SHORT SOLUTIONS FOR 18.102 FINAL EXAM, SPRING 2015

Problem 1

Consider the subspace $H \subset \mathcal{C}[0, 2\pi]$ consisting of those continuous functions on $[0, 2\pi]$ which satisfy

$$u(x) = \int_0^x U, \ \forall \ x \in [0, 2\pi] \tag{1}$$

for some $U \in L^2(0, 2\pi)$ (depending on u of course). Show that the function U is determined by u (given that it exists) and that

$$\|u\|_H^2 = \int_{(0,2\pi)} |U|^2 \tag{2}$$

turns H into a Hilbert space.

Solution: If $U \in L^2([0, 2\pi]$ then the integral (1) defines a continuous function since

$$|u(x) - u(y)| \le \int_y^x |U| \le |x - y|^{\frac{1}{2}} \|U\|_{L^2}, \ \sup |u| \le (2\pi)^{\frac{1}{2}} \|U\|_{L^2}$$

so in fact $I : L^2[0, 2\pi] \longrightarrow \mathcal{C}([0, 2\pi])$ is a bounded linear map. To say that U, if it exists, is determined by u is to say that this map in injective. The vanishing of u means precisely that $\langle \chi_{[0,x]}, U \rangle_{L^2} = 0$. Taking linear combination, this means that U is orthogonal to all step functions. However the step functions are dense in $\mathcal{C}([0, 2\pi])$ in the supremum norm and hence in $L^2[0, 2\pi]$, so this imples $U = 0$ in L^2. Since I is injective, it is a bijection onto its range, H and this gives a bijection to $L^2[0, 2\pi]$, making H into a Hilbert space.

Other arguments that work include computing the Fourier coefficients of U to shows that they are determined by u. In general a measurable set (where $U > 0$ for instance) does not contain a close measurable set of positive measure, so that sort of approach is hard.

Problem 2

Consider the space of those complex-valued functions on $[0, 1]$ for which there is a constant $C \ge 0$ (depending on the function) such that

$$|u(x) - u(y)| \le C|x - y|^{\frac{1}{2}} \ \forall \ x, y \in [0, 1]. \tag{3}$$

Show that this is a Banach space with norm

$$\|u\|_{\frac{1}{2}} = \sup_{[0,1]} |u(x)| + \inf_{(3) \text{ holds}} C. \tag{4}$$

Solution: These are the Hölder-$\frac{1}{2}$ functions, $\mathcal{C}^{\frac{1}{2}}[0,1]$. If (3) holds for some constant $C \geq 0$ then

$$\|u\|' = \sup_{x \neq y \in [0,1]} \frac{|u(x) - u(y)|}{|x-y|^{\frac{1}{2}}} < \infty$$

is the smallest such constant and the putative norm is

$$\|u\|_{\frac{1}{2}} = \sup_{[0,1]} |u(x)| + \|u\|'.$$

I expected you to quickly check that this is a norm and that the space of functions $\mathcal{C}^{\frac{1}{2}}[0,1]$ is linear. The inequality (3) implies that the elements of $\mathcal{C}^{\frac{1}{2}}$ are continuous and if u_n is a Cauchy sequence it follows that it is Cauchy with respect to the supremum norm, $\|u\|_\infty \leq \|u\|_{\frac{1}{2}}$ by definition. Since this space is complete, $u_n \to u$ uniformly with $u : [0,1] \longrightarrow \mathbb{C}$ continuous. A Cauchy sequence is bounded in norm so

$$|u_n(x) - u_n(y)| \leq C|x-y|^{\frac{1}{2}}$$

with C independent of n. Passing to the limit $n \to \infty$ shows that $u \in \mathcal{C}^{\frac{1}{2}}$. The Cauchy condition itself implies that given $\epsilon > 0$ there exists N such that

$$|(u_n(x) - u_m(x)) - (u_n(y) - u_m(y))| \leq C\epsilon|x-y|^{\frac{1}{2}} \ \forall \ n, m > N.$$

Taking $m \to \infty$ and using the convergence in supremum norm it follows that $\|u - u_n\|_{\frac{1}{2}} \to 0$.

Generally well done.

Problem 3

Let $A_j \subset \mathbb{R}$ be a sequence of subsets with the property that the characteristic function, χ_j of A_j, is integrable for each j. Show that the characteristic function of $\mathbb{R} \setminus A$, where $A = \bigcup_j A_j$ is locally integrable.

Solution: Since for each j, $\chi_j \in L^1(\mathbb{R})$ are real functions it follows that $\chi_{[k]}$, the characteristic function of $\bigcup_{j \leq k} A_j$ is in $L^1(\mathbb{R})$ as the supremum of a finite number of L^1 functions and so is $\chi_{[-R,R]}\chi_{[k]}$ for each $R > 0$. The L^1 integral of this increasing sequence if bounded by $4R$ so by Monotone Convergence, $\chi_{[-R,R]}\chi_A \in L^1(\mathbb{R})$ where χ_A is the characteristic function of $A = \bigcup_j A_j$. The difference $\chi_{[-R,R]}(1 - \chi_{[\infty]})$ is therefore also integrable and this is $\chi_{[-R,R]}\chi_B$ where $B = \mathbb{R} \setminus A$, so χ_B is locally integrable.

