

PROBLEM SET 6, FOR 18.102, FALL 2007
DUE THURSDAY 8 NOV, AT 2:30 IN CLASS (2-102).

All homework should either be submitted to me by email (it does not have to be in TeX, scanned written pages are fine) or on paper of reasonable quality, so it goes through the scanner! If you need decent paper I am happy to supply some.

- (1) From the open mapping theorem proved in class, deduce that if $T : X \rightarrow Y$ is a continuous linear map between Banach spaces which is a linear isomorphism (meaning injective and surjective) then its inverse is continuous and hence it is a topological isomorphism.
- (2) Show that if X and Y are normed spaces then the ‘direct product’ $X \times Y = \{(x, y); X \in X, y \in Y\}$ is a normed space by combining the norms on X and Y . Show that if X and Y are Banach spaces then $X \times Y$ is a Banach space.
- (3) Show that the projection map $X \times Y \ni (x, y) \mapsto x \in X$ is continuous.
- (4) If $T : X \rightarrow Y$ is a map between sets, its graph is

$$G(T) = \{(x, y) \in X \times Y; y = T(x)\}.$$

Show that if T is a continuous linear map between normed spaces (i.e. bounded) then $G(T)$ is a closed linear subspace of $X \times Y$.

- (5) Suppose $T : X \rightarrow Y$ is a linear map, show that the map

$$\tilde{T} : X \rightarrow G(T), \tilde{T}(x) = (x, Tx) \in G(T)$$

is a linear isomorphism, meaning it is injective and surjective.

- (6) Now, suppose that $T : X \rightarrow Y$ is a linear map between Banach spaces with the property that its graph $G(T) \subset X \times Y$ is closed. Show that $G(T)$ is a Banach space and (by applying the earlier parts) that \tilde{T} in the previous question is a topological isomorphism and hence is continuous.
- (7) Finally, deduce the Closed Graph Theorem. If $T : X \rightarrow Y$ is a linear map between Banach spaces with $G(T)$ closed then T is bounded. [Hint:- Write T as the composite of \tilde{T} and a continuous map.]
- (8) Debnath and Mikusiński, 3rd Edition p.211 no 5
- (9) Debnath and Mikusiński, 3rd Edition p.211 no 6
- (10) Debnath and Mikusiński, 3rd Edition p.211 no 10