18.102 QUESTIONS FOR FINAL EXAM, SPRING 2015

Problem 1

Let $T : H_1 \longrightarrow H_2$ be a continuous linear map between two Hilbert spaces and suppose that T is both surjective and injective.

(1) Let $A_2 \in \mathcal{K}(H_2)$ be a compact linear operator on H_2, show that there is a compact linear operator $A_1 \in \mathcal{K}(H_1)$ such that

$$A_2 T = T A_1.$$

(2) If A_2 is self-adjoint (as well as being compact) and H_1 is infinite dimensional, show that A_1 has an infinite number of linearly independent eigenvectors.

Problem 2

Let a be a continuous function on the square $[0, 2\pi]^2$. Show that $[0, 2\pi] \ni x \longmapsto a(x, \cdot) \in \mathcal{C}^0([0, 2\pi])$ is a continuous map into the continuous functions with supremum norm. Using this, or otherwise, show that the Fourier coefficients with respect to the second variable

$$c_k(x) = \frac{1}{2\pi} \int_0^{2\pi} a(x,t) e^{ikt} dt \tag{1}$$

are continuous functions on $[0, 2\pi]$.

Problem 3

Let H_i, $i = 1, 2$ be two Hilbert spaces with inner products $(\cdot, \cdot)_i$ and suppose that $I : H_1 \longrightarrow H_2$ is a continuous linear map between them. Suppose that the range of I is dense and that I is injective.

(1) Show that there is a continuous linear map $Q : H_2 \longrightarrow H_1$ such that $(u, I(f))_2 = (Qu, f)_1 \ \forall\ f \in H_1$.
(2) Show that as a map from H_1 to itself, $Q \circ I$ is bounded, self-adjoint and injective.

Problem 4

Let $u_n : [0, 2\pi] \longrightarrow \mathbb{C}$ be a sequence of continuously differentiable functions which is uniformly bounded, with bounded derivatives i.e. $\sup_n \sup_{x \in [0,2\pi]} |u_n(x)| < \infty$ and $\sup_n \sup_{x \in [0,2\pi]} |u_n'(x)| < \infty$. Show that u_n has a subsequence which converges in $L^2([0, 2\pi])$.

Problem 5

Consider the subspace $H \subset \mathcal{C}[0,2\pi]$ consisting of those continuous functions on $[0,2\pi]$ which satisfy

$$u(x) = \int_0^x U(x),\ \forall\ x \in [0,2\pi] \tag{2}$$

for some $U \in L^2(0,2\pi)$ (depending on u of course). Show that the function U is determined by u (given that it exists) and that

$$\|u\|_H^2 = \int_{(0,2\pi)} |U|^2 \tag{3}$$

turns H into a Hilbert space.

Problem 6

Consider the space of those complex-valued functions on $[0,1]$ for which there is a constant C (depending on the function) such that

$$|u(x) - u(y)| \le C|x-y|^{\frac{1}{2}}\ \forall\ x,y \in [0,1]. \tag{4}$$

Show that this is a Banach space with norm

$$\|u\|_{\frac{1}{2}} = \sup_{[0,1]} |u(x)| + \inf_{(4)\text{ holds}} C. \tag{5}$$

Problem 7

Let $A_j \subset \mathbb{R}$ be a sequence of subsets with the property that the characteristic function, χ_j of A_j, is integrable for each j. Show that the characteristic function of $\mathbb{R} \setminus A$, where $A = \bigcup_j A_j$ is locally integrable.

Problem 8

Let A be a Hilbert-Schmidt operator on a separable Hilbert space H, which means that for some orthonormal basis $\{e_i\}$

$$\sum_i \|Ae_i\|^2 < \infty. \tag{6}$$

Using Bessel's identity to expand $\|Ae_i\|^2$ with respect to another orthonormal basis $\{f_j\}$ show that $\sum\limits_j \|A^* f_j\|^2 = \sum\limits_i \|Ae_i\|^2$. Conclude that the sum in (6) is independent of the othornormal basis used to define it and that the Hilbert-Schmidt operators form a Hilbert space.

Problem 9

Let B_n be a sequence of bounded linear operators on a Hilbert space H such that for each u and $v \in H$ the sequence $(B_n u, v)$ converges in $\mathbb{C}$. Show that there is a uniquely defined bounded operator B on H such that

$$(Bu, v) = \lim_{n\to\infty} (B_n u, v)\ \forall\ u, v \in H.$$

Problem 10

Suppose $P \subset H$ is a closed linear subspace of a Hilbert space, with $\pi_P : H \longrightarrow P$ the orthogonal projection onto P. If H is separable and A is a compact self-adjoint operator on H, show that there is a complete orthonormal basis of H each element of which satisfies $\pi_P A \pi_P e_i = t_i e_i$ for some $t_i \in \mathbb{R}$.

Problem 11

Let $e_j = c_j C^j e^{-x^2/2}$, $c_j > 0$, where $j = 1, 2, \dots$, and $C = -\frac{d}{dx} + x$ is the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. You may assume completeness in $L^2(\mathbb{R})$ and use the facts established in class that $-\frac{d^2 e_j}{dx^2} + x^2 e_j = (2j+1)e_j$, that $c_j = 2^{-j/2}(j!)^{-\frac{1}{2}}\pi^{-\frac{1}{4}}$ and that $e_j = p_j(x)e_0$ for a polynomial of degree j. Compute Ce_j and Ae_j in terms of the basis and hence arrive at a formula for de_j/dx. Use this to show that the sequence $j^{-\frac{1}{2}}\frac{de_j}{dx}$ is bounded in $L^2(\mathbb{R})$. Conclude that if

$$H^1_{\text{iso}} = \{u \in L^2(\mathbb{R}); \sum_{j \ge 1} j|(u, e_j)|^2 < \infty\} \tag{7}$$

then there is a uniquely defined operator $D : H^1_{\text{iso}} \longrightarrow L^2(\mathbb{R})$ such that $De_j = \frac{de_j}{dx}$ for each j.

Problem 12

Suppose that $f \in \mathcal{L}^1(0, 2\pi)$ is such that the constants $c_k = \int_{(0,2\pi)} f(x)e^{-ikx}$, $k \in \mathbb{Z}$, satisfy $\sum\limits_{k\in\mathbb{Z}} |c_k|^2 < \infty$. Show that there is an element $g \in L^2([0, 2\pi])$ such that $\int (f - g)e^{ikx}dx = 0$ for all $k \in \mathbb{Z}$.

Problem 13

Let A be a compact self-adjoint operator on a separable Hilbert space and suppose that for *any* orthonormal basis

$$\sum_i |(Ae_i, e_i)| < \infty. \tag{8}$$

Show that the eigenvalues of A, if infinite in number, form a sequence in l^1.

Problem 14

For $u \in L^2(0, 1)$ show that

$$Iu(x) = \int_0^x u(t)dt, \ x \in (0, 1)$$

is a bounded linear operator on $L^2(0, 1)$. If $V \in \mathcal{C}([0, 1])$, is real-valued and $V \ge 0$, show that there is a bounded linear operator B on $L^2(0, 1)$ such that

$$B^2 u = u + I^* M_V I u \ \forall \ u \in L^2(0, 1) \tag{9}$$

where M_V denotes multiplication by V.