Problem 4

Let A be a Hilbert-Schmidt operator on a separable Hilbert space H, which means that for some orthonormal basis $\{e_i\}$

$$\|A\|^2_{\mathrm{HS}} = \sum_i \|Ae_i\|^2 < \infty. \tag{5}$$

Using Bessel's identity to expand $\|Ae_i\|^2$ with respect to another orthonormal basis $\{f_j\}$ show that $\sum\limits_j \|A^* f_j\|^2 = \sum\limits_i \|Ae_i\|^2$. Conclude that the sum in (5) is independent of the othornormal basis used to define it and that the Hilbert-Schmidt operators form a Hilbert space.

Solution: Everyone got the proof that the Hilbert-Schmidt norm is independent of the onb. I expected you to quickly check linearity and the norm properties.

Taking a unit vector u and an orthonormal basis e_i and orthonormalizing the sequence $u, e_1 \dots,$ gives an orthonormal sequence with first element u. Thus

$$\|Au\| \le \|A\|_{\mathrm{HS}} \Longrightarrow \|A\| \le \|A\|_{\mathrm{HS}}$$

So, if A_n is Cauchy with respect to the Hilbert-Schmidt norm it is Cauchy in the norm on $\mathcal{B}$, which is complete, so $A_n \to A$ in norm. A Cauchy sequence is bounded in norm so for any finite M it follows that

$$\sum_{i<M} \|A_n e_i\|^2 \le \sup \|A_n\|_{\mathrm{HS}} \le C < \infty.$$

Passing to the limit as $n \to \infty$ using norm convergence and then letting $M \to \infty$ it follows that A is Hilbert-Schmidt and then the Cauchy condition shows that given $\epsilon > 0$ there exists N such that $n, m > N$ implies

$$\sum_{i<M} \|A_n e_i - A_m e_i\|^2 \le \epsilon^2 \ \forall\ M.$$

Taking $m \to \infty$ then $M \to \infty$ it follows that $A_n \to A$ in the Hilbert-Schmidt norm.

Problem 5

Let A be a compact self-adjoint operator on a separable Hilbert space and suppose that for *every* orthonormal basis

$$\sum_i |(Ae_i, e_i)| < \infty. \tag{6}$$

Show that the eigenvalues of A, if infinite in number, form a sequence in l^1. Solution:

Every compact self-adjoint operator has an orthonormal basis of eigenvectors so if the eigenvalues are listed with multiplicity then

$$\sum_i |\lambda_i| = \sum_i |\langle Ae_i, e_i\rangle| < \infty$$

from (6). If the eigenvalues are listed without multiplicity, the sum is smaller so still in l^1. [Either interpretation is acceptable.]

Problem 6

For $u \in L^2(0, 1)$ show that

$$Iu(x) = \int_0^x u(t)dt,\ x \in (0, 1)$$

is a bounded linear operator on $L^2(0,1)$. If $V \in \mathcal{C}([0,1])$, is real-valued and $V \geq 0$, show that there is a bounded linear operator B on $L^2(0,1)$ such that

$$B^2 u = u + I^* M_V I u \ \forall\ u \in L^2(0,1) \tag{7}$$

where M_V denotes multiplication by V.

Solution: Iu is continuous if $u \in L^2(0,1)$ (see Problem 1 ...) since

$$|u(x) - u(y)| \leq \int_x^y |u| \leq |x-y|^{\frac{1}{2}} \|u\|_{L^2}$$

by Cauchy-Schwartz. By the linearity of the integral, this is a linear may from $L^2(0,1)$ to $\mathcal{C}([0,1])$ and

$$\|Iu\|_{L^2} \leq \sup |u| \leq \|u\|_{L^2}$$

so it is bounded on L^2. The image of the unit ball in $L^2(0,1)$ is a uniformly bounded and equicontinuous set in $\mathcal{C}(0,1)$ so has compact closure by Arscoli-Arzela. The image under the inclusion into $L^2(0,1)$ is therefore also precompact and hence I is a compact operator.

Multiplication by a continuous function $V \geq 0$ gives a bounded and self-adjoint operator on $L^2(0,1)$,

$$\|M_V\| \leq \sup V,\ \langle M_v u, v\rangle = \int V u \overline{v} = \langle u, M_V v\rangle$$

so $I^* M_V I$ is compact (since the compact operators form a $*$-ideal) and self-adjoint, since $(ABC)^* = C^* B^* A^*$. It follows that $L^2(0,1)$ has an orthornomal basis of eigenfunctions e_i for $I^* M_V I$ with eigenvalues

$$\lambda_i = \langle I^* M_V I e_i, e_i\rangle = \langle M_V I e_i, I e_i\rangle \geq 0$$

by the positivity of V. So

$$B e_i = (1 + \lambda_i)^{\frac{1}{2}} e_i$$

defines, by continuous extension, a bounded operator on $L^2(0,1)$ such that

$$B^2 = \mathrm{Id} + I^* M_V I.$$

Or, without the compactness of I (which can also be proved by checking tails in the Fourier basis) one needs to show that the spectrum of $A = \mathrm{Id} + I^* M_V I$ is contained in $[0, \|A\|]$. This is not quite obvious, but follows from the positivity. Namely the operator $A - \frac{1}{2}\|A\| \,\mathrm{Id}$ satisfies

$$\frac{1}{2}\|A\| \|u\| \geq \langle (A - \frac{1}{2}\|A\| \,\mathrm{Id}) u, u\rangle \geq -\frac{1}{2}\|A\|$$

so its spectrum is contained in $[-\frac{1}{2}\|A\|, \frac{1}{2}\|A\|]$. Then B is well-defined by the functional calculus.