18.102: PROBLEMS FOR TEST1 – 11 OCTOBER, 2007
FIRST CORRECTED VERSION, THANKS TO URS NIESEN

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These 10 questions will reappear on the test on Thursday october 11 when you will be asked to prove 3 of them (my choice, not yours), without using notes or the book. However you may use the earlier problems to prove the later ones – even in the test itself. I will not change this except to make corrections if any are needed.

Recall that the objective is to give an answer which is clear, concise and complete!

(1) Recall Lebesgue's Dominated Convergence Theorem and use it to show that if $u \in L^2(\mathbb{R})$ and $v \in L^1(\mathbb{R})$ then

$$\lim_{N\to\infty}\int_{|x|>N}|u|^2=0,\ \lim_{N\to\infty}\int|C_Nu-u|^2=0,$$
$$\lim_{N\to\infty}\int_{|x|>N}|v|=0 \text{ and } \lim_{N\to\infty}\int|C_Nv-v|=0. \tag{1}$$

where

$$C_Nf(x)=\begin{cases}N & \text{if } f(x)>N\\ -N & \text{if } f(x)<-N\\ f(x) & \text{otherwise.}\end{cases} \tag{2}$$

(2) Show that step functions are dense in $L^1(\mathbb{R})$ and in $L^2(\mathbb{R})$ (Hint:- Look at Q1 and think about $f-f_N$, $f_N=C_Nf\chi_{[-N,N]}$ and its square. So it suffices to show that f_N is the limit in L^2 of a sequence of step functions. Show that if g_n is a sequence of step functions converging to f_N in L^1 then $C_N\chi_{[-N,N]}$ is converges to f_N in L^2.) and that if $f\in L^1(\mathbb{R})$ then there is a sequence of step functions u_n and an element $g\in L^1(\mathbb{R})$ such that $u_n\to f$ a.e. and $|u_n|\le g$.

(3) Show that $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ are separable, meaning that each has a countable dense subset.

(4) Show that the minimum and the maximum of two locally integrable functions is locally integrable.

(5) A subset of $\mathbb{R}$ is said to be (Lebesgue) measurable if its characteristic function is locally integrable. Show that a countable union of measurable sets is measurable. Hint: Start with two!

(6) Define $L^\infty(\mathbb{R})$ as consisting of the locally integrable functions which are bounded, $\sup_{\mathbb{R}}|u|<\infty$. If $\mathcal{N}_\infty\subset L^\infty(\mathbb{R})$ consists of the bounded functions which vanish outside a set of measure zero show that

$$\|u+\mathcal{N}_\infty\|_{L^\infty}=\inf_{h\in\mathcal{N}_\infty}\sup_{x\in\mathbb{R}}|u(x)+h(x)| \tag{3}$$

is a norm on $\mathcal{L}^\infty(\mathbb{R})=L^\infty(\mathbb{R})/\mathcal{N}_\infty$.

(7) Show that if $u \in L^\infty(\mathbb{R})$ and $v \in L^1(\mathbb{R})$ then $uv \in L^1(\mathbb{R})$ and that

$$|\int uv| \le \|u\|_{L^\infty}\|v\|_{L^1}. \tag{4}$$

(8) Show that each $u \in L^2(\mathbb{R})$ is continuous in the mean in the sense that $T_z u(x) = u(x-z) \in L^2(\mathbb{R})$ for all $z \in \mathbb{R}$ and that

$$\lim_{|z|\to 0}\int |T_z u - u|^2 = 0. \tag{5}$$

(9) If $\{u_j\}$ is a Cauchy sequence in $L^2(\mathbb{R})$ show that both (5) and (1) are uniform in j, so given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int |T_z u_j - u_j|^2 < \epsilon,\ \int_{|x|>1/\delta} |u_j|^2 < \epsilon\ \forall\ |z| < \delta \text{ and all } j. \tag{6}$$

(10) Construct a sequence in $L^2(\mathbb{R})$ for which the uniformity in (6) does not hold.

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