18.102: TEST2 – TAKE HOME

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Your work on this take-home exam should be in my hands or inbox by 2:30 on Tuesday, November 20, 2007.

(1) Recall the discussion of the Dirichlet problem for d^2/dx^2 from class and carry out an analogous discussion for the Neumann problem to arrive at a complete orthonormal basis of $L^2([0,1])$ consisting of $\psi_n \in \mathcal{C}^2$ functions which are all eigenfunctions in the sense that

$$\frac{d^2\psi_n(x)}{dx^2} = \gamma_n \psi_n(x)\ \forall\ x \in [0,1],\ \frac{d\psi_n}{dx}(0) = \frac{d\psi_n}{dx}(1) = 0. \tag{NeuEig}$$

This is actually a little harder than the Dirichlet problem which I did in class, because there is an eigenfunction of norm 1 with $\gamma = 0$. Here are some individual steps which may help you along the way!

What is the eigenfunction with eigenvalue 0 for (NeuEig)?

What is the operator of orthogonal projection onto this function?

What is the operator of orthogonal projection onto the orthocomplement of this function?

The crucual part. Find an integral operator $A_N = B - B_N$, where B is the operator from class,

$$(Bf)(x) = \int_0^x (x-s)f(s)ds \tag{B-Def}$$

and B_N is of finite rank, such that if f is continuous then $u = A_N f$ is twice continuously differentiable, satisfies $\int_0^1 u(x)dx = 0$, $A_N 1 = 0$ (where 1 is the constant function) and

$$\begin{gathered}\int_0^1 f(x)dx = 0 \Longrightarrow \\ \frac{d^2u}{dx^2} = f(x)\ \forall\ x \in [0,1],\ \frac{du}{dx}(0) = \frac{du}{dx}(1) = 0.\end{gathered} \tag{GI}$$

Show that A_N is compact and self-adjoint.

Work out what the spectrum of A_N is, including its null space.

Deduce the desired conclusion.

(2) Show that these two orthonormal bases of $L^2([0,1])$ (the one above and the one from class) can each be turned into an orthonormal basis of $L^2([0,\pi])$ by change of variable.

(3) Construct an orthonormal basis of $L^2([-\pi,\pi])$ by dividing each element into its odd and even parts, resticting these to $[0,\pi]$ and using the Neumann basis above on the even part and the Dirichlet basis from class on the odd part.

(4) Prove the basic theorem of Fourier series, namely that for any function $u \in L^2([-\pi, \pi])$ there exist unique constants $c_k \in \mathbb{C}$, $k \in \mathbb{Z}$ such that

$$u(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx} \text{ converges in } L^2([-\pi, \pi]) \tag{FS}$$

and give an integral formula for the constants.

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