) = e^{2\pi ix}\,\mathrm{Id}$ and the matrix

$$\begin{pmatrix} e^{4\pi ix}\,\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{5.352}$$

Then on $H \oplus l_2(H)$ we can deform from

$$\begin{pmatrix} e^{4\pi ix}\,\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \text{ to } \begin{pmatrix} e^{4\pi ix}\,\mathrm{Id} & 0 \\ 0 & \tilde{G}(0,x) \end{pmatrix} \tag{5.353}$$

with $\tilde{G}(0,x)$ in (5.349). However we can then identify

$$H \oplus l_2(H) = l_2(H),\ (u,v) \longmapsto w = \{w_j\},\ w_1 = u,\ w_{j+1} = v_j,\ j \geq 1. \tag{5.354}$$

This turns the matrix of operators in (5.353) into $\tilde{G}(0,x)^{-1}$. Now, we can apply the same construction to deform this curve to the identity. Notice that this really does ultimately give an homotopy, which we can renormalize to be on $[0,1]$ if you insist, of curves of operators on H – at each stage we transfer the homotopy back to H.

Bibliography

[1] W.W.L Chen, *http://rutherglen.science.mq.edu.au/wchen/lnlfafolder/lnlfa.html* Chen's notes
[2] B. S. Mitjagin, *The homotopy structure of a linear group of a Banach space*, Uspehi Mat. Nauk **25** (1970), no. 5(155), 63–106. MR 0341523 (49 #6274a)
[3] W. Rudin, *Principles of mathematical analysis*, 3rd ed., McGraw Hill, 1976.
[4] T.B. Ward, *http://www.mth.uea.ac.uk/∼h720/teaching/functionalanalysis/materials/FAnotes.pdf*
[5] I.F. Wilde, *http://www.mth.kcl.ac.uk/∼iwilde/notes/fa1/*.